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SOME HIGHER-ORDER METHODS FOR THE SIMULTANEOUS APPROXIMATION OF MULTIPLE POLYNOMIAL ZEROS

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Abstract—Applying Newton's and Halley's corrections, some modified methods of higher order for the simultaneous approximation of multiple zeros of a polynomial are derived. Further acceleration of convergence of these methods is performed by approximating all zeros in a serial fashion using the new approximations as they become available. Faster convergence is attained without additional calculations. Lower bounds of the R-order of convergence for the serial (single-step) methods are given.

1. INTRODUCTION

Iterative methods for the simultaneous determination of multiple zeros of a polynomial, most frequently developed as modifications of the known methods for simple zeros, have been investigated rarely in the literature. A classical Weierstrass result [1] was applied in Refs [2, 3] for determining multiple polynomial zeros $r_1, \ldots, r_v$, when the order of multiplicity of these zeros $\mu_1, \ldots, \mu_v$ is known. The corresponding methods have quadratic convergence, but their efficiency is low because besides the evaluation of the values of polynomial P, they require additional evaluation of the values of the derivatives $P', \ldots, P^{(\mu-1)}$, where

$$\mu = \max_i \mu_i .$$

The iterative formulas of Weierstrass' type have been also studied in Refs [4–6]. These formulas are considerably simpler than the previously mentioned ones, but a requirement for the extraction of the μ_ith root of a complex number in determining an approximation to the zero r_i of the multiplicity μ_i leads to the problem of choice of the appropriate value of a root (among μ_i values).

The iterative methods developed using the logarithmic derivatives of a polynomial [7–9], are more suitable for the simultaneous determination of multiple zeros of a polynomial. Some of them have been considered in Refs [4, 6, 9].

Over the last decade, a lot of attention has been paid to formulating the iterative methods for the simultaneous inclusion of complex zeros of a polynomial using circular arithmetic. Some results, presented in Refs [9–12], related to multiple polynomial zeros, have been used for deriving the simultaneous methods for multiple zeros in ordinary ('point') arithmetic.

In this paper some modifications of the known simultaneous methods of high order, which provide the acceleration of convergence of the basic methods using Newton's and Halley's corrections and the Gauss–Seidel approach, are presented. The new formulas are analyzed using the results given in Refs [13–15].

2. BASIC METHODS

Consider a monic polynomial P of degree $n \geqslant 3$,

$$P(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{v} (z - r_j)^{\mu_j} \quad (a_i \in C)$$

with real or complex zeros $r_1, \ldots, r_\nu$ having the order of multiplicity $\mu_1, \ldots, \mu_\nu$ respectively, where $\mu_1 + \cdots + \mu_\nu = n$ ($\nu > 1$). Let $z_1, \ldots, z_\nu$ be distinct reasonably good approximations to these zeros, and $\hat{z}_i$ be the next approximation to r_i obtained by using some iterative scheme. Let us define the rational function $z \mapsto f_k(z)$ by

$$f_k(z) = \frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left(\frac{P'(z)}{P(z)} \right) \quad (k = 1, 2, \ldots).$$

It is easy to show that

$$f_k(z) = \sum_{j=1}^{\nu} \mu_j (z - r_j)^{-k}. \tag{1}$$

For $z = z_i$ in equation (1), we find

$$r_i = z_i - \frac{(\mu_i)^{1/k}}{\left[f_k(z_i) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - r_j)^{-k} \right]_*^{1/k}}. \tag{2}$$

The symbol $*$ in equation (2) denotes that the 'appropriate' (among k-values) kth root of a complex number has to be chosen. Setting $r_i := z_i$ and $r_j := z_j$ ($j \neq i$) in equation (2), we obtain the total-step iterative processes (TS) of root type for the simultaneous determination of multiple polynomial zeros [9]:

$$\hat{z}_i = z_i - \frac{(\mu_i)^{1/k}}{\left[f_k(z_i) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - z_j)^{-k} \right]_*^{1/k}} \quad (i = 1, \ldots, \nu). \tag{3}$$

From a practical point of view, the iterative formulas

$$\hat{z}_i = z_i - \frac{\mu_i}{f_1(z_i) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - z_j)^{-1}} \quad (i = 1, \ldots, \nu), \tag{4}$$

and

$$\hat{z}_i = z_i - \frac{(\mu_i)^{1/2}}{\left[f_2(z_i) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - z_j)^{-2} \right]_*^{1/2}} \quad (i = 1, \ldots, \nu), \tag{5}$$

obtained from equation (3) for $k = 1$ and $k = 2$, have the greatest importance. The functions f_1 and f_2 in the above formulas are given by

$$f_1(z) = \frac{P'(z)}{P(z)} \quad \text{and} \quad f_2(z) = \frac{P'(z)^2 - P(z)P''(z)}{P(z)^2}.$$

Iterative method (4) has a convergence order equal to 3 and can be obtained from the iterative interval method introduced by Gargantini [10] (see also Ref. [4]). Let us note that, if all zeros are simple, then formula (4) defines the iterative method which is often called Ehrlich's method [16], although this formula appeared earlier in papers by Maehly [8] and Börsch-Supan [17] (see also Refs [7, 18]).

Iterative method (5) can be obtained from the interval method for multiple zeros, presented in Ref. [11]. The convergence order of this method is 4. The symbol $*$ in formula (5) denotes that one of two values of the square root has to be chosen. A criterion for the choice of appropriate values of the square root can be established following Gargantini [11].

3. SELECTION OF THE VALUES OF THE ROOT

Introducing the notations

$$\epsilon_i = |z_i - r_i|, \quad \epsilon = \max_{1 \leq i \leq \nu} \epsilon_i,$$

$$d = \min_{\substack{i,j \\ i \neq j}} (|z_i - z_j| - \epsilon_j)$$

and

$$Q_i = \mu_i [f_2(z_i) - \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - r_j)^{-2}] \quad (i = 1, \ldots, \nu),$$

and considering the identity (2) for $k = 2$, we observe that, if all roots but r_i are known, then the unknown zero r_i is equal to one of two values of

$$z_i - \mu_i / Q_i^{1/2}. \tag{6}$$

A criterion for the selection of the appropriate value of the square root is given in the following lemma.

Lemma 1

If $d > (n-1)\epsilon$, then the value of the square root to be chosen in expression (6) is that which satisfies

$$|f_1(z_i) - Q_i^{1/2}| \leq \frac{n - \mu_i}{d}.$$

Proof. We first prove that $P'(z_i) \neq 0$ for $i = 1, \ldots, \nu$. Using equation (1) for $k = 1$ and the inequalities $d > (n-1)\epsilon$ and $|z_i - r_j| \geq d$, we obtain

$$|f_1(z_i)| = \left| \frac{P'(z_i)}{P(z_i)} \right| = \left| \sum_{j=1}^{\nu} \mu_j (z_i - r_j)^{-1} \right| \geq \mu_i |z_i - r_i|^{-1} - \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j |z_i - r_j|^{-1}$$

$$\geq \frac{\mu_i}{\epsilon} - \frac{n - \mu_i}{d} = \frac{d\mu_i - (n - \mu_i)\epsilon}{\epsilon d} > 0,$$

wherefrom it follows that $P'(z_i) \neq 0$ for $i = 1, \ldots, \nu$.

Denoting the value of $Q_i^{1/2}$ equal to $\mu_i (z_i - r_i)^{-1}$ with $\{q_i^{(1)}\}^{1/2}$; the other value of $Q_i^{1/2}$ is, obviously, $\{q_i^{(2)}\}^{1/2} = -\{q_i^{(1)}\}^{1/2}$. Since $P'(z_i) \neq 0$ and $|z_i - r_j| \geq d$, we find

$$|f_1(z_i) - \{q_i^{(1)}\}^{1/2}| = \left| \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - r_j)^{-1} \right| \leq \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j |z_i - r_j|^{-1} \leq \frac{n - \mu_i}{d}.$$

For the other value, $\{q_i^{(2)}\}^{1/2} = -\{q_i^{(1)}\}^{1/2}$, we have

$$|f_1(z_i) - (-\{q_i^{(1)}\}^{1/2})| = \left| 2\mu_i (z_i - r_i)^{-1} + \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j (z_i - r_j)^{-1} \right|$$

$$\geq 2\mu_i |z_i - r_i|^{-1} - \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \mu_j |z_i - r_j|^{-1} \geq \frac{2\mu_i}{\epsilon} - \frac{n - \mu_i}{d} = \frac{2\mu_i d - (n - \mu_i)\epsilon}{\epsilon d}.$$

To prove Lemma 1, it is sufficient to show that the inequality

$$\frac{2\mu_i}{\epsilon} - \frac{n - \mu_i}{d} > \frac{n - \mu_i}{d}$$

holds. Using the inequality $d > (n-1)\epsilon$, we find

$$\frac{2\mu_i}{\epsilon} - \frac{n-\mu_i}{d} > \frac{2\mu_i(n-1)}{d} - \frac{n-1}{d} = (2\mu_i - 1)\frac{(n-1)}{d} \geqslant \frac{n-\mu_i}{d}.$$

Let $w_i^{(1)}$ and $w_i^{(2)}$ be the values of the square root of Q_i in the (practical) case when the zeros r_j on the r.h.s. of equation (2) are substituted by their approximations. According to Lemma 1, the selection of the 'proper' value of the square root, $w_i^{(1)}$ or $w_i^{(2)}$, is reduced to the choice of the value which minimizes the expression

$$|f_1(z_i) - w_i^{(\lambda)}| \quad (\lambda \in \{1, 2\}).$$

Note that, if all zeros of polynomial P are real, then the criterion established by Lemma 1 gives the choice of sign of (real value) $P(z_i)P'(z_i)$.

Finally, we note that the iterative method (5) is a modification of the square root method

$$\hat{z}_i = z_i - \frac{1}{[f_2(z_i)]^{1/2}}$$

with cubic convergence. This method was analyzed in detail by Ostrowski [19], and is often called Ostrowski's method. Accordingly, the function $f_2(z)$ will be called Ostrowski's function in the following.

4. MODIFIED METHODS

Let μ be the multiplicity of the zero r of P. By means of

$$f_1(z) = \frac{P'(z)}{P(z)} \quad \text{and} \quad g(z) = \frac{P''(z)}{P'(z)}$$

we define

$$f_2(z) = f_1(z)[f_1(z) - g(z)] \quad \text{(Ostrowski's function)},$$

$$N(z) = -\frac{\mu}{f_1(z)} \quad \text{(Newton's correction)} \tag{7}$$

and

$$H(z) = 2\left[g(z) - \left(1 + \frac{1}{\mu}\right)f_1(z)\right]^{-1} \quad \text{(Halley's correction).} \tag{8}$$

We recall that the correction terms (7) and (8) appear in the iterative formulas

$$\hat{z} = z_i + N(z_i) \quad \text{(Schröders [20] modifications of Newton's method for multiple zeros)} \tag{9}$$

and

$$\hat{z} = z_i + H(z_i) \quad \text{(modification of Halley's method, introduced by Hansen and Patrick [21] for multiple zeros),} \tag{10}$$

with the convergence orders 2 and 3, respectively. We note that the order of multiplicity in the iterative formulas (9) and (10) takes the values $\mu_1, \ldots, \mu_v$.

Setting $r_i := \hat{z}_i$ in equation (2) and taking some approximations of r_j on the r.h.s. of the identity (2) (for $k = 1$ and $k = 2$), some modified iterative methods for the simultaneous determination of multiple complex zeros of a polynomial can be obtained. The convergence order of these methods is higher compared to that of the basic methods and is attained without additional calculations.

(1) Taking $r_j := \hat{z}_j (j < i)$ and $r_j := z_j (j > i)$ in equation (2) (for $k = 1$ and $k = 2$), we get the single-step method (SS):

$$\hat{z}_i = z_i - \frac{(\mu_i)^{1/k}}{\left[f_k(z_i) - \sum_{j<i} \mu_j (z_i - \hat{z}_j)^{-k} - \sum_{j>i} \mu_j (z_i - z_j)^{-k}\right]^{1/k}} \quad (i = 1, \ldots, v;\ k = 1, 2). \tag{11}$$

(2) Putting $r_j := z_j + N(z_j)$ $(i \neq j)$ in equation (2), where $N(z_j)$ is Newton's correction given by equation (7), we obtain the total-step method with Newton's correction (TSN):

$$\hat{z}_i = z_i - \frac{(\mu_i)^{1/k}}{\left\{f_k(z_i) - \sum_{j \neq i} \mu_j [z_i - z_j - N(z_j)]^{-k}\right\}^{1/k}} \quad (i = 1, \ldots, v; k = 1, 2). \tag{12}$$

(3) The iterative process (12) can be accelerated by approximating all zeros in a serial fashion, i.e. using new approximations as soon as they become available (the so-called Gauss–Seidel approach). In this way, substituting $r_j := \hat{z}_j (j < i)$, $r_j := z_j + N(z_j)$ $(j > i)$ in equation (2), we derive the single-step method with Newton's correction (SSN):

$$\hat{z}_i = z_i - \frac{(\mu_i)^{1/k}}{\left\{f_k(z_i) - \sum_{j<i} \mu_j (z_i - \hat{z}_j)^{-k} - \sum_{j>i} \mu_j [z_i - z_j - N(z_j)]^{-k}\right\}^{1/k}}$$

$$(i = 1, \ldots, v; k = 1, 2). \tag{13}$$

(4) As for the TSN method, we can apply Halley's correction (8) for multiple zeros. Taking $r_j := z_j + H(z_j)$ $(j \neq i)$ in equation (2) for $k = 2$, we obtain the total-step method with Halley's correction (TSH):

$$\hat{z}_i = z_i - \frac{\sqrt{\mu_i}}{\left\{f_2(z_i) - \sum_{j \neq i} \mu_j [z_i - z_j - H(z_j)]^{-2}\right\}^{1/2}} \quad (i = 1, \ldots, v). \tag{14}$$

(5) Finally, setting $r_j := \hat{z}_j (j < i)$, $r_j := z_j + H(z_j)$ $(j > i)$ in equation (2) for $k = 2$, we obtain the single-step method with Halley's correction (SSH):

$$\hat{z}_i = z_i - \frac{\sqrt{\mu_i}}{\left\{f_2(z_i) - \sum_{j<i} \mu_j (z_i - \hat{z}_j)^{-2} - \sum_{j>i} \mu_j [z_i - z_j - H(z_j)]^{-2}\right\}^{1/2}} \quad (i = 1, \ldots, v). \tag{15}$$

Note that an application of Halley's correction in the basic methods (4) decreases the efficiency of the modified method. Namely, to evaluate $H(z_j)$ it is necessary to evaluate the second derivative of the polynomial, which is not required for the basic method (8). Therefore, Halley's correction has been used only in the case of the square root method (5), where the functions $f_2(z)$ and the correction $H(z)$ are evaluated using the same functions $f_1(z)$ and $g(z)$.

5. CONVERGENCE ANALYSIS OF MODIFIED METHODS

In this section we will consider the convergence order of the modified iterative schemes, formulated in Section 4. For the single-step methods, where new approximations are used immediately they become available, the definition of the R-order of convergence [22] is applied. The R-order of convergence of the iterative process IP with the limit point $\mathbf{r} = [r_1 \ldots r_v]^T$, where $r_1, \ldots, r_v$ are the zeros of a polynomial, will be denoted by $O_R(\mathrm{IP}, \mathbf{r})$.

Let $u_i^{(m)}$ be a multiple of $|z_i^{(m)} - r_i|$ $(i = 1, \ldots, v)$, where $m = 0, 1, \ldots$ is the iteration index. The following relations can be derived for a class of iterative simultaneous methods which includes the algorithms (11)–(15):

$$u_i^{(m+1)} \leqslant \frac{1}{n-1} u_i^{(m)p} \left(\alpha \sum_{j<i} u_j^{(m+1)} + \sum_{j>i} u_j^{(m)q} \right) \quad (i = 1, \ldots, v; p, q \in N). \tag{16}$$

For brevity, we introduce the ordered triplet

$$U(\mathrm{IP}) = (p, \alpha, q) \quad (p, q \in N; \alpha \in \{0, 1\})$$

as a characteristics of the relations (16) for the iterative process IP. The components p and q of $U(\mathrm{IP})$ are equal to the exponents of $u_i^{(m)}$ and $u_j^{(m)}$, respectively. We observe that $\alpha = 0$ in the case of a TS method and $\alpha = 1$ for an SS method.

Theorem 1

Assume that the starting approximations $z_1^{(0)}, \dots, z_\nu^{(0)}$ are chosen sufficiently close to the zeros $r_1, \dots, r_\nu$ so that

$$u_i^{(0)} \leqslant u = \max_{1 \leqslant i \leqslant \nu} u_i^{(0)} < 1. \tag{17}$$

Then, for the iterative process IP with $U(\mathrm{IP}) = (p, \alpha, q)$ we have

$$O_R(\mathrm{IP}, \mathbf{r}) = p + q \quad \text{if } \alpha = 0 \quad \text{(total-step methods)},$$

$$O_R(\mathrm{IP}, \mathbf{r}) \geqslant p + t_\nu \quad \text{if } \alpha = 1 \quad \text{(single-step methods)},$$

where t_ν is the unique positive root of the equation

$$t^\nu - tq^{\nu-1} - pq^{\nu-1} = 0. \tag{18}$$

Proof. Taking into consideration the condition (17), from the inequalities (16) we find that the sequences $\{u_i^{(m)}\}$ $(i = 1, \dots, \nu)$ converge to 0 when $m \to +\infty$. Hence, $z_i^{(m)} \to r_i$ when $m \to +\infty$, which means that the iterative processes, characterized by relations (16) and (17), are convergent.

For the total-step methods ($\alpha = 0$), from relations (16) we obtain

$$u_i^{(m+1)} \leqslant \frac{1}{n-1} u_i^{(m)p} \sum_{j \neq i} u_j^{(m)q} \quad (i = 1, \dots, \nu). \tag{19}$$

Let $u_j^{(m)} = c_{ji}^{(m)} u_i^{(m)} \leqslant c_{ji} u_i^{(m)}$, where c_{ij} is a postive constant independent on the iteration index m. From inequalities (19) it follows that

$$u_i^{(m+1)} \leqslant d_i u_i^{(m)p+q},$$

where

$$d_i = \frac{1}{n-1} \sum_{j \neq i} (c_{ji})^q.$$

Thus, the order of convergence of the total-step methods IP, with $U(\mathrm{IP}) = (p, 0, q)$, is $p + q$.

Following Alefeled and Herezberger [23, 24], it can be shown from relations (16) and (17) that

$$u_i^{(m+1)} \leqslant u^{s_i^{(m+1)}} \quad (i = 1, \dots, \nu;\ m = 0, 1, \dots).$$

The vectors $\mathbf{s}^{(m)} = [s_1^{(m)} \dots s_\nu^{(m)}]^{\mathrm{T}}$ are successively computed by

$$\mathbf{s}^{(m+1)} = \mathbb{A}_\nu(p, q)\, \mathbf{s}^{(m)} \quad (m = 0, 1, \dots) \tag{20}$$

starting with $\mathbf{s}^{(0)} = [1 \dots 1]^{\mathrm{T}}$. The matrix $\mathbb{A}_\nu(p, q)$ in equation (20) is given by

$$\mathbb{A}_\nu(p, q) = \begin{bmatrix} p & q & & & & & & \\ & p & q & & & & & \\ & & p & q & & & & \\ & & & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & p & q \\ p & q & 0 & 0 & \dots & & 0 & p \end{bmatrix} \quad (p, q \in N).$$

The characteristic polynomial of the matrix $\mathbb{A}_v(p,q)$ is

$$v(\lambda) = (\lambda - p)^v - (\lambda - p)\, q^{v-1} - pq^{v-1}.$$

Substituting $t = \lambda - p$, we obtain

$$y(t) = v(t+p) = t^v - tq^{v-1} - pq^{v-1}.$$

It is easy to see that the graph of the function $y_1(t) = \; = t^v$ and the straight line $y_2(t) = tq^{v-1} + pq^{v-1}$ intersect only at one point for $t > 0$. Since $y(q) = -pq^{v-1} < 0$, $y(p+q) = (p+q)^v - 2pq^{v-1} - q^v > 0$, we conclude that the intersecting point belongs to the interval $(q, p+q)$. Therefore, the equation

$$t^v - tq^{v-1} - pq^{v-1} = 0$$

has the unique positive root $t_v > q$. The corresponding (positive) eigenvalue of the matrix $\mathbb{A}_v(p,q)$ is $p + t_v$. The matrix $\mathbb{A}_v(p,q)$ is nonnegative and its directed graph is strongly connected [25, p. 20], i.e. $\mathbb{A}_v(p,q)$ is irreducible. By the Perron–Frobenius theorem [25, p. 30] it follows that $\mathbb{A}_v(p,q)$ has a positive eigenvalue equal to its spectral radius $\rho\,[\mathbb{A}_v(p,q)]$. According to the analysis presented in Ref. [23, pp. 240–241] it can be shown that the lower bound of the R-order of convergence of the SS method IP, for which the inequalities (16) and (17) are valid, is given by the spectral radius $\rho\,[\mathbb{A}_v(p,q)]$. Therefore, for the SS methods IP with $U(p,1,q)$, we have

$$O_R(\mathrm{IP}, \mathbf{r}) \geqslant \rho\,[\mathbb{A}_v(p,q)] = p + t_v,$$

where t_v is the unique positive root of the equation

$$t^v - tq^{v-1} - pq^{v-1} = 0.$$

The following theorem gives a narrower interval of the lower bound of the R-order of convergence compared to $(q, p+q)$ and the dependence of this bound on the number of different zeros v.

Theorem 2

The lower bound of the R-order of convergence of the iterative method IP with $U(p,1,q)$ is higher as the number of different zeros of a polynomial v becomes smaller, and is bounded by

$$p + q = \rho\,[\mathbb{A}_\infty(p,q)] < \rho\,[\mathbb{A}_v(p,q)] \leqslant \rho\,[\mathbb{A}_2(p,q)] = p + q + \frac{2p}{1 + \sqrt{1 + \dfrac{4p}{q}}}. \tag{21}$$

Proof. Since $\rho\,[\mathbb{A}_v(p,q)] = p + t_v$, where t_v is the unique positive zero of the polynomial

$$y(t) = t^v - tq^{v-1} - pq^{v-1}, \tag{22}$$

it is sufficient to prove that the sequence $\{t_v\}(v = 2, 3, \ldots)$ of the zeros of polynomial (22) is monotonically decreasing and that

$$q < t_v \leqslant + \frac{2p}{1 + \sqrt{1 + \dfrac{4p}{q}}}$$

holds for each $v \geqslant 2$.

Equation (18) can be rearranged in the form

$$x - \left(x + \frac{p}{q}\right)^{1/v} = 0, \tag{23}$$

where $x = t/q$. Let

$$a(v, x) = x - \left(x + \frac{p}{q}\right)^{1/v}.$$

Denote by x_ν the zero of the function $a(\nu, x)$ in the case when ν is fixed. Since $q < t_\nu < p + q$ and $t_\nu = qx_\nu$, it follows $1 < x_\nu < 1 + p/q$. For fixed ν, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} a(\nu, x) = 1 - \frac{1}{\nu\left(x + \frac{p}{q}\right)^{1-1/\nu}} > 0,$$

which means that $x \mapsto a(\nu, x)$, for fixed ν, is a monotonically increasing function for $x > 1$. Further, since

$$\frac{\mathrm{d}}{\mathrm{d}y} a(y, x) = \frac{1}{y^2}\left(x + \frac{p}{q}\right)^{1/y} \ln\left(x + \frac{p}{q}\right) > 0 \quad (\text{for fixed } x > 1),$$

we conclude that

$$a(\nu + 1, x) > a(\nu, x) \quad \text{for all } x > 1 \quad \text{and} \quad \nu = 2, 3, \ldots. \tag{24}$$

According to the monotonicity of the function $x \mapsto a(\nu, x)$ (ν is fixed) and inequality (24), it follows that

$$x_{\nu+1} < x_\nu \quad (\nu = 2, 3, \ldots),$$

wherefrom

$$t_{\nu+1} < t_\nu \quad (\nu = 2, 3, \ldots).$$

Thus, the sequence $\{t_\nu\}$ is monotonically decreasing. The upper bound of this sequence is the positive solution of equation $t^2 - tq - pq = 0$, i.e.

$$t_2 = \frac{q + \sqrt{q^2 + 4pq}}{2} = q + \frac{2p}{1 + \sqrt{1 + \frac{4p}{q}}}.$$

From equation (23) we have

$$\ln x_\nu = \frac{1}{\nu} \ln\left(x_\nu + \frac{p}{q}\right) \to 0 \quad \text{when } \nu \to +\infty,$$

which means that $x_\nu \to 1$ and $t_\nu = qx_\nu \to q$ when $\nu \to +\infty$. Taking into consideration the monotonocity of the sequence $\{t_\nu\}$, we obtain

$$q < t_\nu \leqslant q + \frac{2p}{1 + \sqrt{1 + \frac{4p}{q}}},$$

wherefrom inequality (21) is obtained.

On the basis of the results concerning the total-step methods for finding simple zeros of a polynomial, presented in Refs [13] (TSN for $k = 1$) and [15] (TSN and TSH for $k = 2$), one obtains

$$U(\text{TSN}) = (k + 1, 0, 2) \quad (k = 1, 2)$$

and

$$U(\text{TSH}) = (3, 0, 3).$$

According to this and Theorem 1, we have the following assertion.

Theorem 3

The convergence order is equal to $k + 3$ ($k = 1, 2$) for algorithm TSN (12) and 6 for algorithm TSH (14).

Using the results concerning the single-step methods for finding simple zeros of a polynomial, given in Refs [14] (SS for $k = 1$ and $k = 2$), [13] (SSN for $k = 1$) and [15] (SSN and SSH for $k = 2$), we find

$$U(\text{SS}) = (k + 1, 1, 1) \quad (k = 1, 2)$$

$$U(\text{SSN}) = (k + 1, 1, 2) \quad (k = 1, 2)$$

and

$$U(\text{SSH}) = (3, 1, 3).$$

On the basis of this and Theorem 1, Theorem 4 follows.

Theorem 4

The R-order of convergence of the single-step methods SS, SSN and SSH [algorithms (11), (13) and (15), respectively] is given by

$$O_R(\text{SS}, \mathbf{r}) \geqslant k + 1 + t_v \quad (k = 1, 2),$$
$$O_R(\text{SSN}, \mathbf{r}) \geqslant k + 1 + x_v \quad (k = 1, 2)$$

and

$$O_R(\text{SSH}, \mathbf{r}) \geqslant 3 + y_v,$$

where t_v, x_v and y_v are the unique positive roots of the equations

$$t^v - t - k - 1 = 0 \quad (k = 1, 2),$$
$$x^v - x \cdot 2^{v-1} - (k + 1) \cdot 2^{v-1} = 0 \quad (k = 1, 2)$$

and

$$y^v - y \cdot 3^{v-1} - 3^v = 0,$$

respectively.

The values of the lower bounds of the R-order of convergence in the case of the single-step methods [algorithms (11), (13) (for $k = 1, 2$) and (15) (for $k = 2$)] for $v = 2(1)10$ are displayed in Table 1. These values (for $v > 2$) have been determined solving the equations of the form (18). Note that, sometimes, in determining the lower bound of the R-order of convergence (given by the spectral radius) it is more suitable to use a method for finding the dominant eigenvalue of the matrix, such as the power method, instead of a procedure for finding the dominant zero of the corresponding characteristic polynomial. In the case of equation (18) we know the interval which contains the wanted dominant zero (see Theorem 2) and that the characteristic polynomial is monotonic over the mentioned interval $[q, q + 2p/(1 + \sqrt{1 + 4p/q})]$ [namely, $y'(t) = vt^{v-1} - q^{v-1} > 0$ for $t > q$].

The acceleration of convergence of the single-step methods [algorithms (11), (13) and (15)], compared with the corresponding total-step methods [algorithms (3), (12) and (14)], is greater if the number of different zeros of a polynomial v is smaller (Theorem 2). This acceleration is attained without additional calculations; moreover, the single-step methods require less computer storage (because the calculated approximations immediately take the positions of the former ones).

6. NUMERICAL RESULTS

In practice, it is suitable to use a three-stage globally convergent composite algorithm [4] consisting of:

(1) Find an inclusion region of the complex plane which contains all the zeros of a polynomial (see, for example, [26–33]).

Table 1

Method	v = 2	3	4	5	6	7	8	9	10
SS (11), $k = 1$	4.000	3.521	3.353	3.267	3.215	3.180	3.154	3.135	3.121
SS (11), $k = 2$	5.303	4.672	4.453	4.341	4.274	4.229	4.196	4.172	4.153
SSN (13), $k = 1$	5.236	4.649	4.441	4.335	4.269	4.226	4.194	4.170	4.152
SSN (13), $k = 2$	6.646	5.862	5.585	5.443	5.357	5.299	5.257	5.225	5.200
SSH (15), $k = 2$	7.854	6.974	6.662	6.502	6.404	6.338	6.291	6.255	6.227

(2) Apply a slowly convergent search algorithm to obtain starting approximations to the zeros and calculate their multiplicities. The multiplicities of these approximations can be estimated by [34]

$$\mu_i = \frac{1}{\lim\limits_{z_i \to r_i} \left[1 - \dfrac{g(z_i)}{f_1(z_i)}\right]}.$$

(See also [35, 36].)

(3) Improve these approximations by applying a rapidly convergent iterative method, for example, by using any of the algorithms (11)–(15), to any required accuracy.

In order to test the presented iterative schemes, a Fortran routine was realized on a HONEYWELL 66 system in double-precision arithmetic (about 18 significant decimal digits). In realizing the TSN, SSN, TSH and SSH methods with Newton's and Halley's corrections, before calculating new approximations $z_i^{(m+1)}$ the values $f_1(z_i^{(m)})$ and $g(z_i^{(m)})$ $(m = 0, 1, \dots)$ were calculated. The same values are used for calculating Newton's correction

$$N(z_i^{(m)}) = -\frac{\mu_i}{f_1(z_i^{(m)})} \quad (\text{for } k = 1 \text{ and } k = 2),$$

Ostrowski's function

$$f_2(z_i^{(m)}) = f_1(z_i^{(m)})[f_1(z_i^{(m)}) - g(z_i^{(m)})] \quad (\text{for } k = 2)$$

and Halley's correction

$$H(z_i^{(m)} = 2\left[g(z_i^{(m)}) - \left(1 + \frac{1}{\mu_i}\right) f_1(z_i^{(m)})\right]^{-1} \quad (\text{for } k = 2).$$

Thus, the proposed iterative methods with Newton's and Halley's correction terms require slightly more numerical operations in relation to the basic methods, algorithms (4) and (5). Taking into account the significantly increased order of convergence, it is obvious that the proposed methods have a greater efficiency.

Table 2

	i	$\mathcal{Re}\{z_i^{(2)}\}$	$\mathcal{Im}\{z_i^{(2)}\}$
	1	0.999999853800923892	2.000000112716998844
TS	2	0.999999826741999847	−2.000000351383949125
(5)	3	−0.999999859207295616	-8.18×10^{-7}
	4	3.000000527270300803	-3.48×10^{-8}
	1	0.999999939617346251	1.999999964305993363
SS	2	1.000000861310650873	−2.000000509862992614
(11)	3	−0.999999999709498985	1.35×10^{-9}
	4	3.000000000000030662	7.16×10^{-14}
	1	0.999999455077856744	2.000000212961094747
TSN	2	1.000000018147137107	−2.000000068835695135
(12)	3	−0.999999974528732211	3.43×10^{-8}
	4	3.000000722708680682	-9.58×10^{-8}
	1	0.999999894885117145	2.000000042747320793
SSN	2	0.999999994177457521	−2.000000000709903145
(13)	3	−1.000000000007845003	3.82×10^{-11}
	4	2.999999999999997525	-6.58×10^{-15}
	1	1.000000000098386276	1.999999999890580897
TSH	2	1.000000000450329186	−2.000000000521585396
(14)	3	−0.999999999986166747	-2.93×10^{-12}
	4	3.000000000368704406	-6.92×10^{-10}
	1	1.000000000032764666	2.000000000002146278
SSH	2	1.000000000000674921	−1.999999999997025086
(15)	3	−1.000000000000001322	2.74×10^{-15}
	4	3.000000000000000383	-2.01×10^{-16}

In order to illustrate numerically the efficiency of the modified methods, the algorithms TS (5), SS (11), TSN (12), SSN (13), TSH (14) and SSH (15) of square root type were applied for the determination of zeros of the polynomial

$$P(z) = z^9 - 7z^8 + 20z^7 - 28z^6 - 18z^5 + 110z^4 - 92z^3 + 44z^2 + 345z + 225.$$

The exact zeros of this polynomial are $r_1 = 1 + 2i$, $r_2 = 1 - 2i$, $r_3 = -1$ and $r_4 = 3$, with the multiplicities $\mu_1 = 2$, $\mu_2 = 2$, $\mu_3 = 3$ and $\mu_4 = 2$. As initial approximations to these zeros the following complex numbers were taken:

$$z_1^{(0)} = 1.8 + 2.7i, z_2^{(0)} = 1.8 - 2.7i, z_3^{(0)} = -0.3 - 0.8i, z_4^{(0)} = 2.3 - 0.7i.$$

In spite of crude initial approximations

$$(\min_i |z_i^{(0)} - r_i| \cong 1),$$

the modified methods demonstrate very fast convergence. Numerical results, obtained in the second iteration, are given in Table 2.

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