FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. 1 (1986), 7—11

AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS. II

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Abstract. Let W_n be the set of all algebraic polynomials of exact degree n, whose coefficients are all nonnegative. For the norm in $L^2[0, \infty)$ with Freud's weight function, the extremal problem (1. 2) is considered.

1. Introduction

In a previous paper G. V. Milovanović found a complete solution of the following problem of A. K. Varma [2]:

Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which nonnegative, i.e.,

$$W_n = \left\{ P_n \,|\, P_n(x) = \sum_{k=0}^{n} a_k x^k,\ a_k \geqq 0 \ \ (k = 0, 1, \ldots, n-1),\ a_n > 0 \right\} \tag{1.1}$$

and let $||f||^2 = (f, f)$, where

$$(f, g) = \int_0^\infty w(x) f(x) g(x)\, dx \qquad (f, g \in L^2[0, \infty)),$$

with generalized Laguerre weight function $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$.

Determine the best constant in the inequality

$$||P_n'||^2 \leqq C_n(\alpha) ||P_n||^2 \qquad (P_n \in W_n),$$

i.e.,

$$C_n(\alpha) = \sup_{P_n \in W_n} \frac{||P_n'||^2}{||P_n||^2}. \tag{1.2}$$

Namely, Milovanović [1] proved the following result:

Received December 8, 1985.

Theorem A. *The best constant* $C_n(\alpha)$ *defined in* (1.2) *is*

$$C_n(\alpha)=\begin{cases}\dfrac{1}{(2+\alpha)(1+\alpha)} & (-1<\alpha\leqq\alpha_n),\\[2mm] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & (\alpha_n\leqq\alpha<+\infty),\end{cases}$$

where

$$\alpha_n=\frac{1}{2(n+1)}\left((17n^2+2n+1)^{1/2}-3n+1\right).$$

From Theorem A we can see:

(a) $C_n(\alpha_n-0)=C_n(\alpha_n+0)$;

(b) $C_{n+1}(\alpha)\geqq C_n(\alpha)$;

(c) The sequence (α_n) is decreasing.

Remark 1. The statement of Theorem A holds if W_n is the set of all algebraic polynomials $P\,(\not\equiv 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients.

In this note we consider the extremal problem (1.2) with Freud's weight function

$$w(x)=x^{\alpha}e^{-x^s}\quad(\alpha>-1,\ s>0) \tag{1.3}$$

on $[0,\infty)$. The corresponding best constant we will denote by $C_n(\alpha;s)$.

2. Extremal problem with Freud's weight

Let the set W_n be defined by (1.1) and the weight function $x\mapsto w(x)$ by (1.3) The subset of W_n for which $a_0=0$ (i.e. $P_n(0)=0$) we denote by W_n^0.

By a simple application of integration by parts we can prove:

Lemma 1. *If* $P\in W_n^0$, *then for the inner products*

$$J_n(\alpha;s)=(P_n',P_n')=\int_0^\infty x^\alpha e^{-x^s}P_n(x)^2\,dx,$$

$$I_{n,i}(\alpha;s)=(P_n,P_n^{(i)})=\int_0^\infty x^\alpha e^{-x^s}P_n(x)P_n^{(i)}(x)\,dx\quad(i=0,1,2)$$

the following recurrence relations hold

$$2I_{n,1}(\alpha;s)=sI_{n,0}(\alpha+s-1;s)-\alpha I_{n,0}(\alpha-1;s)\quad(\alpha>-2),$$

$$I_{n,2}(\alpha;s)=sI_{n,1}(\alpha+s-1;s)-\alpha I_{n,1}(\alpha-1;s)-J_n(\alpha;s)\quad(\alpha>-1).$$

In [1] Milovanović proved an interesting inequality for $P_n\in W_n$. Namely, for every $x\geqq 0$ the inequality

$$x\left(P_n'(x)^2-P_n(x)P_n''(x)\right)\leqq P_n'(x)P_n(x)$$

holds.

From this inequality and Lemma 1 we obtain:

Lemma 2. *If $P_n \in W_n^0$, then for $\alpha > -1$ and $s > 0$*

$$J_n(\alpha; s) \leqq \frac{1}{4}\{s^2 I_{n,0}(\alpha + 2s - 2; s) + s(2 - 2\alpha - s) I_{n,0}(\alpha + s - 2; s) + (\alpha - 1)^2 I_{n,0}(\alpha - 2; s)\}.$$

Since the supremum in (1.2) attained for some $P_n \in W_n^0$ (see [1]), we will consider only such polynomials, i.e., $P_n(x) = \sum_{k=1}^{n} a_k x^k$ ($a_n > 0$ and other $a_k \geqq 0$). Then

$$P_n(x)^2 = \sum_{k=2}^{2n} b_k x^k \quad (b_{2n} > 0 \text{ and other } b_k \geqq 0)$$

and

$$\|P_n\|^2 = I_{n,0}(\alpha; s) = \frac{1}{s} \sum_{k=2}^{2n} b_k \Gamma\left(\frac{\alpha + k + 1}{s}\right), \tag{2.1}$$

where Γ is the gamma function.

Using the same method as in the paper [1] we find

$$\|P_n'\|^2 = J_n(\alpha; s) = (P_n', P_n') \leqq \frac{1}{s} \sum_{k=2}^{2n} H_k(\alpha; s) b_k \Gamma\left(\frac{\alpha + k + 1}{s}\right), \tag{2.2}$$

where

$$H_k(\alpha; k) = \frac{k^2}{4} \cdot \frac{\Gamma\left(\frac{\alpha + k - 1}{s}\right)}{\left(\frac{\alpha + k + 1}{s}\right)}. \tag{2.3}$$

According to (2.1) and (2.2) we have

$$\|P_n'\|^2 \leqq \left(\max_{2 \leqq k \leqq 2n} H_k(\alpha; s)\right) \|P_n\|^2,$$

and then

$$C_n(\alpha; s) \leqq \max_{2 \leqq k \leqq 2n} H_k(\alpha; s).$$

The case $s = 1$ is solved in [1].

For $s = 2$ we get a simple result:

Theorem 1. *The best constant $C_n(\alpha; 2)$ is given by*

$$C_n(\alpha; 2) = \begin{cases} \dfrac{2}{\alpha + 1}, & -1 < \alpha \leqq -\dfrac{n-1}{n+1}, \\[2ex] \dfrac{2n^2}{2n + \alpha - 1}, & -\dfrac{n-1}{n+1} \leqq \alpha < +\infty. \end{cases}$$

Proof. In this case, (2.3) reduces to $H_k(\alpha;\, 2)=\dfrac{k^2}{2(\alpha+k-1)}$. Determining the maximum of $f(x)=\dfrac{x^2}{x+\alpha-1}$ on the interval $[2,\, 2n]$, we find that

$$\max_{2\leqq k\leqq 2n} H_k(\alpha;\, 2)=\begin{cases} H_2(\alpha;\, 2) & \text{if} \quad -1<\alpha\leqq\alpha_n,\\ H_{2n}(\alpha;\, 2) & \text{if} \quad \alpha_n\leqq\alpha<+\infty,\end{cases}$$

where $\alpha_n=-\dfrac{n-1}{n+1}$.

Similarly as in the paper [1] we show that $C_n(\alpha;\, 2)=\max\limits_{2\leqq k\leqq 2n} H_k(\alpha;\, 2)$, The polynomial $x\mapsto x^n$ is an extremal polynomial for $\alpha\geqq\alpha_n$. If $-1<\alpha\leqq\alpha_n$, there exists a sequence of polynomials, for example, $p_{n,k}(x)=x^n+kx$, $k=1, 2, \ldots$, for which $\lim\limits_{k\to\infty}\dfrac{\|p'_{n,k}\|^2}{\|p_{n,k}\|^2}=C_n(\alpha;\, 2)$.

Remark 2. The statement of Theorem 1 holds if W_n is a set as in Remark 1. In that case, if $-1<\alpha\leqq\alpha_n$, we can see that $x\mapsto\lambda x\ (\lambda>0)$ is an extremal polynomial.

From Theorem 1 we obtain the following inequality

$$\int_0^\infty e^{-t^2} P_n'(t)^2\, dt\leqq\frac{2n^2}{2n-1}\int_0^\infty e^{-t^2} P(t)^2\, dt$$

for each $P_n\in W_n$.

The case when s is an arbitrary positive number is more complicated. We state the following conjecture:

Conjecture. *Let $s\geqq 1$ and let $\alpha_n\,(>-1)$ be the unique root of the equation*

$$\frac{\Gamma\left(\frac{\alpha+1}{s}\right)}{\Gamma\left(\frac{\alpha+3}{s}\right)}=n^2\,\frac{\Gamma\left(\frac{\alpha+2n-1}{s}\right)}{\Gamma\left(\frac{\alpha+2n+1}{s}\right)}.$$

The best constant $C_n(\alpha;\, s)$ is given by

$$C_n(\alpha;\, s)=\begin{cases} H_2(\alpha;\, s), & -1<\alpha\leqq\alpha_n,\\ H_{2n}(\alpha;\, s), & \alpha_n\leqq\alpha<+\infty.\end{cases}$$

At the end we give more one result:

Theorem 2. *If $\alpha>1$ and $s>0$ we have*

$$\|P_n'\|_\alpha\leqq n\,\|P_n\|_{\alpha-2} \qquad (P_n\in W_n),$$

where $\|f\|_\alpha=\left(\int_0^\infty x^\alpha e^{-x^s} f(x)^2\, dx\right)^{1/2}$.

This result follows immediately from (2.1), (2.2), and (2.3).

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EKSTREMALNI PROBLEM ZA POLINOME SA NENEGATIVNIM KOEFICIJENTIMA. II

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Neka je W_n skup svih algebarskih polinoma egzaktnog stepena n čiji su koeficijenti nenegativni. U radu se razmatra ekstremalni problem (1.2) za normu u $L^2[0, \infty)$ sa Freudovom težinom.