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NUMERICAL DIFFERENTIATION OF ANALYTIC FUNCTIONS USING QUADRATURES ON THE SEMICIRCLE

FRANCA CALIO', MARCO FRONTINI*
Dipartimento di Matematica, Politecnico di Milano
I-20133 Milano, Italy
GRADIMIR V. MILOVANOVIĆ**
Faculty of Electronic Engineering, University of Niš
18000 Niš, Yugoslavia

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Abstract—Using quadrature formulas on the semicircle of Gauss-Christoffel type and an integral representation of derivatives, we consider differentiation formulas for higher derivatives of an analytic function. An error analysis and some numerical experiments are included.

1. INTRODUCTION

Recently Gautschi and Milovanović [1] introduced a new type of orthogonality: orthogonality on the semicircle, with respect to nonhermitian inner product

$$(f,g) = \int_0^\pi f(e^{i\theta})g(e^{i\theta})\,d\theta.$$

A general case with the weight function $z \mapsto w(z)$,

$$(f,g) = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta,$$

was considered by Gautschi, Landau and Milovanović [2]. Some applications of such orthogonal polynomials in numerical integration and numerical approximation of the first derivative of an analytic function were given by Milovanović [3].

In this paper we consider the extensions of these results to the approximation of higher derivatives. In Section 2 we derive such differentiation formulas and we give the corresponding error terms, using Gegenbauer weight function. Some improvements of these results will be given for real-valued analytic functions in Section 3. Section 4 contains some numerical results.

We mention that the numerical differentiation of analytic functions is considered in a few papers written by Lyness and Moler [4], Lyness [5], Marshal Ash and Jones [6], Tošić [7], Tošić and Elbahi [8], etc. The corresponding differentiation formulas are obtained mostly by Cauchy's integral formula and by applying the trapezoidal rule.

2. DIFFERENTIATION FORMULAE FOR HIGHER DERIVATIVES

Let $w(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$, and let

$$\int_0^\pi f(e^{i\theta})w(e^{i\theta})\,d\theta = \sum_{\nu=1}^{n} \sigma_\nu f(\zeta_\nu) + R_n(f) \tag{2.1}$$

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be the Gauss-Christoffel quadrature formula over the semicircle, which is exact for all polynomials of degree at most $2n-1$. A construction of such formula is given in [3]. The nodes ζ_ν are the zeros of the polynomial π_n, which is orthogonal on the semicircle with respect to the Gegenbauer weight function w, i.e., they are the eigenvalues of the Jacobi matrix

$$J_n = \begin{bmatrix} i\alpha_0 & \theta_0 & & & O \\ \theta_0 & i\alpha_1 & \theta_1 & & \\ & \theta_1 & i\alpha_2 & \ddots & \\ & & \ddots & \ddots & \theta_{n-2} \\ O & & & \theta_{n-2} & i\alpha_{n-1} \end{bmatrix},$$

where

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad k \geq 1,$$

$$\theta_k = \frac{1}{\lambda + k} \cdot \frac{\Gamma((k+2)/2)\Gamma(\lambda + (k+1)/2)}{\Gamma((k+1)/2)\Gamma(\lambda + (k/2))}.$$

The weights σ_ν can be obtained by an adaptation of the procedure of Golub and Welsch [9] (see [1] and [3]).

Let f be an analytic function on some domain containing the point a and a circular neighborhood of a with radius r. Using the central difference operator δ_h defined by

$$\delta_h f(a) = \frac{1}{h}\left(f\left(a + \frac{h}{2}\right) - f\left(a - \frac{h}{2}\right)\right),$$

we can find

$$\delta_h^m f(a) = \delta_h\left(\delta_h^{m-1} f(a)\right),$$

i.e.,

$$\delta_h^m f(a) = \frac{1}{h^m}\sum_{k=0}^{m}(-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2}h\right). \tag{2.2}$$

Putting $he^{i\theta}$ instead of h, where h is such that $|a + \frac{mh}{2}e^{i\theta}| < r$, and integrating (2.2) over the semicircle, we obtain:

LEMMA 2.1. *The formula*

$$\int_0^\pi \delta_{he^{i\theta}}^m f(a) w(e^{i\theta})\, d\theta = \pi f^{(m)}(a) \tag{2.3}$$

holds.

PROOF. Expanding f in Taylor series

$$f\left(a + \frac{m-2k}{2}he^{i\theta}\right) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}\left(\frac{m-2k}{2}\right)^j (he^{i\theta})^j,$$

we have

$$\int_0^\pi \delta_{he^{i\theta}}^m f(a) w(e^{i\theta})\, d\theta = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} S_j^{(m)} \mathcal{I}_{j-m} h^{j-m},$$

where

$$S_j^{(m)} = \sum_{k=0}^{m}(-1)^k \binom{m}{k}\left(\frac{m-2k}{2}\right)^j \tag{2.4}$$

and

$$\mathcal{I}_k = \int_0^\pi e^{ik\theta} w(e^{i\theta})\, d\theta. \tag{2.5}$$

We note that $S_j^{(m)} = 0$ for $j < m$, and for every $j = m + 2p + 1$ with $p = 0, 1, \ldots$. Also, $S_m^{(m)} = m!$. So, we have

$$\int_0^\pi \delta_{he^{i\theta}}^m f(a) w(e^{i\theta})\, d\theta = \sum_{p=0}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} S_{m+2p}^{(m)} \mathcal{I}_{2p} h^{2p}.$$

Since the integrals (2.5) are

$$\mathcal{I}_0 = \pi \quad \text{and} \quad \mathcal{I}_{2p} = 0, \qquad p > 0,$$

we obtain (2.3). ∎

Applying the Gauss-Christoffel quadrature formula on the semicircle (2.1) to the integral on the left side in (2.3), we use the following differentiation formula to obtain higher derivatives

$$f^{(m)}(a) \approx D_{n,h}^m f(a) = \frac{1}{\pi} \sum_{\nu=1}^{n} \sigma_\nu \delta_{h\zeta_\nu}^m f(a),$$

i.e.,

$$D_{n,h}^m f(a) = \frac{1}{\pi h^m} \sum_{\nu=1}^{n} \frac{\sigma_\nu}{\zeta_\nu^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2} h\zeta_\nu\right). \tag{2.6}$$

Regarding the truncation error we can give the following result:

Theorem 2.2. *The error of the differentiation formula (2.6) for analytical functions is given by*

$$R_{n,h}^m f(a) = f^{(m)}(a) - D_{n,h}^m f(a) = \frac{1}{\pi} \sum_{p=n}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} S_{m+2p}^{(m)} R_n(z^{2p}) h^{2p}, \tag{2.7}$$

where $S_j^{(m)}$ and $R_n(z^{2p})$ are defined by (2.4) and (2.1), respectively. The dominant error term is

$$\frac{S_{m+2n}^{(m)}}{\pi(m+2n)!} \left(\frac{\Gamma((n+1)/2)\,\Gamma(\lambda + n/2)}{\Gamma(\lambda+n)} \right)^2 f^{(m+2n)}(a) h^{2n}. \tag{2.8}$$

Proof. Expanding f in Taylor series at $z = a$, we find

$$D_{n,h}^m f(a) = \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} \sum_{\nu=1}^{n} \sigma_\nu (h\zeta_\nu)^{j-m} S_j^{(m)}.$$

Regarding to the values of $S_i^{(m)}$, we have

$$\begin{aligned} D_{n,h}^m f(a) &= \frac{1}{\pi} \sum_{p=0}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} \sum_{\nu=1}^{n} \sigma_\nu (h\zeta_\nu)^{2p} S_{m+2p}^{(m)} \\ &= f^{(m)}(a) - \frac{1}{\pi} \sum_{p=1}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} S_{m+2p}^{(m)} R_n(z^{2p}) h^{2p}, \end{aligned}$$

because $\sum_{\nu=1}^{n} \sigma_\nu \zeta_\nu^{2p} = -R_n(z^{2p})$. Since $R_n(z^{2p}) = 0$ for $p < n$, we obtain (2.7).

Finally, for $p = n$, we find (2.8) because (see [3])

$$R_n(z^{2n}) = \|\pi_n\|^2 = \left(\frac{\Gamma((n+1)/2)\,\Gamma(\lambda + n/2)}{\Gamma(\lambda+n)} \right)^2.$$

∎

In our investigation we will use $\lambda = 1$ because the sequence θ_k in the Jacobi matrix is constant, i.e., $\theta_k = 1/2$ for $k \geq 0$. Constants in the dominant error term $C_{n,m} f^{(m+2n)}(a) h^{2n}$ are computed and given in Table 2.1 for $n = 2, 5, 10, 20$ and $m = 1(1)10$. Numbers in parenthesis indicate decimal exponents.

Table 2.1. The constant $C_{n,m}$ in the dominant error term for $n = 2, 5, 10, 20$ and $m = 1(1)10$.

m	$n = 2$	$n = 5$	n=10	$n = 20$
1	3.26(−5)	2.39(−14)	1.78(−32)	2.47(−74)
2	1.74(−4)	4.08(−12)	1.70(−27)	1.29(−63)
3	4.23(−4)	6.10(−11)	8.28(−25)	1.12(−57)
4	7.81(−4)	3.67(−10)	5.16(−23)	1.20(−53)
5	1.25(−3)	1.39(−9)	1.09(−21)	1.23(−50)
6	1.82(−3)	3.98(−9)	1.20(−20)	2.93(−48)
7	2.51(−3)	9.52(−9)	8.57(−20)	2.62(−46)
8	3.30(−3)	2.00(−8)	4.50(−19)	1.16(−44)
9	4.20(−3)	3.82(−8)	1.88(−18)	3.04(−43)
10	5.21(−3)	6.79(−8)	6.61(−18)	5.31(−42)

Some considerations for $n = 2$ and $m = 1$ regarding to λ are given in [3].

For real-valued analytic functions the formula (2.6) can be simplified. Namely, when n is even and $\operatorname{Re}\zeta_\nu > 0$, for $\nu = 1, 2, \ldots, n/2$, one finds

$$D^m_{n,h} f(a) = \frac{2}{\pi h^m} \sum_{\nu=1}^{n/2} \operatorname{Re}\left\{ \frac{\sigma_\nu}{\zeta_\nu^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2} h\zeta_\nu\right) \right\}. \tag{2.9}$$

In the simplest case when $n = 2$, we have

$$\zeta_{1,2} = \frac{1}{4}\left(\pm\sqrt{3} + i\right), \qquad \sigma_{1,2} = \frac{\pi}{2}\left(1 \pm i\frac{\sqrt{3}}{3}\right).$$

Then, the corresponding differentiation formula (2.9) reduces to

$$D^m_{2,h} f(a) = \frac{2}{\pi h^m} \operatorname{Re}\left\{ \frac{\sigma_1}{\zeta_1^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2} h\zeta_1\right) \right\}. \tag{2.10}$$

Its error is $O(h^4)$.

Similarly as in conventional formulas of numerical differentiation based on divided differences, the small errors in the function values can be amplified and they can produce an incorrect result when h tends to zero. In order to estimate the roundoff error behavior for the formula (2.10), we put

$$e_k = f(a + \frac{m-2k}{2} h\zeta_1) - \tilde{f}_k,$$

where $\tilde{f}_k$ is the computed value of the exact function value $f\left(a + \frac{m-2k}{2} h\zeta_1\right)$.

If the errors e_k do not exceed E in magnitude, i.e., $|e_k| \leq E$, we have in the worst case

$$\begin{aligned} E^m_{2,h} f(a) \equiv \left| D^m_{2,h} f(a) - D^m_{2,h} \tilde{f}(a) \right| &\leq \frac{2}{\pi h^m} \left| \frac{\sigma_1}{\zeta_1^m} \right| \cdot \left| \sum_{k=0}^{m} (-1)^k \binom{m}{k} e_k \right| \\ &\leq \frac{2}{\sqrt{3}} \left(\frac{2}{h}\right)^m \sum_{k=0}^{m} \binom{m}{k} |e_k|, \end{aligned}$$

i.e.,

$$E_{2,h}^{m} f(a) \le \frac{2}{\sqrt{3}} \left(\frac{2}{h}\right)^{m} 2^{m} E. \tag{2.11}$$

Note that for small h, this bound can become strongly magnified.

Using an estimate of the truncation error and (2.11) we find the following bound of the total error (a combination of the truncation error and roundoff error)

$$|R_T| \le C_{2,m} |f^{(m+4)}(a)| h^4 + \frac{2}{\sqrt{3}} \left(\frac{2}{h}\right)^{m} 2^{m} E,$$

from which we obtain an optimal value of h

$$h^* = \left(\frac{2^{2m-1} m E}{\sqrt{3}\, C_{2,m} |f^{(m+4)}(a)|} \right)^{1/(m+4)}, \tag{2.12}$$

where $C_{2,m}$ is given in Table 2.1.

In real cases, it is very difficult to predict quantitatively the roundoff error $E_{2,h}^{m} f(a)$. But, in any case we can expect that there is an optimal value of h, usually less than h^* given by (2.12), for which the total error achieves the minimum.

EXAMPLE. Let $f(z) = e^z$, $a = 0$, $m = 3$. We consider the differentiation formula (2.10) on the MICROVAX 3400 using VAX FORTRAN Ver. 5.3 in E-, D-, and Q-arithmetics, with machine precisions $\approx 1.19 \times 10^{-7}$, $\approx 2.76 \times 10^{-17}$, and $\approx 1.93 \times 10^{-34}$, respectively. Taking $h = 2^{-k}$, $k = 1, 2, \ldots$, we obtain results presented in Table 2.2. We stop this process when the total error begins to increase.

Table 2.2. The total error of the formula (2.10) for $m = 3$ and $h = 2^{-k}$, $k = 1, 2, \ldots$, using $E-$, $D-$, and $Q-$arithmetics.

k	Error (E)	Error (D)	Error (Q)
1	−1.53(−5)	−2.65(−5)	−2.65(−5)
2	1.22(−4)	−1.65(−6)	−1.65(−6)
3		−1.03(−7)	−1.03(−7)
4		−6.46(−9)	−6.46(−9)
5		−3.98(−10)	−4.04(−10)
6		−4.37(−11)	−2.52(−11)
7		−4.66(−10)	−1.58(−12)
8			−9.85(−14)
9			−3.85(−16)
11			−2.41(−17)
12			−1.50(−18)
13			−9.40(−20)
14			−3.85(−16)
15			−1.86(−20)
16			6.78(−20)

Applying the Fibonnaci minimizing procedure on the last intervals $(h, 2h) = (2^{-k}, 2^{1-k})$, where $k = 2$, 7, and 16, we find the optimal values h_E, h_D, and h_Q of h in E-, D-, and Q-arithmetics, respectively.

Thus,

$$h_E \approx 0.3690948, \qquad h_D \approx 0.01256087, \qquad h_Q \approx 0.49151797(-4).$$

The values of h^* obtained by formula (2.12) are something greater then the above computed values. It is interesting that we obtain the machine-zero for the coresponding errors E_E, E_D, and E_Q.

Table 3.1. The constant $\tilde{C}_{n,m}$ in the dominant error term for $n = 2, 5, 10, 20$ and $m = 1(1)10$.

m	$n=2$	$n=5$	$n=10$	$n=20$
1	4.84(−8)	9.57(−18)	2.20(−36)	8.56(−79)
2	7.75(−7)	5.60(−15)	7.69(−31)	1.71(−67)
3	3.31(−6)	1.64(−13)	7.76(−28)	3.19(−61)
4	8.78(−6)	1.53(−12)	7.93(−26)	5.82(−57)
5	1.83(−5)	7.98(−12)	2.43(−24)	8.88(−54)
6	3.31(−5)	2.94(−11)	3.58(−23)	2.92(−51)
7	5.41(−5)	8.62(−11)	3.23(−22)	3.41(−49)
8	8.27(−5)	2.15(−10)	2.07(−21)	1.90(−47)
9	1.20(−4)	4.76(−10)	1.03(−20)	6.04(−46)
10	1.67(−4)	9.60(−10)	4.19(−20)	1.25(−44)

3. AN IMPROVEMENT OF ACCURACY FOR REAL-VALUED ANALYTIC FUNCTIONS

The formula (2.6) for real-valued analytic functions can be improved with a little change. Namely, if we put $he^{i\alpha}$ instead of h in Lemma 2.1, where α is an arbitrary real parameter, and applying again Gauss-Christoffel formula (2.1), we obtain the following differentiation formula

$$f^{(m)}(a) \approx D^m_{n,h,\alpha} f(a) = \frac{1}{\pi} \sum_{\nu=1}^{n} \sigma_\nu \delta^m_{he^{i\alpha}\zeta_\nu} f(a).$$

Similar to the above investigation we find an expression for the error, depending on the real parameter α

$$R^m_{n,h,\alpha} f(a) = f^{(m)}(a) - D^m_{n,h,\alpha} f(a) = \frac{1}{\pi} \sum_{p=n}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} S^{(m)}_{m+2p} R_n(z^{2p}) e^{i2p\alpha} h^{2p}. \tag{3.1}$$

Since the derivative $f^{(m)}(a)$ is real for real a and real-valued functions the parameter α can be chosen such that the dominant error term in the last expressions be purely imaginary. Then, for such functions, the dominant error term in (3.1), i.e.,

$$\frac{1}{\pi(m+2n)!} f^{(m+2n)}(a) S^{(m)}_{m+2n} R_n(z^{2n}) e^{i2n\alpha} h^{2n},$$

becomes purely imaginary. This can be achieved for $\alpha = \pi/4n$. In that case, the dominant error term for real-valued functions becomes the real part of the term in (3.1) for $p = n+1$. So we have the following result:

THEOREM 3.1. *The dominant error term of the differentiation formula*

$$f^{(m)}(a) \approx \operatorname{Re}\left\{D^m_{n,h,\pi/4n} f(a)\right\}, \qquad a \in \mathbf{R}, \tag{3.2}$$

for real-valued analytic functions is given by

$$-\frac{\sin(\pi/2n)}{\pi(m+2n+2)!} S^{(m)}_{m+2n+2} R_n(z^{2n+2}) f^{(m+2n+2)}(a) h^{2n+2}, \tag{3.3}$$

where $R_n(g)$ is defined in (2.1).

REMARK. Putting $\alpha = 3\pi/4n$ in (3.1) we also obtain a rule of degree precision $2n+2$. Then, in the dominant error term (3.3), the factor $-\sin(\pi/2n)$ should be replaced by $\sin(3\pi/2n)$.

Constants in the dominant error term (3.3), $-\tilde{C}_{m,n} f^{(m+2n+2)}(a) h^{2n+2}$, are given in Table 3.1, for the same values of n and m as in Table 2.1. The case $m = 1$ was considered by Milovanović [10].

4. NUMERICAL RESULTS

Using the previous differentiation formulae for $n = 2$, in this section we give some numerical results. In our considerations we take

$$f(z) = \frac{e^z}{\sin^3 z + \cos^3 z}, \quad a = 0.$$

This example was used by Lyness and Moler [4] and Milovanović [3].

The exact values of $f^{(m)}(0)$, for $m = 1, \dots, 6$, are

$$f'(0) = 1, \quad f''(0) = f'''(0) = 4, \quad f^{(4)}(0) = 28, \quad f^{(5)}(0) = -164, \quad f^{(6)}(0) = 64,$$

respectively.

Applying the formula (2.10), with $h = 2^{-k}$, $k = 1(1)9$ in double precision, we obtain the approximations for $f^{(m)}(0)$, $m = 2,3,4,5$, which are given in Table 4.1.

Table 4.1. Approximations for $f^{(m)}(0)$, $m = 2,3,4,5$, using the formula (2.10), with $h = 2^{-k}$, $k = 1(1)9$.

k	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$
1	3.99874547229	4.39538759148	24.10674877036	−90.47081117719
2	3.99994766806	4.02280224303	27.82640940590	−159.33453144447
3	3.99999714764	4.00139277235	27.99052498512	−163.73068567739
4	3.99999982828	4.00008654043	27.99942951942	−163.98353600502
5	3.99999998937	4.00000540083	27.99996468434	−163.99897655845
6	3.99999999934	4.00000033739	27.99999779815	−163.99993181229
7	3.99999999996	4.00000002189	27.99999986216	−163.99984741211
8	4.00000000000	4.00000000093	27.99999998696	−163.99536132813
9	4.00000000006	4.00000000745	28.00000007451	−164.00000000000

In Table 4.2 we give the corresponding relative errors for $m = 1(1)6$, $n = 2$, $h = 2^{-k}$, $k = 1(1)10$, in Q-precision.

Table 4.2. Relative errors for $m = 1(1)6$, $n = 2$, $h = 2^{(-k)}$, $k = 1(1)10$, and $\alpha = 0$.

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	−3.34(−04)	1.74(−04)	−8.83(−02)	7.86(−02)	3.13(−01)	6.86(+00)
2	−2.09(−05)	1.09(−05)	−5.53(−03)	5.14(−03)	2.55(−02)	1.53(+00)
3	−1.30(−06)	6.78(−07)	−3.45(−04)	3.22(−04)	1.59(−03)	9.61(−02)
4	−8.15(−08)	4.25(−08)	−2.16(−05)	2.01(−05)	9.94(−05)	6.02(−03)
5	−5.09(−09)	2.65(−09)	−1.35(−06)	1.26(−06)	6.22(−06)	3.75(−04)
6	−3.18(−10)	1.66(−10)	−8.43(−08)	7.86(−08)	3.89(−07)	2.34(−05)
7	−1.99(−11)	1.04(−11)	−5.28(−09)	4.93(−09)	2.43(−08)	1.47(−06)
8	−1.24(−12)	6.48(−13)	−3.30(−10)	3.07(−10)	1.52(−09)	9.17(−08)
9	−7.77(−14)	4.05(−14)	−2.06(−11)	1.92(−11)	9.51(−11)	5.73(−09)
10	−4.86(−15)	2.53(−15)	−1.29(−12)	1.20(−12)	5.94(−12)	3.58(−10)

We can obtain better results using the improved formula (3.2), with $\alpha = \pi/8$, where $n = 2$. The corresponding relative errors are presented in Table 4.3. Similar results can be obtain using $\alpha = 3\pi/8$.

In conclusion it should be mentioned that our methods for numerical differentiation of analytic functions are attractive in D-arithmetics, and especially in Q-arithmetics.

Table 4.3. Relative errors for $m = 1(1)6$, $n = 2$, $h = 2^{-k}$, $k = 1(1)10$, and $\alpha = 0$.

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	7.16(−06)	−9.93(−05)	7.45(−03)	−4.21(−02)	−6.83(−02)	−9.55(+00)
2	1.12(−07)	−1.58(−06)	1.22(−04)	−7.43(−04)	−2.09(−03)	−2.41(−01)
3	1.75(−09)	−2.47(−08)	1.92(−06)	−1.17(−05)	−3.39(−05)	−3.91(−03)
4	2.73(−11)	−3.85(−10)	3.00(−08)	−1.83(−07)	−5.30(−07)	−6.13(−05)
5	4.27(−13)	−6.03(−12)	4.68(−10)	−2.86(−09)	−8.29(−09)	−9.56(−07)
6	6.67(−15)	−9.43(−14)	7.30(−12)	−4.46(−11)	−1.30(−10)	−1.49(−08)
7	1.04(−16)	−1.47(−15)	1.14(−13)	−6.96(−13)	−2.02(−12)	−2.33(−10)
8	1.63(−18)	−2.30(−17)	1.79(−15)	−1.09(−14)	−3.16(−14)	−3.66(−12)
9	2.54(−20)	−3.60(−19)	2.80(−17)	−1.70(−16)	−4.95(−16)	−5.70(−14)
10	3.97(−22)	−5.63(−21)	4.35(−19)	−2.66(−18)	−7.62(−18)	5.77(−15)

REFERENCES

1. W. Gautschi and G.V. Milovanović, Polynomials orthogonal on the semicircle, *J. Approx. Theor.* **46**, 230–250 (1986).
2. W. Gautschi, H. Landau and G.V. Milovanović, Polynomials orthogonal on the semicircle II, *Constr. Approx.* **3**, 389–404 (1987).
3. G.V. Milovanović, Complex orthogonality on the semicircle with respect to Gegenbauer weight: Theory and applications, In *Topics in Mathematical Analysis*, (Edited by Th.M. Rassias), pp. 695–722, World Scientific, Singapore, (1989).
4. J.N. Lyness and C.B. Moler, Numerical differentiation of analytic functions, *SIAM J. Numer. Anal.* **4**, 202–210 (1967).
5. J.N. Lyness, Differentiation formulas for analytic functions, *Math. Comp.* **22**, 352–362 (1968).
6. J. Marshal Ash and R.L. Jones, Optimal numerical differentiation using three function evaluations, *Math. Comp.* **37**, 159–167 (1981).
7. D.Đ. Tošić, Numerical diferentiation of analytic functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No **678**–No **715**, 173–182 (1980).
8. D.Đ. Tošić and A.A. Elbahi, Optimal numerical diferentiation of real-valued analytic functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No **735**–No **762**, 122–126 (1982).
9. G.H. Golub and J.H. Welsch, Calculation of Gauss quadrature rules, *Math. Comp.* **23**, 221–230 (1969).
10. G.V. Milovanović, Some applications of the polynomials orthogonal on the semicircle, *Numerical Methods*, (Miskolc, 1986), Colloq. Math. Soc. János Bolyai, Vol. 50, pp. 625–634, North-Holland, Amsterdam/New York, (1987).