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A GENERALIZATION OF A RESULT OF A. MEIR FOR NON-DECREASING SEQUENCES

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1. In [3], the following result is given.

THEOREM A. *Let* $0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$ *and* $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$, *satisfying* $a_i - a_{i-1} \leqq p_i$ $(i = 1, 2, \ldots, n)$. *If* $r \geqq 1$ *and* $s + 1 \geqq 2(r + 1)$, *then*

$$(1.1)\quad \left((s + 1)\sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2}\right)^{1/(s+1)} \leqq \left((r + 1)\sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2}\right)^{1/(r+1)}.$$

In this paper we shall prove an inequality which is stronger than inequality (1.1). Also, we show a generalization of Theorem A.

THEOREM 1. *Let* $0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$ *and* $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$, *satisfying* $a_i - a_{i-1} \leqq p_i (i = 1, 2, \ldots, n)$. *If* $r \geqq 1$ *and* $s + 1 \geqq 2(r + 1)$, *then*

$$(1.2)\quad \begin{aligned} &(s + 1) \sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2} + \frac{(s + 1)(s - r)}{8} \sum_{i=1}^{n-1} (p_{i+1}^2 - p_i^2) a_i^{s-1} \\ &\leqq \left((r + 1)\sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2}\right)^{(s+1)/(r+1)}. \end{aligned}$$

2. PROOF. Since $x \mapsto x^r$ $(r \geqq 1)$ is a convex function on $[0, \infty)$, the inequality

$$\sum_{i=1}^{j} \int_{a_{i-1}}^{a_i} x^r dx \leqq \sum_{i=1}^{j} (a_i - a_{i-1}) \frac{a_i^r + a_{i-1}^r}{2} \quad (1 \leqq j \leqq n),$$

holds, wherefrom, according to the condition $a_i - a_{i-1} \leqq p_i$, we obtain

$$(2.1)\quad \frac{1}{r + 1} a_j^{r+1} \leqq \sum_{i=1}^{j} p_i \frac{a_i^r + a_{i-1}^r}{2}.$$

For $q_j = \sum_{i=1}^{j} a_{i0}^r \frac{p_i + p_{i+1}}{2}$, the inequality (2.1) becomes

$$\frac{1}{r + 1} a_j^{r+1} \leqq q_j - \frac{1}{2} p_{j+1} a_j^r = q_{j-1} + \frac{1}{2} p_j a_j^r,$$

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i.e.,

$$a_j^{r+1} \leqq (r + 1)\, c_j, \tag{2.2}$$

where $c_j = q_j - p_{j+1}\, a_j^r/2 = q_{j-1} + p_j\, a_j^r/2$.

If we take $k = (s + 1)/(r + 1)$, the inequality (2.2) becomes

$$a_j^{s-r} \leqq (r + 1)^{k-1}\, c_j^{k-1}. \tag{2.3}$$

Note that $q_{j-1} \leqq c_j \leqq q_j$ $(j = 1, \ldots, n)$. Using a generalization of Hadamard's integral inequality for convex functions, which is proved in **[2]**, we find that the inequality

$$c_j^{k-1} + (k - 1)\, c_j^{k-2}\left(\frac{q_j + q_{j-1}}{2} - c_j\right) \leqq \frac{q_j^k - q_{j-1}^k}{k(q_j - q_{j-1})},$$

i.e.,

$$k\, \frac{p_j + p_{j+1}}{2}\, a_j^r\, c_j^{k-1} + k\,(k - 1)\, \frac{p_{j+1}^2 - p_j^2}{8}\, a_j^{2r}\, c_j^{k-2} \leqq q_j^k - q_{j-1}^k$$

is valid. Whence, after summing for $j = 1, \ldots, n - 1$ and using (2.2) and (2.3), we obtain the inequality (1.2).

Since

$$(s + 1)\,(s - r) \sum_{i=1}^{n-1} (p_{i+1}^2 - p_i^2)\, a_i^{s-1} \geqq 0,$$

we conclude that the inequality (1.2) is stronger than inequality (1.1).

For $p_1 = \cdots = p_n = 1$, the inequality (1.2) is reduce to the inequality proved in **[1]**.

3. Similary, as in Theorem 1 (also, see **[4]**), the following result can be proved.

THEOREM 2. *Let f and g be differentiable functions on* $[0, \infty)$ *satisfying* $f(0) = f'(0) = g(0) = g'(0) = 0$. *Suppose that f and g are convex and increasing on* $[0, \infty)$. *Set $h(x) = g(f(x))$. Then for any finite sequences (a_i), (p_i) such that $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$ and $0 \leqq p_1 \leqq \cdots \leqq p_n$, which satisfy $a_i - a_{i-1} \leqq p_i$ $(i = 1, \ldots, n)$, we have*

$$h^{-1}\left(\sum_{i=1}^{n-1} \frac{p_i + p_{i+1}}{2}\, h(a_i)\right) \leqq f^{-1}\left(\sum_{i=1}^{n-1} \frac{p_i + p_{i+1}}{2}\, f(a_i)\right). \tag{3.1}$$

COROLLARY. *For $h(x) = x^{s+1}$ and $f(x) = x^{r+1}$ the Meir's result is obtained from the inequality* (3.1).

EXAMPLE. Functions $f(x) = x^2$ and $g(x) = x^3 e^x$ satisfy the conditions of the above theorem. This shows that potential functions are not the only ones which satisfy the conditions of the Theorem 2.

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