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ON THE ZEROS OF POLYNOMIALS AND RELATED REGULAR FUNCTIONS OF A QUATERNIONIC VARIABLE

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Abstract. In this paper, we establish a ring-shaped region containing all the zeros of a unilateral polynomial with quaternionic coefficients located on only one side of the powers of the quaternionic variable. We shall also obtain zero-free regions for the related subclass of regular power series.

1. Introduction and preliminaries

A classical study in geometric function theory is to locate the zeros of a polynomial in the plane using various approaches and techniques. This kind of study is considered to be very significant and has deeply influenced the development of mathematics and its application areas, such as physical systems. This study, in addition to having multiple applications, has inspired much more research, both theoretically and practically. A classical result of practical interest, giving the upper bound for the moduli of the zeros of a complex coefficient polynomial due to Cauchy [2], is as follows:

THEOREM A. *Let $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. Then all the zeros of $P(z)$ lie in*

$$|z|\le 1+\max_{0\le\nu\le n-1}\left|\frac{a_\nu}{a_n}\right|.$$

Many similar studies, which shed light on the zero bounds of a complex coefficient polynomial in the plane, have since appeared in the literature as a result of this elegant result, for instance, see [19]. Using the classical Schwarz lemma, Mohammad [23] obtained the following upper bound for the zeros of $P(z)$.

THEOREM B. *Let $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. Then all the zeros of $P(z)$ lie in $|z|\le M/|a_n|$ if $|a_n|\le M$, where*

$$M=\max_{|z|=1}\left|a_{n-1}z^{n-1}+\cdots+a_0\right|=\max_{|z|=1}\left|a_0 z^{n-1}+\cdots+a_{n-1}\right|.$$

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In order to estimate better and sharper zero bounds, it is desirable to put some restrictions on the coefficients of the polynomial. One of the most known results about the distribution of zeros of a complex polynomial with important applications in geometric function theory is the Eneström-Kakeya theorem [15].

THEOREM C (Eneström-Kakeya Theorem). *If* $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \quad (z \in \mathbb{C})$ *is a polynomial of degree* n*, with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $T(z)$ *lie in* $|z| \leq 1$.

We refer the reader to the comprehensive books of Marden [15] and Milovanović et al. [20] for an exhaustive survey of extensions and refinements of this well-known result. We get the following equivalent form of Theorem C by applying it to the polynomial $z^n T(1/z)$.

THEOREM D. *If* $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \quad (z \in \mathbb{C})$ *is a polynomial of degree* n*, with real coefficients and satisfying*

$$a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0,$$

then $T(z)$ *does not vanish in* $|z| < 1$.

The Eneström-Kakeya theorem is generally seen as an important addition to this field of study and has been the subject of a substantial amount of scholarly discourse. The extension of Theorem B to a class of related analytic functions was established by Aziz and Mohammad [1] in the form of the following result:

THEOREM E. *Let* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \not\equiv 0$ *be an analytic function in* $|z| \leq t$, $t > 0$. *If*

$$a_\nu > 0 \quad \textit{and} \quad a_{\nu-1} - t a_\nu \geq 0, \quad \nu = 1, 2, 3, \ldots,$$

then $f(z)$ *does not vanish in* $|z| < t$.

By $\mathbb{H}$ we denote the noncommutative division ring of quaternions. It consists of elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, with $x_0, x_1, x_2, x_3 \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i, ki = -ik = j$. Every element $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = x_0$ and the imaginary part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} : |q| < r\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centered at the origin, i.e.,

$$\mathbb{B} = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Notice that if $I \in \mathbb{S}$, then $I^2 = -1$. Thus, for any fixed $I \in \mathbb{S}$, we define

$$\mathbb{C}_I = \{x + Iy : \ x, y \in \mathbb{R}\},$$

which can be identified with a complex plane. The real axis belongs $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and so a real quaternion $q = x_0$ belongs to $\mathbb{C}_I$ for any $I \in \mathbb{S}$. For any non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique $x, y \in \mathbb{R}$ with $y > 0$ and $I \in \mathbb{S}$ such that $q = x + Iy$.

We refer the reader to [4], [6], [8]–[10], [14] and the reference therein, for definitions and properties of quaternions and many aspects of the theory of quaternionic regular functions. The following definition of regularity for functions of a quaternionic variable was introduced in [9] by Gentili and Struppa, who were inspired by a work of Cullen [5] on analytic intrinsic functions of quaternions:

DEFINITION 1. Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is said to be left slice regular or simply as slice regular if, for every $I \in \mathbb{S}$, its restriction f_I of f to the complex plane $\mathbb{C}_I$ satisfies

$$\overline{\partial}_I f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0.$$

Since for all $n \geq 1$ and for all $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(x+Iy)^n = 0,$$

it follows by definition that the monomial $P(q) = q^n$ is regular. Because addition and right multiplication by a constant preserves regularity, all polynomials of the form

$$T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}, \quad a_{\nu} \in \mathbb{H}, \quad \nu = 0, 1, 2, \ldots, n, \tag{1}$$

with coefficients on the right and indeterminate on the left are regular.

Given two quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ and $g(q) = \sum_{\nu=0}^{\infty} q^{\nu} b_{\nu}$ with radii of convergence greater than R, we define the regular product of f and g as the series

$$(f * g)(q) = \sum_{\nu=0}^{\infty} q^{\nu} c_{\nu},$$

where $c_{\nu} = \sum_{k=0}^{\nu} a_k b_{\nu-k}$ for all ν. Further, as observed in [6] and [9] for each quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$, there exists a ball $B(0,R) = \{q \in \mathbb{H} : |q| < R\}$ such that f converges absolutely and uniformly on each compact subset of $B(0,R)$ and where the sum function of f is regular.

Polynomials with quaternionic coefficients located on only one side of the variable were also investigated in [12] and [13]. It is observed (e.g., see [6], [12]) that the zeros of a polynomial of type (1) are either isolated or spherical. Gentili and Stoppato [10] (see also [8]) provided a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the form of the following outcome by utilising some helpful tools from the theory of regular functions:

THEOREM F. *Let* $f(q)=\sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ *be a given quaternionic power series with radius of convergence* R*, and let* $p \in B(0,R)$. *Then* $f(p)=0$ *if and only if there exists a quaternionic power series* $g(q)$ *with radius of convergence* R *such that*

$$f(q)=(q-p)*g(q).$$

In the form of the following conclusion, Gentili and Struppa [9] developed a maximum modulus theorem for regular functions, which encompasses convergent power series and polynomials:

THEOREM G (Maximum Modulus Theorem). *Let* $B=B(0,r)$ *be a ball in* $\mathbb{H}$ *with centre* 0 *and radius* $r>0$*, and let* $f: B \rightarrow \mathbb{H}$ *be a regular function. If* $|f|$ *has a relative maximum at a point* $a \in B$*, then* f *is a constant on* B.

Recently, Gardner and Taylor [7] used Theorem G and extended Schwarz's lemma from the complex to the quaternionic setting as follows:

THEOREM H *Let* $f(q)=\sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ *and* $f: B(0,R) \rightarrow \mathbb{H}$ *be regular, where the coefficients* a_{ν}*,* $0 \leq \nu \leq n$*, and variable* q *are quaternions. Suppose* $f(0)=0$*, then*

$$|f(q)| \leq \frac{M|q|}{R} \quad \text{for} \quad |q| \leq R,$$

where $M=\max_{|q|=R}|f(q)|$.

In the quaternionic environment, the counterpart of the aforementioned Eneström-Kakeya theorem and its different variants were examined in fairly recent articles (cf. [3], [11], [16], [17], [18], [21], [22]). This paper aims to extend some classical results by deriving bounds for the moduli of all zeros of a unilateral polynomial of type (1). First, we will get a more general conclusion: a ring shaped region containing all zeros of the polynomial. We also get the quaternionic counterpart of Theorem B as an application. Additionally, a zero-free region is established for the relevant subclass of regular power series.

2. Main results

In this section, we state our main results. Their proofs are provided in the next section. We begin by obtaining a ring shaped region containing all zeros of a unilateral polynomial of type (1).

THEOREM 1. *Let* $T(q)=\sum_{\nu=0}^{n}q^{\nu}a_{\nu}$ *be a polynomial of degree* n $($*where* q *is a quaternionic variable*$)$ *with quaternionic coefficients* a_{ν}, $\nu=0,1,2,\ldots,n$. *If for some* $t>0$, *we have*

$$\max_{|q|=t}\left|q^{n}a_{n}+q^{n-1}a_{n-1}+\cdots+qa_{1}\right|\leq M_{1}, \tag{2}$$

and

$$\max_{|q|=t}\left|q^{n}a_{0}+q^{n-1}a_{1}+\cdots+qa_{n-1}\right|\leq M_{2}, \tag{3}$$

then all the zeros of $T(q)$ *lie in the region*

$$\min\left(\frac{t|a_{0}|}{M_{1}},t\right)\leq|q|\leq\max\left(\frac{M_{2}}{t|a_{n}|},\frac{1}{t}\right).$$

We get the following outcomes as special instances from Theorem 1:

COROLLARY 1. *Let* $T(q)=\sum_{\nu=0}^{n}q^{\nu}a_{\nu}$ *be a polynomial of degree* n $($*where* q *is a quaternionic variable*$)$ *with quaternionic coefficients* a_{ν}, $\nu=0,1,2,\ldots,n$. *If for some* $t>0$, *we have*

$$\max_{|q|=t}\left|q^{n}a_{n}+q^{n-1}a_{n-1}+\cdots+qa_{1}\right|\leq|a_{0}|,$$

then $T(q)$ *does not vanish in* $|q|<t$.

COROLLARY 2. *Let* $T(q)=\sum_{\nu=0}^{n}q^{\nu}a_{\nu}$ *be a polynomial of degree* n $($*where* q *is a quaternionic variable*$)$ *with quaternionic coefficients* a_{ν}, $\nu=0,1,2,\ldots,n$. *If for some* $t>0$, *we have*

$$\max_{|q|=t}\left|q^{n}a_{0}+q^{n-1}a_{1}+\cdots+qa_{n-1}\right|\leq|a_{n}|,$$

then $T(q)$ *has all its zeros in* $|q|\leq 1/t$.

Next, we use Theorem 1 to prove the following result, which in particular provides an extension of Theorem B to a polynomial with quaternionic coefficients.

THEOREM 2. *Let* $T(q)=\sum\limits_{\nu=0}^{n}q^{\nu}a_{\nu}$ *be a polynomial of degree* n $($*where* q *is a quaternionic variable*$)$ *with quaternionic coefficients* a_{ν}, $\nu=0,1,2,\ldots,n$. *If for some* $t>0$, *we have*

$$\max_{|q|=t}\left|q^{n-1}a_{0}+q^{n-2}a_{1}+\cdots+a_{n-1}\right|\leq M, \tag{4}$$

then all the zeros of $T(q)$ *lie in*

$$|q|\leq\max\left(\frac{M}{|a_{n}|},\frac{1}{t}\right).$$

In Theorem 2, if $t=1$, we have the following:

COROLLARY 3. *Let* $T(q)=\sum_{\nu=0}^{n}q^{\nu}a_{\nu}$ *be a polynomial of degree* n (*where* q *is a quaternionic variable*) *with quaternionic coefficients* a_{ν}, $\nu=0,1,2,\ldots,n$, *and*

$$\max_{|q|=1}\left|q^{n-1}a_0+q^{n-2}a_1+\cdots+a_n\right|=\max_{|q|=1}\left|q^{n-1}a_{n-1}+q^{n-2}a_{n-2}+\cdots+a_0\right|\le M,$$

then all the zeros of $T(q)$ *lie in*

$$|q|\le\max\left(\frac{M}{|a_n|},1\right).$$

Corollary 3 provides the quaternionic analogue of Theorem B, specifically for $|a_n|\le M$.

We will now examine the zero-free regions for the relevant subclass of power series that are regular in the ball $B(0,R)$, $R>0$. In this direction, we prove the following result, which, as a consequence, gives the quaternionic analogue of Theorem E.

THEOREM 3. *Let* $f:B(0,R)\to\mathbb{H}$ *be a regular power series in the quaternionic variable* q, *i.e.,* $f(q)=\sum_{\nu=0}^{\infty}q^{\nu}a_{\nu}$, *for all* $q\in B(0,R)$. *If* a_{ν}, $\nu=0,1,2,\ldots$, *are quaternionic coefficients such that for some positive real number* t *with* $t<R$, *we have*

$$\max_{|q|=t}\left|(\rho a_0-ta_1)+\sum_{\nu=2}^{\infty}q^{\nu-1}(a_{\nu-1}-ta_{\nu})\right|\le M, \tag{5}$$

where $\rho\ge1$, *then* $f(q)$ *does not vanish in*

$$|q|<\frac{t|a_0|}{(\rho-1)|a_0|+M}.$$

REMARK 1. Let $f(q)=\sum_{\nu=0}^{\infty}q^{\nu}a_{\nu}$ be a power series (where q is a quaternionic variable) regular in $B(0,R)$ with real and positive coefficients satisfying

$$\rho a_0\ge ta_1\ge t^2a_2\ge\cdots,$$

for some $\rho\ge1$ and $0<t<R$. Then

$$\begin{aligned}\max_{|q|=t}\left|(\rho a_0-ta_1)+\sum_{\nu=2}^{\infty}q^{\nu-1}(a_{\nu-1}-ta_{\nu})\right|&\le(\rho a_0-ta_1)+\sum_{\nu=2}^{\infty}t^{\nu-1}(a_{\nu-1}-ta_{\nu})\\&=\rho|a_0|\\&=M\ \ (\text{say}).\end{aligned}$$

It follows from Theorem 3 that $f(q)$ does not vanish in

$$|q|<\frac{t}{2\rho-1}.$$

For $\rho=1$, we get from Theorem 3 the following result:

COROLLARY 4. *Let $f: B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$, for all $q \in B(0,R)$. If a_{ν}, $\nu = 0,1,2,\ldots$, are quaternionic coefficients such that for some positive real number t with $t < R$, we have*

$$\max_{|q|=t} \left| \sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - t a_{\nu}) \right| \leq M,$$

then $f(q)$ does not vanish in

$$|q| < \frac{t|a_0|}{M}.$$

REMARK 2. Let $f(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ be a power series (where q is a quaternionic variable) regular in $B(0,R)$ with real coefficients satisfying

$$0 \neq a_0 \leq t a_1 \leq \cdots \leq t^{\lambda} a_{\lambda} \geq t^{\lambda+1} a_{\lambda+1} \geq \cdots,$$

for $0 < t < R$ and $\lambda \geq 0$. Then

$$\begin{aligned}
\max_{|q|=t} &\left| \sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - t a_{\nu}) \right| \\
&\leq \sum_{\nu=1}^{\infty} t^{\nu-1} |a_{\nu-1} - t a_{\nu}| \\
&\leq \sum_{\nu=1}^{\infty} t^{\nu-1} \big| t|a_{\nu}| - |a_{\nu-1}| \big| + \sum_{\nu=1}^{\infty} t^{\nu-1} \big| t(a_{\nu} - |a_{\nu}|) - (a_{\nu-1} - |a_{\nu-1}|) \big| \\
&= \sum_{\nu=1}^{\lambda} t^{\nu-1} \big(t|a_{\nu}| - |a_{\nu-1}| \big) + \sum_{\nu=\lambda+1}^{\infty} t^{\nu-1} \big(|a_{\nu-1}| - t|a_{\nu}| \big) \\
&\qquad + \sum_{\nu=1}^{\infty} t^{\nu-1} \big| t(a_{\nu} - |a_{\nu}|) - (a_{\nu-1} - |a_{\nu-1}|) \big| \\
&= 2t^{\lambda} |a_{\lambda}| - |a_0| + \sum_{\nu=1}^{\infty} t^{\nu-1} \big| t(a_{\nu} - |a_{\nu}|) - (a_{\nu-1} - |a_{\nu-1}|) \big| \\
&\leq 2t^{\lambda} |a_{\lambda}| - |a_0| + 2 \sum_{\nu=0}^{\infty} t^{\nu} \big| a_{\nu} - |a_{\nu}| \big| \\
&= |a_0| \left(2t^{\lambda} \left| \frac{a_{\lambda}}{a_0} \right| - 1 + \frac{2}{|a_0|} \sum_{\nu=0}^{\infty} t^{\nu} \big| a_{\nu} - |a_{\nu}| \big| \right) \\
&= M \quad \text{(say)}.
\end{aligned}$$

It follows from Corollary 4 that $f(q)$ does not vanish in

$$|q| < \frac{t}{2t^{\lambda} \left| \frac{a_{\lambda}}{a_0} \right| - 1 + \frac{2}{|a_0|} \sum_{\nu=0}^{\infty} t^{\nu} \big| a_{\nu} - |a_{\nu}| \big|}.$$

REMARK 3. If in Remark 2, we suppose $a_{\nu} > 0$, $\nu = 0,1,2,\ldots$, and take $\lambda = 0$, we get the quaternionic analogue of Theorem E.

3. Proofs of the main results

Proof of Theorem 1. We have

$$T(q) = a_0 + qa_1 + q^2a_2 + \cdots + q^{n-1}a_{n-1} + q^na_n,$$

so that

$$|T(q)| \geq |a_0| - |P(q)|, \tag{6}$$

where

$$P(q) = qa_1 + q^2a_2 + \cdots + q^{n-1}a_{n-1} + q^na_n.$$

Clearly $P(0) = 0$ and by (2), $|P(q)| \leq M_1$ for $|q| = t$.

Therefore, it follows by Theorem H, that

$$|P(q)| \leq \frac{M_1|q|}{t} \quad \text{for} \quad |q| \leq t,$$

which implies by (6), that

$$|T(q)| \geq |a_0| - \frac{M_1|q|}{t} \quad \text{for} \quad |q| \leq t.$$

Hence, if

$$|q| < \min\left(\frac{t|a_0|}{M_1}, t\right),$$

then $T(q) \neq 0$.

In other words, all the zeros of $T(q)$ lie in

$$|q| \geq \min\left(\frac{t|a_0|}{M_1}, t\right). \tag{7}$$

Now let

$$\psi(q) = q^n * T(1/q) = q^na_0 + q^{n-1}a_1 + \cdots + qa_{n-1} + a_n,$$

so that

$$|\psi(q)| \geq |a_n| - |H(q)|, \tag{8}$$

where

$$H(q) = qa_{n-1} + q^2a_{n-2} + \cdots + q^{n-1}a_1 + q^na_0.$$

Clearly, $H(0) = 0$ and by (3), $|H(q)| \leq M_2$ for $|q| = t$.

Therefore, it follows by Theorem H, that

$$|H(q)| \leq \frac{M_2|q|}{t} \quad \text{for} \quad |q| \leq t,$$

which implies by (8), that

$$|\psi(q)| \geq |a_n| - \frac{M_2|q|}{t} \quad \text{for} \quad |q| \leq t.$$

Hence, if

$$|q| < \min\left(\frac{t|a_n|}{M_2}, t\right),$$

then $\psi(q) \neq 0$.

In other words, all the zeros of $\psi(q)$ lie in

$$|q| \geq \min\left(\frac{t|a_n|}{M_2}, t\right).$$

As $T(q) = q^n * \psi(1/q)$, it follows that all the zeros of $T(q)$ lie in

$$|q| \leq \max\left(\frac{M_2}{t|a_n|}, \frac{1}{t}\right). \tag{9}$$

Combining (7) and (9), the desired result follows. □

Proof of Theorem 2. We have from (3) and (4), that

$$\begin{aligned}\max_{|q|=t}\left|q^n a_0 + q^{n-1}a_1 + \cdots + qa_{n-1}\right| &= t\max_{|q|=t}\left|q^{n-1}a_0 + q^{n-2}a_1 + \cdots + a_{n-1}\right| \\ &\leq tM = M_2.\end{aligned}$$

Therefore, it follows from Theorem 1 by replacing M_2 by tM, that all the zeros of $T(q)$ lie in

$$|q| \leq \max\left(\frac{M}{|a_n|}, \frac{1}{t}\right).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Consider the power series

$$F(q) = (t-q) * f(q) = ta_0 + (\rho - 1)qa_0 - \psi(q),$$

where

$$\psi(q) = q(\rho a_0 - ta_1) + q\sum_{\nu=2}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_\nu).$$

Clearly, $\psi(0) = 0$ and by (5), $|\psi(q)| \leq tM$ for $|q| = t$.

Since $\psi(q)$ is regular in $B(0,R)$, it follows by Theorem H, that

$$|\psi(q)| \leq M|q| \quad \text{for} \quad |q| \leq t,$$

which implies

$$\begin{aligned}|F(q)| &= |ta_0 + (\rho - 1)qa_0 - \psi(q)| \\ &\geq t|a_0| - (\rho - 1)|q||a_0| - M|q| \quad \text{for} \quad |q| \leq t.\end{aligned}$$

Hence, if

$$|q| < \min\left(\frac{t|a_0|}{(\rho-1)|a_0|+M}, t\right),$$

then $F(q) \neq 0$.

In other words, all the zeros of $F(q)$ lie in

$$|q| \geq \min\left(\frac{t|a_0|}{(\rho-1)|a_0|+M}, t\right). \tag{10}$$

Since

$$\begin{aligned}
\rho|a_0| &= \left|(\rho a_0 - ta_1) + \sum_{\nu=2}^{\infty} t^{\nu-1}(a_{\nu-1} - ta_\nu)\right| \\
&\leq \max_{|q|=t}\left|(\rho a_0 - ta_1) + \sum_{\nu=2}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_\nu)\right| \\
&= M \quad \text{(by Theorem G)},
\end{aligned}$$

it follows that for $\rho \geq 1$,

$$|a_0| \leq \rho|a_0| \leq (\rho-1)|a_0| + M,$$

and, hence

$$\min\left(\frac{t|a_0|}{(\rho-1)|a_0|+M}, t\right) = \frac{t|a_0|}{(\rho-1)|a_0|+M}.$$

Using this in (10), it follows that $F(q)$ does not vanish in

$$|q| < \frac{t|a_0|}{(\rho-1)|a_0|+M}. \tag{11}$$

By Theorem F, the only zeros of $F(q)$ are $q = t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in the disk defined by (11). This completes the proof of Theorem 3. $\square$

4. Conclusion

The historical Cauchy's and the Eneström-Kakeya theorems form an essential part of the classical content of the geometric function theory. They are equally important in modern papers dealing with the regional location of zeros in regular functions of a quaternionic variable. Here, we establish a ring-shaped region with all the zeros of a unilateral polynomial with quaternionic coefficients located on only one side of the quaternionic variable. A zero-free region is also established for the relevant subclass of regular power series.

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