Simple Optimization Method of One-Dimensional M-PAM Constellations for the AWGN Channels

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Abstract— **This paper presents the simple method optimization of one-dimensional M-ary Pulse Amplitude Modulation (M-PAM) constellations for the Additive White Gaussian Noise (AWGN) channels when the error probability is minimized under the power constraint. This method is independent of the distribution information source, and adapts the given source M-PAM signal constellation to transmission the AWGN channel.**

Keywords— **Optimization, Error probability, M-PAM constellation, AWGN channel, average power.**

I. Introduction

It is proved that, for the optimization of any chosen parameter of the communications system, there is a possibility of dividing the problem of the optimal shaping of signals set, by which the information can be transmitted, into the two basic problems: the code choice (the vectors set above some alphabet), and the mapping function choice of this code onto the set of allowed signals. These two problems can be considered separately without making worse the system performances. In this paper, we will discuss the setting of the optimal mapping function of the code words set (or the source symbols) onto the set of allowed signals, by the optimization of the signals set elements, i.e. by defining the optimal signal set (the optimal constellation of signals) for the observed source of information. The problem of designing optimal one-dimensional M-PAM constellations of signals for the AWGN channel with the constraint of the average power is a mathematical problem of non-linear programming (NP-problem). In all previous papers, which discuss the constellation designing, the equiprobable transmit of symbols (i.e. probabilities of appearing the different signals are same) is discussed.

In this paper, the generalized case, the so-called unequiprobable transmit of symbols (with different probabilities of the signal points) will be discussed [1]. Even the papers discussing the unequiprobable transmit do not consider the problem of designing the signal constellations to the unequiprobable transmit, but on the basis of existing constellations they determine the optimal probabilities of the constellation points. Also, in previous papers discussing the designing of the signal constellations, the designing is done on the basis of the continuous approximation [2], or on the basis of some other approximations. By applying these methods, the improvement is got, but the problem of getting the optimal constellations is not solved. So, in this paper, a method for getting the optimal one-dimensional constellations of signals is given. This method is more complex than the existing ones for designing the constellations, but it is one of the simplest methods for getting the optimal solving of the given non-linear problem.

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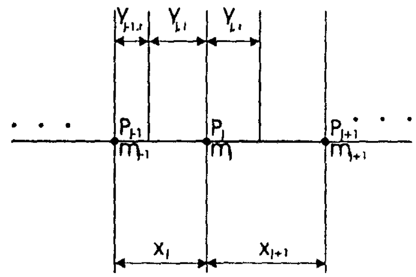

Fig. 1. An example of decision regions one-dimensional signal constellation

II. Description and Optimization

Symmetrical M-PAM constellations (in relation to the coordinate beginning) are better than no-symmetrical ones, $m_{-k} = m_k$, $k = 1, \ldots, L$, $m_0 = 0$ (m_k is the kth constellation point on the positive part of the real axis) for $M = 2L + 1$ and $m_{-k} = m_k$, $k = 1, \ldots, L$, for $M = 2L$. Therefore, in determining the optimal arrangement of constellation points on the real axis, there is a need for observing only the positive part of the real axis.

The problem of determining the optimal arrangement of constellation points with the constraint of average power is actually the problem of minimization of the average probability of the error P_e with the constraint of the average power of the signal P_{av}. This is a problem of nonlinear programming and it can be presented in the following way:

Minimize

$$P_e(x_1, \ldots, x_L) = 2\sum_{j=1}^{L} \frac{P_j}{\sqrt{2\pi}\,\sigma_n} \left(\int_{-\infty}^{-y_{j,l}} e^{-r^2/(2\sigma_n^2)} dr + \int_{y_{j,r}}^{+\infty} e^{-r^2/(2\sigma_n^2)} dr \right),$$

under constraints

$$g(x_1, \ldots, x_L) = 2\sum_{j=1}^{L} P_j \left(\sum_{i=1}^{j} x_i \right)^2 - P_{av} \le 0, \quad (1)$$

$$x_j \ge 0,$$

where σ_n^2 is a average power of the channel noise, m_j is the jth constellation point, P_j is a probability of the jth constellation point $M = 2L$ or $M = 2L+1$ is the number of constellation points, $y_{j,l}$ is a distance between the jth constellation point and the left threshold of deciding for that point, $y_{j,r}$ is a distance between the right threshold of deciding the jth point and the jth constellation point. The left and right threshold of deciding the jth point can be calculated on the basis of MAP principle of detection in the following way (see Fig. 1):

$$y_{1l} = 0, \quad y_{Lr} = +\infty,$$

$$y_{jl} = \frac{\Delta m_{j-1}}{2} + \frac{\sigma_n^2}{\Delta m_{j-1}} \ln \frac{P_j}{P_{j-1}} = \frac{x_j}{2} + \frac{\sigma_n^2}{x_j} \ln \frac{P_j}{P_{j-1}},$$

$$y_{jr} = \frac{\Delta m_j}{2} + \frac{\sigma_n^2}{\Delta m_j} \ln \frac{P_j}{P_{j+1}} = \frac{x_{j+1}}{2} + \frac{\sigma_n^2}{x_{j+1}} \ln \frac{P_j}{P_{j+1}},$$

where $\Delta m_j = m_{j+1} - m_j$ and $j = 1, \ldots, L$. Here, we suppose that $\sigma_n^2 |\ln(P_j/P_{j-1})| \ll x_j^2/2$ for each j.

Before we describe the minimization procedure, we prove the problem of minimization of the $P_e(x_1, \ldots, x_L)$ is a convex programming problem. This follows directly from the following auxiliary result:

Lemma 1: The function $P_e(x_1, \ldots, x_L)$ is a convex function and constraints $g(x_1, \ldots, x_L) \leq 0$ and $x_j \geq 0$ form a convex set.

Proof: In order to prove that $P_e(x_1, \ldots, x_L)$ is a convex function and that the constraints $g(x_1, \ldots, x_L) \leq 0$ and $x_j \geq 0$ form the convex set it is sufficient to prove that the Hessian matrices of P_e and g are positive semi-definite [3, p. 27].

Since

$$\begin{aligned}\frac{\partial P_e}{\partial x_k} &= -\frac{2P_k}{\sqrt{2\pi}\,\sigma_n} e^{-y_{kl}^2/(2\sigma_n^2)} \left(\frac{1}{2} - \frac{\sigma_n^2}{x_k^2} \ln \frac{P_k}{P_{k-1}}\right) \\ &\quad - \frac{2P_{k-1}}{\sqrt{2\pi}\,\sigma_n} e^{-y_{k-1,r}^2/(2\sigma_n^2)} \left(\frac{1}{2} + \frac{\sigma_n^2}{x_k^2} \ln \frac{P_k}{P_{k-1}}\right),\end{aligned}$$

and from MAP principal detection it follows that

$$\frac{P_k}{\sqrt{2\pi}\,\sigma_n} e^{-y_{kl}^2/(2\sigma_n^2)} = \frac{P_{k-1}}{\sqrt{2\pi}\,\sigma_n} e^{-y_{k-1,r}^2/(2\sigma_n^2)},$$

we obtain

$$\frac{\partial P_e}{\partial x_k} = -\frac{2P_k}{\sqrt{2\pi}\,\sigma_n} e^{-y_{kl}^2/(2\sigma_n^2)}$$

and

$$\frac{\partial^2 P_e}{\partial x_k^2} = \frac{P_k y_{k,l}}{\sqrt{2\pi}\,\sigma_n^3} e^{-y_{kl}^2/(2\sigma_n^2)} \left(1 - \frac{2\sigma_n^2}{x_k^2} \ln \frac{P_k}{P_{k-1}}\right) > 0,$$

$$\frac{\partial^2 P_e}{\partial x_k \partial x_j} = 0 \qquad (k \neq j).$$

This means that Hessian matrix of P_e is positive definite. On the other hand, for the constraint we have

$$\frac{\partial g}{\partial x_k} = 4 \sum_{j=k}^{L} P_j \left(\sum_{i=1}^{j} x_i\right), \quad \frac{\partial^2 g}{\partial x_k \partial x_l} = 4 \sum_{j=\max\{k,l\}}^{L} P_j.$$

For each k, we put $a_k = P_k + P_{k+1} + \cdots + P_L$, i.e., $a_k - a_{k+1} = P_k > 0$. Then, the corresponding Hessian matrix becomes

$$4 \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_L \\ a_2 & a_2 & a_3 & \ldots & a_L \\ a_3 & a_3 & a_3 & \ldots & a_L \\ \vdots & & & & \\ a_L & a_L & a_L & \ldots & a_L \end{bmatrix},$$

and we see very easy that it is a positive definite matrix, because its principal minors are

$$D_k = 4^k \left(\sum_{i=k}^{L} P_i\right) \prod_{i=1}^{k-1} P_i > 0 \qquad (k = 1, \ldots, L).$$

This completes the proof. ∎

Also, it should be mentioned that the local minimum is at the same time the global minimum for this kind of problems [3, p. 27]. In our case, variables in the objective function P_e are mutually dependent, so the solution in the closed form, using the Lagrange's multipliers, is not possible. From that reason, we are going to consider the possibility of applying a method of nonlinear programming for convex problems.

Also, it should be mentioned that the direct application of a gradient method is not possible, but the application of some modified gradient method is possible. In general, all gradient methods have the same approach to searching for an optimal solution. First, choose the starting point $\boldsymbol{X}^{(0)}$. A new point is obtained by moving for a step h in the direction of the highest decrease of the given function. Differences of some gradient methods depend on the way of calculating the gradient, and also on the value of moving toward the requested solution along the gradient direction. Generally, in a gradient approach, we start from the point $\boldsymbol{X}^{(0)}$ and we are approaching, iteratively, the optimal solution $\boldsymbol{X}^*$. The new value $\boldsymbol{X}^{(i+1)}$ from $\boldsymbol{X}^{(i)}$ is defined by $\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} + \boldsymbol{d}^{(i)} h^{(i)}$, i.e.,

$$x_k^{(i+1)} = x_k^{(i)} + d_k^{(i)} h^{(i)},$$

where $\boldsymbol{d}^{(i)}$ (the unit vector of the moving destination) is got on the basis of the gradient $\nabla F(\boldsymbol{X}^{(i)})$. The simplest approach is if we take $d_k^{(i)} = \partial F(\boldsymbol{X})/\partial x_k^{(i)}$ and $h = \text{const}$.

It is known (see [4])) that good methods for solving the convex programs are those of permissible unit vectors of the movement, when the permissible unit vector $(\boldsymbol{d})$ from a point is defined by solving the linear program:

$$\begin{aligned} & \max \alpha \\ \text{subject} \quad & \\ & \nabla P_e(x_1^*, \ldots, x_L^*)\,\boldsymbol{d} + \alpha \leq 0, \\ & \nabla g(x_1^*, \ldots, x_L^*)\,\boldsymbol{d} + \alpha \leq 0. \end{aligned} \tag{2}$$

In order to avoid the solving of linear system in each iteration, the permissible unit vector of movement would consist only of two coordinates. This can be done by moving one constellation point for Δx_j, and that should be the one which gives the greatest decrease of the error probability, and the movement of some other point in the opposite direction for Δx_k should be done in order to enable the average power to stay unchangeable. For the small step value of moving, the total differentials of P_e and g can be approximated for the previously assumed unit vector of movement:

$$\Delta P_e \approx \frac{\partial P_e}{\partial x_j}\Delta x_j + \frac{\partial P_e}{\partial x_k}\Delta x_k < 0 \ (\Delta x_j > 0,\ \Delta x_k < 0), \quad (3)$$

$$\Delta g \approx \frac{\partial g}{\partial x_j}\Delta x_j + \frac{\partial g}{\partial x_k}\Delta x_k < 0 \quad (\Delta x_j > 0,\ \Delta x_k < 0). \quad (4)$$

In order to find the most efficient movement, the quantity $|\Delta P_e|$ should be maximized. From (4), we get $\Delta_k = -\frac{\partial g/\partial x_j}{\partial g/\partial x_k}\Delta x_j$, and by replacing it in (3), we get

$$\Delta P_e \approx \frac{\partial P_e}{\partial x_j}\Delta x_j - \frac{\partial P_e}{\partial x_k}\frac{\partial g/\partial x_j}{\partial g/\partial x_k}\Delta x_j < 0.$$

Thus,

$$\max_{x_j,x_k}\left|\frac{\partial P_e/\partial x_j}{\partial g/\partial x_j} - \frac{\partial P_e/\partial x_k}{\partial g/\partial x_k}\right|,$$

because for the defined problem, $\partial P_e/\partial x_k < 0$ and $\partial g/\partial x_k > 0$ and the maximum is got for

$$\max_{x_i}\left|\frac{\partial P_e/\partial x_i}{\partial g/\partial x_i}\right| \quad \text{and} \quad \max_{x_j \neq x_i}\left|\frac{\partial P_e/\partial x_j}{\partial g/\partial x_j}\right|.$$

The previously explained method has the great similarity with the method of the reduced gradient which is used in the approximate linear programming (nonlinear problem is approximated by the linear one for the small step of movement-approximation of the total increments of the function and constraints) [4].

In order to avoid the search of the maximum and minimum for each iteration or for solving the linear problem, principles which enable getting, approximately, the direction of the fastest descent, are introduced. The first principle lies in the following: *If $|\partial P_e/\partial x_k| > \partial g/\partial x_k$, then $d_i > 0$, and if $|\partial P_e/\partial x_k| < \partial g/\partial x_k$, then $d_i < 0$.* By applying the first principle, we should determine the coordinates directions in the permissible unit vector. According to the second principle, we should have in mind the influence of the quotient value $\left|\frac{\partial P_e/\partial x_i}{\partial g/\partial x_i}\right|$ on the coordinates value in the permissible unit vector. For the same directions, that coordinate, which quotient value is greater, is more important. The application of these principles, without any comparisons, can be realized by calculating the movement direction coordinate $d_k^{(i)}$ $(k = 1, \ldots, L)$ in the following way:

$$d_k^{(i)} = \frac{\left|\frac{\partial P_e}{\partial x_k} \Big/ \frac{\partial g}{\partial x_k}\right|}{\sum_{j=1}^{L}\left|\frac{\partial P_e}{\partial x_j} \Big/ \frac{\partial g}{\partial x_j}\right|} - \frac{\left|\frac{\partial g}{\partial x_k} \Big/ \frac{\partial P_e}{\partial x_k}\right|}{\sum_{j=1}^{L}\left|\frac{\partial g}{\partial x_j} \Big/ \frac{\partial P_e}{\partial x_j}\right|}.$$

If the medium power is changed by the fixed beforehand value P_{av}, then positions of constellation points, after each iteration, can be calculated as follows:

$$m_{k,\,\text{rescal}}^{(i+1)} = \sqrt{\lambda}\, m_k^{(i+1)} \quad (k = 1, \ldots, L); \quad \lambda = P_{av}/P_{av}^{(i+1)}.$$

The determination of searching direction in the way previously explained is used intuitively in the papers [5], [6].

III. The Algorithm

The algorithm for determining the optimal nonequiprobable nonuniform M-PAM constellation signals for the AWGN channel can be expressed as follows:

Step 1: Initialization. The arrangement of obtained applying quantization methods or any other constellation signals is used as the starting value $(\boldsymbol{X}^{(0)})$. The determination of $h^{(0)}$ is experimentally.

Step 2: Determine the average error probability gradient ∇P_e,

$$\frac{\partial P_e}{\partial x_k} = -\frac{2P_k}{\sqrt{2\pi}\,\sigma_n}\, e^{-y_{kl}^2/(2\sigma_n^2)} \qquad (k = 1, \ldots, L),$$

where $x_k = y_{k-1,r} + y_{k,l}$.

Step 3: Determine $\boldsymbol{X}^{(i+1)}$ from $\boldsymbol{X}^{(i)}$ using the iteration $\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} + \boldsymbol{d}^{(i)}h^{(i)}$, i.e., $x_k^{(i+1)} = x_k^{(i)} + d_k^{(i)}h^{(i)}$, where

$$d_k^{(i)} = \left(\frac{\left|\frac{\partial P_e}{\partial x_k} \Big/ \frac{\partial g}{\partial x_k}\right|}{\sum_{j=1}^{L}\left|\frac{\partial P_e}{\partial x_j} \Big/ \frac{\partial g}{\partial x_j}\right|} - \frac{\left|\frac{\partial g}{\partial x_k} \Big/ \frac{\partial P_e}{\partial x_k}\right|}{\sum_{j=1}^{L}\left|\frac{\partial g}{\partial x_j} \Big/ \frac{\partial P_e}{\partial x_j}\right|}\right)\Bigg|_{\boldsymbol{X}=\boldsymbol{X}^{(i)}}$$

and $k = 1, \ldots, L$.

Step 4: The average power to stay unchangeable calculate $M_{\text{rescal}}^{(i+1)}$ and $\boldsymbol{X}_{\text{rescal}}^{(i+1)}$ from

$$m_{k,\,\text{rescal}}^{(i+1)} = \sum_{j=1}^{k} x_{j,\,\text{rescal}}^{(i+1)}, \qquad x_{j,\,\text{rescal}}^{(i+1)} = \sqrt{\lambda}\, x_j^{(i+1)},$$

where $k = 1, \ldots, L$ and $\lambda = P_{av}/P_{av}^{(i+1)}$.

Step 5: If $P_e(M_{\text{rescal}}^{(i+1)}) - P_e(M_{\text{rescal}}^{(i)}) \geq 0$, the procedure is finished for $h^{(i)}$. Otherwise, set $\boldsymbol{X}^{(i)} = \boldsymbol{X}_{\text{rescal}}^{(i+1)}$ and go to **Step 2**.

Step 6: If $\max|d_k^{(i)}h^{(i)}| \leq \varepsilon$, where ε is the required accuracy, the procedure is finished $(M_{\text{opt}} = M_{\text{rescal}}^{(i)})$. Otherwise, set $h^{(i)} := 0.5h^{(i)}$, $\boldsymbol{X}^{(i)} := \boldsymbol{X}_{\text{rescal}}^{(i)}$ and go to **Step 2**.

It should be mentioned that this optimization method of M-PAM constellation of signals for transmitting information over the AWGN channel is independent of the information source, and it can be applied to any distribution of the symbols (data) source probability. The optimization is especially done for different relations signal-to-noise ratio (SNR).

TABLE I

SIGNAL CONSTELLATION POINTS ARRANGEMENTS BEFORE (THE FIRST ROW) AND AFTER (THE SECOND AND THIRD ROWS) OPTIMIZATION

$M_q^{(I)}$	±0.07419	±0.22258	±0.38187	±0.55242	±0.74221	±0.97729	±1.27027	±1.73338
M_{opt} (SNR = 28 dB)	±0.09472	±0.28405	±0.47531	±0.66910	±0.86663	±1.07125	±1.28350	±1.50839
M_{opt} (SNR = 33 db)	±0.09720	±0.29159	±0.48656	±0.68225	±0.87908	±1.07825	±1.28009	±1.48639

Example 1: For SNR 28 and 33 dB and 16 constellation points, it is got the optimal arrangement presented in Table I. The first row of table presents the arrangement of constellation points got by the quantization of the Gauss source and by the equal probabilities [7]. On SNR=33 dB, the error probability for the no-optimized constellation from the first row of Table I is $P_e = 3.736 \times 10^{-4}$, while, for the optimized constellation, SNR is the same $P_e = 7.218 \times 10^{-6}$.

IV. CONCLUSION

In this paper, simple method for determining the conditional minimum of the average error probability per symbol under the average power constraint is presented. This optimization method is independent of the information source and can be applied for generating the optimal nonequiprobable nonuniform M-PAM signals to transmission AWGN channel. The optimization is especially done for different relations signal-to-noise ratio (SNR).

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