3. Now we shall show how to get the conditions for the inclusion $\overline{\mathscr{L}}_p(\mathbf{r},\mu) \subseteq \overline{\mathscr{L}}_q(\mathbf{r},\mu)$.

THEOREM 4. Let $0 < p,\ q \le \infty$. The inclusion $\overline{\mathscr{L}}_p(\mathbf{r},\mu) \subseteq \overline{\mathscr{L}}_q(\mathbf{r},\mu)$ holds if and only if for each m there exists a k such that $r_m L_p(\mu) \subseteq r_k L_q(\mu)$.

Proof. The sufficiency of the condition is obvious. We shall prove the necessity. Let $\overline{\mathscr{L}}_p(\mathbf{r},\mu) \subseteq \overline{\mathscr{L}}_q(\mathbf{r},\mu)$. We consider a number $0 < s < \min(p, q)$ and we set $p^* = p^*(s)$, $q^* = q^*(s)$. According to property 1) of the operation of taking the dual and (2)

$$\mathscr{L}_{p^*}(1/\mathbf{r},\mu) = (\overline{\mathscr{L}}_p(\mathbf{r},\mu))^*_{L_s(\mu)} \supseteq (\overline{\mathscr{L}}_q(\mathbf{r},\mu))^*_{L_s(\mu)} = \mathscr{L}_{q^*}(1/\mathbf{r},\mu).$$

Now using Theorem 2, we get that for each m there exists a k such that $(1/r_k) L_{q^*}(\mu) \subseteq (1/r_m) L_{p^*}(\mu)$. Again passing to the dual with respect to $L_s(\mu)$, we arrive at the inclusion $r_k L_{q^{**}}(\mu) \supseteq r_m L_{p^{**}}(\mu)$. But $p^{**} = p$ and $q^{**} = q$. Thus the theorem is proved.

LITERATURE CITED

1. B. Subramanian, "On the inclusion $L_p(\mu) \subset L_q(\mu)$," Am. Math. Monthly, 85, No. 6, 479-481 (1978).

INEQUALITIES WITH CONVEX SEQUENCES

G. V. Milovanovich and I. Zh. Milovanovich

UDC 517.51

In this paper we prove some inequalities with mean powers for convex sequences of order k and one inequality of Hölder type.

We give some definitions and theorems, which will be used later in the paper.

Definition. For a positive sequence $\mathbf{a} = (a_1, \ldots, a_n)$ the mean power of order r, $r \in R$, $r = \pm\infty$, is defined by the formula

$$M_n^{[r]}(\mathbf{a},\mathbf{p}) = \begin{cases} \left(\sum_{i=1}^n p_i a_i^r / \sum_{i=1}^n p_i\right)^{1/r}, & r \ne 0,\ |r| < +\infty, \\ \left(\prod_{i=1}^n a_i^{p_i}\right)^{1/P}, & r = 0, \\ \max(a_1, \ldots, a_n), & r = +\infty, \\ \min(a_1, \ldots, a_n), & r = -\infty, \end{cases}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ is a weight sequence, $P = \sum_{i=1}^n p_i$.

Let us assume that S_k is the set of all real convex sequences $\mathbf{a} = (a_1, \ldots, a_n)$ of order k, $1 < k < n$,

$$S_k = \left\{\mathbf{a} \mid \Delta^k a_m = \sum_{i=0}^k (-1)^i \binom{k}{i} a_{m+k-i} \geqq 0,\quad 1 \leqq m \leqq n-k\right\}.$$

We define a sequence $\mathbf{a}^{(r)} = (a_1^{(r)}, \ldots, a_n^{(r)})$ (r is a natural number) as follows:

$$a_m^{(r)} = m^{-1} a_m^{(r-1)},\quad a_m^{(1)} = a_m,\quad a_m^{(r)} = a_m / m^{r-1}.$$

Let $S_k^{(p)} = \{\mathbf{a} \mid \mathbf{a} \in S_k \wedge (\Delta^{k-i} a_1^{(i+1)} \geqq 0,\ i = 1, \ldots, p)\}$, where $p < k$.

In [1] theorems are proved according to which, for each $k \in \{2, 3, \ldots\}$ one has the implications

$$\mathbf{a} \in S_k^{(1)} \Rightarrow \mathbf{a}^{(2)} \in S_{k-1} \text{ and } \mathbf{a} \in S_k^{(k-1)} \Rightarrow \mathbf{a}^{(k)} \in S_1.$$

Yugoslavia. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 34, No. 4, pp. 522-525, July-August, 1982. Original article submitted November 2, 1980.

0041-5995/82/3404-0426$07.50

Using theorems from [1] we shall prove the following theorem.

THEOREM 1. If $p_i > 0$, $i = 1, \ldots, n$, and $\mathbf{x} = (x_1, \ldots, x_n)$ is a positive sequence from $S_k^{(k-1)}$, $n > k$, then for $r \geqq s$

$$M_n^{[r]}(\mathbf{x};\mathbf{p}) \geqq \alpha_k M_n^{[s]}(\mathbf{x};\mathbf{p}) \tag{2}$$*

where α_k is a constant, calculated according to the formula

$$\alpha_k = M_n^{[r]}(\mathbf{a};\mathbf{p})/M_n^{[s]}(\mathbf{a};\mathbf{p}) \geqq 1, \quad \mathbf{a} = (1^{k-1}, \ldots, n^{k-1}).$$

The equality in (2) is achieved for $\mathbf{x} = \mathbf{a}$.

Proof. To prove (2) we first set $p_i = p_i i^{s(k-1)}$, $x_i = x_i / i^{k-1}$, $i = 1, \ldots, n$ in the inequality [2]

$$M_n^{[r]}(\mathbf{x};\mathbf{p}) \geqq M_n^{[s]}(\mathbf{x};\mathbf{p}), \quad r \geqq s. \tag{3}$$

Then, defining for any sequences $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ the sequence $\mathbf{ab} = (a_1 b_1, \ldots, a_n b_n)$, we get

$$M_n^{[r]}(\mathbf{xa}^{-1};\mathbf{pa}^s) \geqq M_n^{[s]}(\mathbf{xa}^{-1};\mathbf{pa}^s), \ \mathbf{a}^t = (1^{t(k-1)}, \ldots, n^{t(k-1)}).$$

To complete the proof we shall show that one has

$$\sum_{i=1}^n p_i x_i^r \sum_{i=1}^n p_i i^{s(k-1)} \geqq \sum_{i=1}^n p_i i^{r(k-1)} \sum_{i=1}^n p_i x_i^r i^{(s-r)(k-1)}.$$

It is obtained from Chebyshev's inequality [3]

$$\sum_{i=1}^n q_i \sum_{i=1}^n q_i u_i v_i \geqq \sum_{i=1}^n q_i u_i \sum_{i=1}^n q_i v_i$$

for $q_i = p_i i^{s(k-1)}$, $u_i = i^{(r-s)(k-1)}$, $v_i = x_i^r / i^{r(k-1)}$, $i = 1, \ldots, n$.

Since the sequence $\mathbf{x} = (x_1, \ldots, x_n)$ belongs to $S_k^{(k-1)}$, $k \geqq 2$, according to a theorem from [1] the sequence $(x_1/1^{k-1}, \ldots, x_n/n^{k-1})$ is nondecreasing.

If in (3) one sets $x_i = i^{k-1}$, then $\alpha_k \geqq 1$.

COROLLARY 1. Since $\alpha_k \geqq 1$, (2) is more precise than (3).

COROLLARY 2. For $k = 2$, from Theorem 1 we get the theorem connected with Theorem 3 of [4].

We note that this theorem is proved in [4], and later also proved in [5].

COROLLARY 3. We introduce $x_i = i^k$, $i = 1, \ldots, n$ in (2). Then $\alpha_{k+1} \geqq \alpha_k$, so (2) becomes more precise with increasing k.

COROLLARY 4. If $p_i = 1$, $i = 1, \ldots, n$, then (2) assumes the form

$$\left(\sum_{i=1}^n x_i^r\right)^{1/r} \geqq M(k)\left(\sum_{i=1}^n x_i^s\right)^{1/s}, \tag{4}$$

where

$$M(k) = \left(\sum_{i=1}^n i^{r(k-1)}\right)^{1/r} \Big/ \left(\sum_{i=1}^n i^{s(k-1)}\right)^{1/s}.$$

Since $\lim\limits_{n\to+\infty}(n^{(r-s)/sr} M(k)) = (s(k-1)+1)^{1/s}/(r(k-1)+1)^{1/r}$, as $n \to +\infty$ from (4) one can get the inequalities for convex functions of order k proved in [6].

Remark. On an integral analog of Theorem 1 cf. [7].

Analogously to Theorem 1, one can prove the following theorem.

THEOREM 2. Let the sequence $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ be such that $p_1 > 0, \ldots, p_n > 0$; $x_1 > 0, \ldots, x_n > 0$; $b_1 > 0, \ldots, b_n > 0$; $b_1 \leqq b_2 \leqq \ldots \leqq b_n$ and $(x_1/b_1, \ldots, x_n/b_n)$, $n > k$ be a sequence from $S_k^{(k-1)}$, $k \geqq 2$. Then for $r \geqq s$ one has

*Numbered as in Russian original – Publisher.

$$M_n^{[r]}(\mathbf{x};\mathbf{p})/M_n^{[r]}(\mathbf{b};\mathbf{p}) \geqq H_k M_n^{[s]}(\mathbf{x};\mathbf{p})/M_n^{[s]}(\mathbf{b};\mathbf{p}), \tag{5}$$

where $H_k = M_n^{[r]}(\mathbf{a};\mathbf{pb}')/M_n^{[s]}(\mathbf{a};\mathbf{pb}') \geqq 1$, and the sequence **a** is defined in Theorem 1. The equality in (5) is achieved for $x_i = b_i i^{k-1}$, $i = 1, \ldots, n$.

COROLLARY 5. In (5) we set $x_i = b_i$, $i = 1, \ldots, n$. We get $H_{k+1} \geqq H_k$, and thus (5) becomes more precise when the order of convexity of the sequence $(x_1/b_1, \ldots, x_n/b_n)$ increases.

COROLLARY 6. For k = 2 from Theorem 2 one can get the theorem connected with Theorem 4, proved in [4].

THEOREM 3. Let the sequence $\mathbf{p} = (p_1, \ldots, p_n)$ be positive. Let the r sequences $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n), \ldots, \mathbf{l} = (l_1, \ldots, l_n)$ be positive and belong to the set $S_k^{(k-1)}$, $n > k$. Then for $0 \leqq m_i \leqq 1$, $i = 1, \ldots, r$ one has

$$\frac{\sum_{i=1}^n p_i a_i b_i \ldots l_i}{\left(\sum_{i=1}^n p_i a_i^{m_1}\right)^{1/m_1} \left(\sum_{i=1}^n p_i b_i^{m_2}\right)^{1/m_2} \cdots \left(\sum_{i=1}^n p_i l_i^{m_r}\right)^{1/m_r}} \geqq Q_r, \tag{6}$$

where

$$Q_r = \frac{\sum_{i=1}^n p_i i^{r(k-1)}}{\left(\sum_{i=1}^n p_i i^{m_1(k-1)}\right)^{1/m_1} \cdots \left(\sum_{i=1}^n p_i i^{m_r(k-1)}\right)^{1/m_r}}.$$

The equality in (6) is achieved for $a_i = b_i = \ldots = l_i = i^{k-1}$, $i = 1, \ldots, n$.

Proof. In [8] for sequences from the set $S_k^{(k-1)}$, $k \geqq 2$ there is proved Chebyshev's inequality

$$\sum_{i=1}^n p_i a_i b_i \ldots l_i \geqq \frac{\sum_{i=1}^n p_i i^{r(k-1)}}{\left(\sum_{i=1}^n p_i i^{k-1}\right)^r} \left(\sum_{i=1}^n p_i a_i\right) \cdots \left(\sum_{i=1}^n p_i l_i\right). \tag{7}$$

From (7) we get

$$\frac{\sum_{i=1}^n p_i a_i \ldots l_i}{\left(\sum_{i=1}^n p_i a_i^{m_1}\right)^{1/m_1} \cdots \left(\sum_{i=1}^n p_i l_i^{m_r}\right)^{1/m_r}} \geqq \frac{\sum_{i=1}^n p_i i^{r(k-1)}}{\left(\sum_{i=1}^n p_i i^{k-1}\right)^r} \frac{\left(\sum_{i=1}^n p_i a_i\right) \cdots \left(\sum_{i=1}^n p_i l_i\right)}{\left(\sum_{i=1}^n p_i a_i^{m_1}\right)^{1/m_1} \cdots \left(\sum_{i=1}^n p_i l_i^{m_r}\right)^{1/m_r}}. \tag{8}$$

Using Theorem 1 we get

$$\sum_{i=1}^n p_i x_i \Big/ \left(\sum_{i=1}^n p_i x_i^{m_i}\right)^{1/m_i} \geqq \sum_{i=1}^n p_i i^{k-1} \Big/ \left(\sum_{i=1}^n p_i i^{(k-1)m_i}\right)^{1/m_i}. \tag{9}$$

From (8) and (9) follows (6).

COROLLARY 7. For k = 2, r = 2, $p_i = 1$, $i = 1, \ldots, n$, from Theorem 3 we get a theorem similar to Theorem 3 of [5].

LITERATURE CITED

1. G. V. Milovanovich and I. Z. Milovanovich, "Neke diskretne nejednakosti za nizove Konveksne reda k," Naucni Podmladak, No. 1/2, 20-26 (1981).
2. D. S. Mitrinovic and P. M. Vasic (eds.), Analytic Inequalities, Springer-Verlag (1970).
3. D. S. Mitrinovich and P. M. Vasic, "History, variations and generalizations of the Cebysev inequality and the question of some priorities," Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 461-497, 1-30 (1974).

4. P. M. Vasic and I. Z. Milovanovic, "On the ratio of means," Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 577-598, 33-37 (1977).
5. R. T. Rakhmail, "Refinement and inversion of certain inequalities for convex and concave sequences," Ukr. Mat. Zh., 30, No. 6, 840-845 (1978).
6. H. Thunsdorff, Konvexe Funktionen und Ungleichungen [in German], Göttingen (1932).
7. Lj. R. Stankovic, "Extension of Fhunsdorff's inequality to the case of convex function of order k," Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 438-541, 145-148 (1975).

SOLUTION OF THE FOKKER–PLANCK–KOLMOGOROV EQUATION FOR NONAUTONOMOUS SYSTEMS SUBJECTED TO PERIODIC AND RANDOM DISTURBANCES

Nguyen Dong An

UDC 517

The effect of a random disturbance on mechanical systems can be properly studied by the method of Fokker–Planck–Kolmogorov (FPK) equations, especially when the latter is combined with the asymptotic method of nonlinear mechanics [1]. In the nonzutonomous case, however, it was noted in [1] that the corresponding FPK equation will be complicated. In this paper we shall solve the FPK equation for an important class of nonautonomous systems. On the basis of [2] we shall seek the solution in the form of a series for the amplitude. We obtain a system of separable differential equations that makes it possible to successively find the series coefficients of any order.

1. Let us consider a nonautonomous mechanical system with one degree of freedom whose equation of motion has the form

$$\ddot{x}+\omega^2 x=\varepsilon f(x,\dot{x})+\varepsilon P\cos\nu t+\sqrt{\varepsilon}\sigma\dot{\xi}(t) \tag{1}$$

in the principal resonance region

$$\omega^2=\nu^2+\varepsilon\Delta, \tag{2}$$

where $\dot{\xi}(t)$ is white noise of unit intensity, and

$$f(x,\dot{x})=\sum_{s=1}^{m}\alpha_s\left(\sum_{i,j=0}^{i+j=s}\gamma_{ij}x^i\dot{x}^j\right),\ \alpha_s,\gamma_{ij}=\text{const} \tag{3}$$

is a polynomial in x and $\dot{x}$.

With the use of (2) let us rewrite (1) in the form

$$\ddot{x}+\nu^2 x=\varepsilon f_1(x,\dot{x},\nu t)+\sqrt{\varepsilon}\sigma\dot{\xi}(t), \tag{4}$$

where

$$f_1(x,\dot{x},\nu t)=f(x,\dot{x})-\Delta x+P\cos\nu t. \tag{5}$$

By a change of variables [1]

$$x=a\cos\psi,\quad \dot{x}=-a\nu\sin\psi,\quad \psi=\nu t+\theta \tag{6}$$

we can transform Eq. (4) with the aid of Ito's formula to standard form

$$\begin{aligned} da&=\left[-\frac{\varepsilon}{\nu}f_1(x,\dot{x},\nu t)\sin\psi+\frac{\varepsilon\sigma}{2\nu^2 a}\cos^2\psi\right]dt-\frac{\sqrt{\varepsilon}\sigma}{\nu}\sin\psi d\xi(t),\\ d\theta&=\left[-\frac{\varepsilon}{a\nu}f_1(x,\dot{x},\nu t)\cos\psi-\frac{\varepsilon\sigma^2}{a^2\nu^2}\sin\psi\cos\psi\right]dt-\frac{\sqrt{\varepsilon}\sigma}{a\nu}\cos\psi d\xi(t). \end{aligned} \tag{7}$$

Vietnam. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 34, No. 4, pp. 525-528, July-August, 1982. Original article submitted July 14, 1981.