

Müntz orthogonal systems and related Gaussian quadrature rules

Aleksandar S. Cvetković and Gradimir V. Milovanović

Abstract

In this paper we give an account on Müntz polynomials orthogonal on $(0, 1)$, the corresponding generalized quadratures of Gaussian type, as well as their numerical construction. Also, we propose an efficient method for calculating the Müntz-Legendre polynomials by using a complex integration.

1 Introduction

Let $\Lambda = \{\lambda_k \mid k \in \mathbb{N}_0\}$ be a sequence of complex numbers. By Λ_n we assume the system of the first $n + 1$ elements of Λ . We adopt the following definition

$$x^{\lambda_k} = e^{\lambda_k \log x}, \quad x \in (0, +\infty), \quad k \in \mathbb{N}_0.$$

Provided $\lambda_k \neq \lambda_n$, $k \neq n$, we define $M_n(\Lambda)$, $n \in \mathbb{N}_0$, to be the (complex) linear span of the set

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

In the case we have some equalities among numbers λ_k , $k \in \mathbb{N}_0$, we need some caution. Let k_ν , $\nu \in \{0, 1, \dots\}$, be all indexes for which some number z appears in the sequence Λ . Here, the set in which ν runs can be finite or infinite. In the sequence of powers we replace every occurrence of $x^z = x^{\lambda_{k_\nu}}$ with $x^{\lambda_{k_0}} \log^{\nu-1} x$, $\nu \in \{0, 1, \dots\}$. For such sequences Λ , which allow a repetition, we define $M_n(\Lambda)$ to be the (complex) linear span of the modified sequence of powers, which construction has been described. For example, the sequence $\lambda_{2k} = \lambda_{2k+1} = k$, $k \in \mathbb{N}_0$, has modified sequence of powers

$$1, \log x, x, x \log x, \dots, x^k, x^k \log x, \dots$$

In order to avoid a constant repetition of this fact we define x^{Λ_n} to represent the general basis which allows some elements of the sequence Λ to be equal.

By x^Λ we represent the union of all x^{Λ_n} , $n \in \mathbb{N}_0$. By $M(\Lambda)$ we denote the union of all sets $M_n(\Lambda)$, $n \in \mathbb{N}_0$. It can be proved easily that x^Λ is a linearly independent system of functions on $(0, 1]$.

Now, we can introduce an inner product on the space $M(\Lambda)$. Given Borel positive measure μ , supported on $[0, 1]$, such that x^Λ are $L^2(\mu)$ -integrable, we define

$$(p, q) = \int_0^1 p(x)\overline{q(x)}d\mu(x), \quad p, q \in M(\Lambda).$$

It is very easy to obtain the following result.

Lemma 1. *The function $(\cdot, \cdot) : M(\Lambda)^2 \mapsto \mathbb{C}$ is an inner product.*

The only thing which may be tricky is to prove $(p, p) > 0$ for $p \neq 0$, $p \in M(\Lambda)$, but this is a consequence of the fact that $|p|^2$ is nonnegative on $(0, 1)$ and the support of the Borel positive measure μ is $[0, 1]$ (see [19, p. 70]).

Using the well-known Gram-Schmidt orthogonalization procedure (see [18]), we can get an orthogonal system of vectors in $M_n(\Lambda)$, starting with the basis x^{Λ_n} .

Definition 1. *Given a sequence Λ and a positive measure μ , supported on the interval $[0, 1]$, the Müntz- μ orthonormal polynomials $p_n(x; \Lambda, \mu)$, $n \in \mathbb{N}_0$, with respect to the measure μ , are the one obtained using Gram-Schmidt procedure for the scalar product*

$$(p, q) = \int_{-1}^1 p(x)\overline{q(x)}d\mu(x), \quad p, q \in M(\Lambda),$$

starting with the basis x^Λ , so that they satisfy

$$(p_n(\cdot; \Lambda, \mu), p_k(\cdot; \Lambda, \mu)) = \delta_{n,k}, \quad n, k \in \mathbb{N}_0.$$

For any sequence $\gamma_k \neq 0$, $k \in \mathbb{N}_0$, the sequence $\gamma_k p_k(\cdot; \Lambda, \mu)$, $k \in \mathbb{N}_0$, is called the sequence of Müntz- μ orthogonal polynomials with respect to the measure μ .

When there is no possibility of confusion we write simply $p_n(\cdot)$ instead of $p_n(\cdot; \Lambda, \mu)$.

It is quite interesting that for the Legendre measure, $d\mu(x) = \chi_{[0,1]}(x)dx$, we can actually express the Müntz-Legendre polynomials in a closed form. At first, we note that the $L^2(\mu)$ -integrability of the basis functions imposes the condition $\text{Re}(\lambda_n) > -1/2$, $n \in \mathbb{N}_0$, since we require $x^{2\text{Re}\lambda_n}$, $n \in \mathbb{N}_0$, to be integrable.

In order to present a closed form of the Müntz-Legendre polynomials, we introduce the following rational functions

$$W_n(s; \Lambda) = \frac{1}{s - \lambda_n} \prod_{k=0}^{n-1} \frac{s + \bar{\lambda}_k + 1}{s - \lambda_k}, \quad n \in \mathbb{N}_0. \quad (1)$$

Empty product in (1) is taken, by definition, to have the value 1. Let simple closed curve Γ has in its interior all λ_k , $k = 0, 1, \dots, n$. Then we define

$$P_n(x) = \frac{1}{2\pi i} \oint_{\Gamma} W_n(s) x^s ds \quad (2)$$

to be the n -th Müntz-Legendre polynomial. Then we have the following result (see [1], [24], [10], [4], [3]):

Theorem 1. *For the Müntz-Legendre polynomials (2), the following orthogonality relation holds*

$$\int_0^1 P_n(x) \overline{P_k(x)} dx = \frac{\delta_{n,k}}{\lambda_k + \bar{\lambda}_k + 1}, \quad n, k \in \mathbb{N}_0. \quad (3)$$

As we can see the sequence of Müntz-Legendre orthogonal polynomials perfectly suites Definition 1. We can also calculate the Müntz-Legendre orthonormal polynomials $p_n(\cdot) = \sqrt{2\operatorname{Re}(\lambda_n) + 1} P_n(\cdot)$, $n \in \mathbb{N}_0$.

Now, it can be easily seen that in the case of all mutually different λ_k , $k \in \mathbb{N}_0$, the Müntz-Legendre polynomial is given by a linear combination of the basis $x^{\lambda_0}, \dots, x^{\lambda_n}$. However, if we have two elements equal $\lambda_k = \lambda_{k+1}$, an application of the Cauchy residue theorem (see [20], [13], [14]) will produce terms x^{λ_k} and $x^{\lambda_k} \log x$.

In the limiting case $\lambda_k = \lambda$, $k \in \mathbb{N}_0$, we get

$$P_n(x) = x^\lambda L_n(-(\lambda + \bar{\lambda} + 1) \log x), \quad n \in \mathbb{N}_0,$$

where $L_n(x)$ is the Laguerre polynomial orthogonal on $(0, +\infty)$, with respect to Laguerre weight e^{-x} (see [12]). In this case $x^\Lambda = \{1, \log x, \dots, \log^k x, \dots\}$.

The other special case is the sequence $\Lambda = \mathbb{N}_0$, in which case the Müntz system becomes $x^\Lambda = \{1, x, x^2, \dots\}$, i.e., the classical polynomial basis. For this special Λ -sequence the Müntz-Legendre polynomials are the ordinary Legendre polynomials shifted to the interval $[0, 1]$ and orthogonal with respect to the constant weight function (see [12]).

The definition of Müntz-Legendre polynomials can be rewritten into the form of the Laplace transform. Namely, we have the following theorem (see [13], [14]):

Theorem 2. *If $G_n(s) = -W_n(-s)$, where W_n is given by (1), we have*

$$P_n(e^{-t}) = \mathcal{L}^{-1}[G_n(\cdot)], \quad t \in (0, +\infty),$$

where \mathcal{L}^{-1} denotes the inverse Laplace transform.

Using this theorem and Bromwich formula for the inverse Laplace transform (see [20, p. 213]), we have

$$P_n(e^{-t}) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} G_n(s) e^{st} ds, \quad (4)$$

where α should be taken to be larger than $1/2$ in order to have all singularities of G_n to the left of the contour of integration.

For the real values of $\lambda_k > -1/2$, $k \in \mathbb{N}_0$, there is also an interesting property of the interlacing of zeros of the P_n . It was proved in [4] that every P_n has exactly n distinct zeros on $(0, 1)$, in every of its zeros it changes sign, and zeros of P_n and P_{n-1} strictly interlace.

There are also some recurrence relations connecting the Müntz-Legendre orthogonal polynomials. The first one connects the Müntz-Legendre orthogonal polynomials with their derivatives, and it is given by

$$xP'_n(x) - xP'_{n-1}(x) = \lambda_n P_n(x) + (1 + \bar{\lambda}_n) P_{n-1}(x), \quad n \in \mathbb{N}. \quad (5)$$

The second one is given by

$$P_n(x) = P_{n-1} - (\lambda_n + \bar{\lambda}_{n-1} + 1)x^{\lambda_n} \int_x^1 t^{-\lambda_n-1} P_{n-1}(t) dt, \quad x \in (0, 1), \quad n \in \mathbb{N}.$$

2 Gaussian quadrature rule

In the rest of this paper we are concerned only with the case all λ_k , $k \in \mathbb{N}_0$, are real and we assume that the sequence λ_k , $k \in \mathbb{N}_0$, is a nondecreasing system of real numbers.

Let f_k , $k \in \mathbb{N}_0$, be linearly independent system of functions on $[a, b]$, usually chosen such that the system is dense in some desirable function space (see [5], [7], [6], [9]). By the generalized Gaussian quadrature rule for the system f_k , $k \in \mathbb{N}_0$, with respect to a non-negative measure σ , supported on $[a, b]$, we mean the quadrature formula

$$\int_a^b f(x) d\sigma(x) = \sum_{\nu=1}^n A_\nu f(x_\nu) + R_n(f), \quad (6)$$

which is exact for each $f = f_k$, $k = 0, 1, \dots, 2n - 1$. In the rest of the paper we assume the measure σ is absolutely continuous.

It was proved (see [8], [2], [7]) that provided the system of continuous functions f_k , $k = 0, 1, \dots, 2n - 1$, is a Chebyshev system on $[a, b]$, the generalized Gaussian quadrature rule (6) always exists uniquely. Also, the coefficients A_ν , $\nu = 1, \dots, n$, in (6) are positive. The first result concerning the Müntz systems x^{λ_k} , $k \in \mathbb{N}_0$, on $[0, 1]$ goes back to Stieltjes (see [22]). He proved the existence of the Gaussian quadrature rule for $0 \leq \lambda_0 < \lambda_1 < \dots$.

A further refinement of the results of existence and uniqueness of the generalized Gaussian quadrature rule was given in [9]. Suppose a system of functions f_k , $k = 0, 1, \dots, 2n - 1$, which are continuous on $(a, b]$, and that there exists a function r integrable and continuous on $(a, b]$, such that $|f_k/r| < +\infty$ as $x \rightarrow a^+$. If f_k/r , $k = 0, 1, \dots, 2n - 1$, constitute the Chebyshev system on $[a, b]$, then the generalized Gaussian quadrature formula (6) exists uniquely. Further, assuming that system of functions f_k , $k = 0, 1, \dots, 2n - 1$, is Chebyshev on the interval $(a, b]$ and is σ -integrable, we have that Gaussian quadrature rule exists uniquely (see [9]).

The most important systems of functions which satisfy the previous condition are the Müntz systems x^Λ for the nondecreasing sequence Λ of real numbers.

A construction of the generalized Gaussian quadrature formula (6) for the Müntz systems was considered in [9]. However, the proposed algorithm appears to be ill-conditioned. In [17], we proposed a quite different numerical algorithm which is numerically stable. We give the very basic analysis of that algorithm in the next section.

3 Numerical construction of the generalized Gaussian quadrature rule

In the very core of the algorithm, presented in [17], stands the fact that the system of functions x^Λ forms a Hermite system on $(0, 1]$. We say that system of functions f_k , $k = 0, 1, \dots, 2n - 1$, forms the Hermite system of functions on $(0, 1]$, provided for every mutually different numbers x_k , $k = 1, \dots, n$, from $(0, 1]$, we have that determinant

$$\begin{vmatrix} f_0(x_1) & \dots & f_0(x_n) & f'_0(x_1) & \dots & f'_0(x_n) \\ f_1(x_1) & & f_1(x_n) & f'_1(x_1) & & f'_1(x_n) \\ \vdots & & & & & \\ f_{2n-1}(x_1) & & f_{2n-1}(x_n) & f'_{2n-1}(x_1) & & f'_{2n-1}(x_n) \end{vmatrix}$$

is different from zero. It can be proved that the Müntz system of functions x^Λ is a Hermite system of functions on $(0, 1]$. The Müntz-Legendre orthogonal polynomials P_n , $n \in \mathbb{N}_0$, given by (2), are linear combinations of the system x^Λ , which means that the Müntz-Legendre polynomials form the Hermite system on $(0, 1]$.

We are interested in the construction of the generalized Gaussian quadrature rule for the Müntz system x^Λ in the following form

$$\int_0^1 P_0(x)P_k(x)dx = \sum_{\nu=1}^n A_\nu P_k(x_\nu), \quad k = 0, 1, \dots, 2n - 1, \quad (7)$$

where P_n are the Müntz-Legendre orthogonal polynomials given by (2). As it can be checked easily our generalized Gaussian quadrature rule is connected with the generalized Gaussian quadrature rule (6), by the substitution $d\sigma(x) = x^{\lambda_0}dx$, where we used the fact that $P_0(x) = x^{\lambda_0}$, which can be obtained from (2).

Using the orthogonality properties of the Müntz-Legendre orthogonal polynomials (3), we have the following system of nonlinear equations

$$\sum_{\nu=1}^n A_\nu P_k(x_\nu) = \frac{\delta_{k,0}}{2\lambda_0 + 1}, \quad k = 0, 1, \dots, 2n - 1. \quad (8)$$

This system of nonlinear equations is the one we solve in order to construct the generalized Gaussian quadrature rule (7).

In [17], we proved the following theorem:

Theorem 3. *The nonlinear system of equations (8) can be solved using the Newton-Kantorovich method.*

This theorem is due to the fact that x^{Λ_n} is a Hermite system of functions in $(0, 1]$. The Jacobian matrix of the nonlinear system (8) can be proved to be of the full rank in the neighborhood of any solution of our system, there is really only one solution according to [9], as it is elaborated in the previous section.

The main problem with the Newton-Kantorovich method is to find a suitable procedure to calculate the starting values, which enable the quadratic convergence. For a given sequence Λ , by $P_n(\cdot; \Lambda)$ we denote the Müntz-Legendre orthogonal polynomial, defined by (2). The problem of determining starting values can be solved using the following result (see [17]):

Theorem 4. *Let $P_k(\cdot; (1 - \lambda)\mathbb{N}_0 + \lambda\Lambda)$, $\lambda \in [0, 1]$, be orthogonal Müntz-Legendre polynomials, for the sequence*

$$(1 - \lambda)\mathbb{N}_0 + \lambda\Lambda = \{(1 - \lambda)0 + \lambda\lambda_0, (1 - \lambda)1 + \lambda\lambda_1, \dots\}.$$

Then the solution of the system (8) depends continuously on $\lambda \in [0, 1]$.

This theorem completely solves the problem of the starting values. We start with the classical Gauss-Legendre quadrature rule shifted on $(0, 1)$, and increase λ from zero slightly, to say $\Delta\lambda$, in order to be able to solve the system of equations (8), for the sequence $(1 - \Delta\lambda)\mathbb{N}_0 + \Delta\lambda\Lambda$. Once we constructed the generalized Gaussian quadrature rule for this sequence we increase λ more and perform the construction again. We perform the same step until λ reaches 1. In practise, to construct the generalized Gaussian quadrature rule for the sequence $\lambda_{3k} = \lambda_{3k+1} = \lambda_{3k+2} = 3k$, $k \in \mathbb{N}_0$, with 100 nodes we need to perform constructions for 0.5, 0.75 and 1.

We can express the Jacobian matrix in the form

$$J_n = \begin{bmatrix} P_0(x_1) & \dots & P_0(x_n) & A_1 P'_0(x_1) & \dots & A_n P'_0(x_n) \\ P_1(x_1) & & P_1(x_n) & A_1 P'_1(x_1) & & A_n P'_1(x_n) \\ \vdots & & & & & \\ P_{2n-1}(x_1) & & P_{2n-1}(x_n) & A_1 P'_{2n-1}(x_1) & & A_n P'_{2n-1}(x_n) \end{bmatrix}$$

The experimental results given in [17] show that the condition number of the Jacobi matrix grows polynomially with n , provided the sequence Λ_n is uniformly distributed on $(-1/2, n)$. This fact enables us to perform the constructions with 100 nodes in the quadrature rule and to achieve precision of the constructed quadrature rules of order 10^{-14} using double precision format (machine precision of approximately 2.2×10^{-16}).

In order to use the Newton-Kantorovich method, we need to calculate the inverse matrix of J_n , which means that we need to calculate actual values of the Müntz-Legendre polynomials $P_k(x_\nu)$, $k = 0, 1, \dots, 2n - 1$, as well as, the values of their derivatives for $x_\nu \in (0, 1)$. A calculation of the derivatives of the Müntz-Legendre polynomials can be performed in the numerically stable fashion using the recurrence relation (5). Thus, the only problem which remains is to calculate the Müntz-Legendre polynomials in a numerically stable way. We discuss this problem in the next section.

4 Numerical calculation of Müntz-Legendre polynomials

Given real nondecreasing sequence Λ we can apply the Cauchy residue theorem to the integral (2), and we can get easily an expression for the Müntz-Legendre orthogonal polynomial P_n , which is a linear combination of the

basis functions x^Λ . However, as it was shown in [13], [14], this expression is heavily unstable for the computational purposes.

A clue, for the way Müntz-Legendre polynomials can be calculated, is the expression (2). As it is shown (Theorem 2), it is equivalent as for saying that the Müntz-Legendre polynomials are inverse Laplace transforms of the rational functions G_n . So, we can apply the methods used for the numerical calculations of the inverse Laplace transform.

We would like to point out that the complexity of our algorithm depends strongly of the complexity of the computation of Müntz-Legendre polynomials. In order to achieve a machine precision (in double precision format approximately 2.2×10^{-16}), we usually need say 7 iterations of the Newton-Kantorovich algorithm, provided starting values are not “very bad”. Every iteration in the Newton-Kantorovich algorithm requires computation of the Müntz-Legendre polynomials P_k , $k = 0, 1, \dots, 2n-1$, in the n different points x_k , $k = 1, \dots, n$. Also, we need the computation of derivatives of the Müntz-Legendre polynomials at the points x_k , $k = 1 \dots, n$, using (5), as well as one solution of the linear system with the matrix of the system given by the Jacobi matrix J_n of dimension $2n$. As is well-known solving system of the linear equations has the complexity $\approx n^3$, if we are using, for example, the Gaussian elimination (see [12]). The complexity of the computation of the derivatives, using (5), is linear in n , i.e., neglectful with respect to the complexity of solving a linear system. Suppose the computation of the polynomial P_{2n-1} requires T time units, then complexity of construction of the Jacobi matrix J_n is of order $\approx n^2T$. It turns out that this quantity is dominant, i.e., the quantity T is significantly bigger than n . This can be understood easily. If we are going to perform a computation using (2), then we are forced to use some quadrature rule with say M points, for every node of the quadrature rule we need n divisions and n multiplications so in total we need nM time units to calculate P_n . In total construction of the Jacobi matrix requires $\approx n^3M$ time units. As we are going to present M is almost always significantly bigger than n .

The most important algorithm used for the calculation of the inverse Laplace transform is the Talbot algorithm (see [23], [21]). However, as it was shown in [16], this algorithm produces completely inaccurate results, provided we are working with Müntz-Legendre polynomials of degree 50 and more. This is the reason why new algorithm has been introduced starting from [13], [14], and is further improved in [16].

Starting with the inverse Laplace transform representation, over the Bromwich contour (4) in [13], [14], [16], the following theorem has been proved:

Theorem 5. For any $\alpha > -\lambda_0$, $x \in (0, 1)$, and $a > 0$, we have

$$\begin{aligned} P_n(x) &= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} x^{-s} G_n(s) ds \\ &= -\frac{x^{-\alpha}}{\pi \log x} \left\{ \operatorname{Re} \int_0^a e^{iu} G_n \left(\alpha - \frac{iu}{\log x} \right) du \right. \\ &\quad \left. - \operatorname{Im} e^{ia} \int_0^{+\infty} e^{-u} G_n \left(\alpha - \frac{ia-u}{\log x} \right) du \right\}. \quad (9) \end{aligned}$$

The main idea in the algorithm is to calculate the actual value of the Müntz-Legendre polynomial P_n , applying the Gaussian quadrature rules on the integrals in (9). For the second integral it is obvious that we are going to apply the Gauss-Laguerre quadrature rule. This is quite natural since we are integrating with respect to the Laguerre measure $e^{-x} dx$ over the interval $(0, +\infty)$. For the first integral we are going to apply the Gauss-Legendre quadrature rule (see [12]). It is well-known that oscillatory integrands can significantly decrease the performance of the Gaussian quadrature rules (see [15]). Due to this fact, it is recommended to divide the integration over the interval $[0, a)$ into the non-overlapping subintervals say $[a_\nu, b_\nu)$, $\nu = 1, \dots, m$, such that $\cup_{\nu=1}^m [a_\nu, b_\nu) = [0, a)$.

The algorithm presented in [13], [14], in order to cope with the oscillatory nature of the integrand e^{iu} , uses the following partition of the interval $[0, a)$

$$a_\nu = (\nu - 1)\pi, \quad \nu = 1, \dots, m,$$

where it is chosen $a = m\pi$. This produces quadrature rules with very high number of evaluations of the function G_n needed to compute $P_n(x)$ to the machine precision (for double precision format approximately 2.2×10^{-16}), it usually results into the couple of thousands evaluations of G_n .

In [16], we tried to reduce the number of evaluations of G_n . We have to report that we are far away from our goal, that is to achieve procedure which will, given x , find autonomously the parameters α , a , as well as the number of subintervals and their placement in the interval $[0, a)$ in order to have as smaller as possible numbers of evaluations of G_n , but to compute P_n with a machine precision.

As it can be checked directly, the function G_n depends on x , which means that different numbers x require different quadrature rules in order to have optimal calculations of $P_n(x)$. This is really the case as it is presented in [16]. Also, the function G_n depends on the sequence Λ which additionally means that the different quadrature rules should be applied for the different sequences Λ . In [16], we limited our attention to the case of uniformly

distributed sequences and we achieved the number of computations to be reduced to approximately 100 evaluations of the function G_n for the polynomial P_{100} .

References

- [1] G. V. Badalyan, *Generalization of Legendre polynomials and some of their applications*, Akad. Nauk. Armyan. SSR Izv. Fiz.-Mat. Estest. Tekhn. Nauk, **8** (5) (1955), 1–28 and **9** (1) (1956), 3–22 (Russian, Armenian summary).
- [2] B. Bojanov, *Gaussian quadrature formulae for Tchebycheff systems*, East J. Approx. **3** (1997), 71–88.
- [3] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics **161**, Springer-Verlag, New York – Berlin – Heidelberg, 1995.
- [4] P. Borwein, T. Erdelyi, J. Zhang, *Müntz systems and orthogonal Müntz-Legendre polynomials*, Trans. Amer. Math. Soc. **342** (1994), 523–542.
- [5] W. Gautschi, *A survey of Gauss-Christoffel quadrature formulae*, In: E. B. Christoffel (P. L. Butzer, F. Fehér, eds.), pp. 72–147, Birkhäuser, Basel, 1981.
- [6] C.G. Harris, W.A.B. Evans, *Extension of numerical quadrature formulae to cater for end point singular behaviours over finite intervals*, Internat. J. Comput. Math. **6** (1977), 219–227.
- [7] S. Karlin, W.J. Studden, *Tchebycheff Systems with Applications in Analysis and Statistics*, Pure and Applied Mathematics, Vol. XV, John Wiley Interscience, New York, 1966.
- [8] M. Krein, *The ideas of P.L. Tchebycheff and A.A. Markov in the theory of limiting values of integrals and their further developments*, Uspehi Fiz. Nauk (1951), 3–120 (Russian) [Amer. Math. Soc. Transl. Ser. 2, **12** (1951), 1–122].
- [9] J. Ma, V. Rokhlin, S. Wandzura, *Generalized Gaussian quadrature rules for systems of arbitrary functions*, SIAM J. Numer. Anal. **33** (1996), 971–996.

- [10] P. C. McCarthy, J. E. Sayre, B. L. R. Shawyer, *Generalized Legendre polynomials*, J. Math. Anal. Appl. **177** (1993), 530–537.
- [11] G. V. Milovanović, *Numerical Analysis, Part I*, Third Edition, Naučna Knjiga, Belgrade, 1991 (Serbian).
- [12] G. V. Milovanović, *Numerical Analysis, Part II*, Third Edition, Naučna Knjiga, Belgrade, 1991 (Serbian).
- [13] G. V. Milovanović, *Müntz orthogonal polynomials and their numerical evaluation*, In: Applications and Computation of Orthogonal Polynomials (W. Gautschi, G.H. Golub, G. Opfer, eds.), pp. 179–202, ISNM, Vol. 131, Birkhäuser, Basel, 1999.
- [14] G. V. Milovanović, *Some generalized orthogonal systems and their connections*, In: Proceedings of the Symposium “Contemporary Mathematics” (Belgrade, 1998) (N. Bokan, ed.), pp. 181–200, Faculty of Mathematics, University of Belgrade, 2000.
- [15] G. V. Milovanović, A. S. Cvetković, *Orthogonal polynomials and Gaussian quadrature rules related to oscillatory weight functions*, J. Comput. Appl. Math. **179** (2005), 263–287.
- [16] G. V. Milovanović, A. S. Cvetković, *Numerical inversion of the Laplace transform*, Facta Univ. Ser. Electr. Energ. (to appear).
- [17] G. V. Milovanović, A. S. Cvetković, *Gaussian type quadrature rules for Müntz systems*, SIAM J. Sci. Comput. (to appear).
- [18] G. V. Milovanović, R. Ž. Djorjdević, *Mathematics for Students of Technical Faculties, Part II*, Cuperak Plavi, Ni, 1996 (Serbian).
- [19] B. Mirković, *Theory of Measures and Integrals*, Naučna knjiga, Beograd, 1990 (Serbian).
- [20] D. S. Mitrinović, *Cauchy Calculus of Residues with Applications*, Matematički problemi i ekspozicije 8, Naučna Knjiga, Beograd, 1991 (Serbian).
- [21] A. Murly, M. Rizzardi, *ALGORITHM 682: Talbot’s method for the Laplace inversion problem*, ACM Trans. Math. Software **16** (1990), 158–168.
- [22] T. J. Stieltjes, *Sur une généralisation de la théorie des quadratures mécaniques*, C.R. Acad. Sci. Paris **99** (1884), 850–851.

- [23] A. Talbot, *The accurate numerical inversion of Laplace transforms*, J. Inst. Math. Appl. **23** (1979), 97-120.
- [24] A. K. Taslakyán, *Some properties of Legendre quasipolynomials with respect to a Müntz system*, Mathematics, Èrevan Univ., Èrevan **2** (1984), 179–189 (Russian, Armenian summary).