A generalized Birkhoff–Young quadrature formula

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ABSTRACT. A generalized $(4n+1)$-point Birkhoff–Young quadrature of interpolatory type with the maximal degree of precision for numerical integration of analytic functions is derived. An explicit form of the node polynomial of such kind of quadratures is obtained. Special cases and an example are presented.

1. INTRODUCTION AND PRELIMINARIES

A quadrature formula for numerical integration of analytic functions in the complex domain $\Omega = \big\{ z \, : \, |z - z_0| \leq r \big\}$, $|h| \leq r$, over the line segment $[z_0 - h, z_0 + h]$,

$$\int_{z_0-h}^{z_0+h} f(z)\mathrm{d}z \approx \frac{h}{15}\Big\{ 24f(z_0) + 4\big[f(z_0 + h) + f(z_0 - h)\big] - \big[f(z_0 + \mathrm{i}h) + f(z_0 - \mathrm{i}h)\big]\Big\},$$

was obtained by Birkhoff and Young [2]. This formula is exact for all algebraic polynomials of degree at most five. Also, Young [22] proved that its error term can be estimated by

$$|R_5^{BY}(f)| \leq \frac{|h|^7}{1890} \max_{z \in S} |f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0 + \mathrm{i}^k h$, $k = 0, 1, 2, 3$ (cf. [4, p. 136]). A similar error estimate was obtained for the so-called extended Simpson rule, with nodes at the points z_0, $z_0 \pm h$, $z_0 \pm 2h$ (cf. [18, p. 124]).

Without loss of generality, Lether [6] transformed the previous Birkhoff–Young fomula from $[z_0 - h, z_0 + h]$ to $[-1, 1]$,

$$(1.1) \qquad I(f) = \int_{-1}^{1} f(z)\mathrm{d}z = \frac{8}{5}f(0) + \frac{4}{15}\big[f(1) + f(-1)\big] - \frac{1}{15}\big[f(\mathrm{i}) + f(-\mathrm{i})\big] + R_5(f),$$

and pointed out that the three point Gauss-Legendre quadrature which is also exact for all polynomials of degree at most five, is more precise than (1.1), and therefore, he recommended it for numerical integration. However, Tošić [19] improved the quadrature (1.1) in a simple way taking its nodes at the points $\pm r$ and $\pm \mathrm{i}r$, with $r \in (0, 1)$, instead of ± 1 and $\pm \mathrm{i}$, respectively, and derived an one-parametric family of quadrature rules in the form

$$
\begin{aligned}
I(f) \;=\; & 2\left(1 - \frac{1}{5r^4}\right)f(0) + \left(\frac{1}{6r^2} + \frac{1}{10r^4}\right)\big[f(r) + f(-r)\big] \\[2mm]
& + \left(-\frac{1}{6r^2} + \frac{1}{10r^4}\right)\big[f(\mathrm{i}r) + f(-\mathrm{i}r)\big] + R_5^T(f; r).
\end{aligned}
$$

It is clear that for $r = 1$ it reduces to (1.1) and for $r = \sqrt{3/5}$ to the three point Gauss-Legendre formula. Moreover, expanding the error-term $R_5^T(f; r)$ in the form

$$R_5^T(f; r) = \left(-\frac{2}{3 \cdot 6!}r^4 + \frac{2}{7!}\right)f^{(6)}(0) + \left(-\frac{2}{5 \cdot 8!}r^4 + \frac{2}{9!}\right)f^{(8)}(0) + \cdots,$$

and putting $r = \sqrt[4]{3/7}$ in order to vanish the first term, Tošić [19] obtained a five-point formula of algebraic degree of precision seven, with the error-term

$$R_5^{MF}(f) = R_5^T(f; \sqrt[4]{3/7}) \approx 1.26 \cdot 10^{-6} f^{(8)}(0).$$

Tošić's formula was extended by Milovanović and Đorđević [16] to the following nine-point quadrature rule of interpolatory type

$$I(f) = Af(0) + B\big[f(x_1) + f(-x_1)\big] + C\big[f(\mathrm{i}x_1) + f(-\mathrm{i}x_1)\big]$$

$$+ D\big[f(x_2) + f(-x_2)\big] + E\big[f(\mathrm{i}x_2) + f(-\mathrm{i}x_2)\big] + R_9(f; x_1, x_2),$$

with $0 < x_1 < x_2 < 1$. Taking

$$x_1 = x_1^* = \sqrt[4]{\frac{63 - 4\sqrt{114}}{143}} \quad \text{and} \quad x_2 = x_2^* = \sqrt[4]{\frac{63 + 4\sqrt{114}}{143}},$$

this formula has the algebraic precision $d = 13$, with the error-term

$$R_9(f; x_1^*, x_2^*) \approx 3.56 \cdot 10^{-14} f^{(14)}(0).$$

This kind of quadrature formulae for analytic functions have been investigated in several papers in different directions (cf. [1], [7], [8], [11], [12], [14]). These formulas can also be used to integrate real harmonic functions (see [2]). In addition, we mention also that Lyness and Moler [9], and later Tošić [20] and Tošić and ElBahi [21] developed formulae for numerical differentiation of complex functions.

In this paper we consider a generalized N-point interpolatory quadrature formula for numerical integration of analytic function

$$(1.2) \qquad I(f) := \int_{-1}^{1} f(z)\mathrm{d}z = Q_N(f) + R_N(f),$$

where $N = 4n + 1$ ($n \in \mathbb{N}$) and the corresponding quadrature sum $Q_N(f)$ has the form

$$(1.3) \qquad Q_N(f) = A_0 f(0) + \sum_{k=1}^{n}\Big\{A_k\big[f(x_k) + f(-x_k)\big] + B_k\big[f(\mathrm{i}x_k) + f(-\mathrm{i}x_k)\big]\Big\},$$

with nodes at the zeros of a monic polynomial of degree $N = 4n + 1$,

$$(1.4) \qquad \omega_N(z) = zp_n(z^4) = z\prod_{k=1}^{n}(z^4 - r_k), \quad 0 < r_1 < \cdots < r_n < 1,$$

i.e., $x_k = \sqrt[4]{r_k}$, $k = 1, \ldots, n$. $R_N(w; f)$ is the corresponding remainder term.

Notice that the $(4n + 1)$-point interpolatory quadrature formula of the form (1.2)–(1.3) has degree of precision at least $4n$ for any distribution of the nodes r_k, $k = 1, \ldots, n$, in (1.4). Precisely, it is one degree more, i.e., $4n + 1$, because this formula is exact for all odd functions.

Our aim is to develop quadrature formulas of type (1.2)–(1.3), with the maximal degree of precision, for arbitrary $n \in \mathbb{N}$. In this way, we will generalize results from [19] ($n = 1$) and [16] ($n = 2$). The corresponding quadrature sums will be denoted by $\widehat{Q}_{4n+1}(f)$.

The paper is organized as follows. The existence and uniquenes of quadrature formula $\widehat{Q}_{4n+1}(f)$, with the maximal degree of precision, as well as an explicit form of its node

polynomial, in notation $\widehat{\omega}_{4n+1}(z) = z\widehat{p}_n(z^4)$, are proved in Section 2. Numerical construction of weight coefficients and some error estimates are presented in Section 3. Finally, a few special cases and a numerical example are presented in Section 4.

2. NUMERICAL CONSTRUCTION OF QUADRATURE FORMULAS

The following theorem gives a characterization of the quadrature formula (1.2) with a maximal degree of precision.

Theorem 2.1. *For each $n \in \mathbb{N}$ there exists the unique interpolatory quadrature formula $\widehat{Q}_{4n+1}(f)$ of the form (1.3), with the maximal degree of precision $d_{\max} = 6n + 1$. The nodes of such a quadrature are zeros of the polynomial $\widehat{\omega}_{4n+1}(z) = z\widehat{p}_n(z^4)$, where*

$$(2.5) \qquad \widehat{p}_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n = 1),$$

with coefficients which satisfy the following system of linear equations

$$(2.6) \qquad \begin{bmatrix} \dfrac{1}{3} & \dfrac{1}{7} & \cdots & \dfrac{1}{4n-1} \\[2mm] \dfrac{1}{5} & \dfrac{1}{9} & & \dfrac{1}{4n+1} \\[2mm] \vdots & & & \\[2mm] \dfrac{1}{2n+1} & \dfrac{1}{2n+5} & & \dfrac{1}{6n-3} \end{bmatrix} \begin{bmatrix} a_0 \\[2mm] a_1 \\[2mm] \vdots \\[2mm] a_{n-1} \end{bmatrix} = - \begin{bmatrix} \dfrac{1}{4n+3} \\[2mm] \dfrac{1}{4n+5} \\[2mm] \vdots \\[2mm] \dfrac{1}{6n+1} \end{bmatrix}.$$

Proof. We start with quadrature sum (1.3), with $N = 4n + 1$ simple nodes at the points 0, $\pm x_k, \pm \mathrm{i} x_k, k = 1, \ldots, n$. Such a quadrature of interpolatory type is exact for polynomials of degree at least $4n$ for any distribution of zeros $r_k, k = 1, \ldots, n$, in (1.4), as well as for all odd polynomials of arbitary degree. Therefore, according to n free parameters $r_k, k = 1, \ldots, n$, in (1.4), we should provide that this quadrature rule be exact for even polynomials z^{2j}, with $4n < 2j \le 6n$, i.e., $2n < j \le 3n$. As we will prove, this expectation is possible, so that the maximal degree of precision will be $d_{\max} = 6n + 1$.

In the sequel we use the following notation $x_k^4 = r_k$, $A_k + B_k = C_k$, $A_k - B_k = D_k$, $k = 1, \ldots, n$, as well as

$$\mu_{2j} = \frac{1}{2} \int_{-1}^{1} z^{2j} \mathrm{d}z = \frac{1}{2j+1}, \quad j = 0, 1, \ldots .$$

Suppose that the maximal degree of precision of our quadrature is $d_{\max} = 6n + 1$, i.e., that this formula is exact for all monomials z^{2j}, $j = 0, 1, \ldots, 3n$. Thus,

$$2\mu_{2j} = A_0 \delta_{j,0} + 2 \sum_{k=1}^{n} \left(A_k + (-1)^j B_k \right) x_k^{2j}, \quad j = 0, 1, \ldots, 3n,$$

where $\delta_{j,0}$ is Kronecker's delta.

According to the previous introduced notations, this system of equations can be devided into three parts:

$1°$ For $j = 0$

$$(2.7) \qquad \frac{1}{2} A_0 + \sum_{k=1}^{n} C_k = 1;$$

2° For odd $j\, (= 2m - 1)$

$$(2.8) \qquad \sum_{k=1}^{n} D_k x_k^{4m-2} = \mu_{4m-2}, \quad m = 1, \ldots, \left[\frac{3n+1}{2}\right];$$

3° For even $j\, (= 2m)$

$$(2.9) \qquad \sum_{k=1}^{n} C_k x_k^{4m} = \mu_{4m}, \quad m = 1, \ldots, \left[\frac{3n}{2}\right].$$

Notice that $\left[\dfrac{3n+1}{2}\right] + \left[\dfrac{3n}{2}\right] = 3n$. From systems of equations (2.8) and (2.9), using certain transformations, we can eliminate D_k and C_k, respectively. In this way, from previous systems of equations we obtain $\left[\dfrac{n+1}{2}\right]$ and $\left[\dfrac{n}{2}\right]$ equations, with only unknowns a_j, $j = 1, \ldots, n$, respectively. Their number is exactly $\left[\dfrac{n+1}{2}\right] + \left[\dfrac{n}{2}\right] = n$.

The elimination process from the system (2.8) can be described in the following way. If we multiply the m-th equation by a_0, the $(m+1)$-st by a_1, ..., and the $(m+n)$-th equation by $a_n\, (= 1)$, and then sum up the resulting equations, we get

$$a_0 \sum_{k=1}^{n} D_k x_k^{4m-2} + a_1 \sum_{k=1}^{n} D_k x_k^{4m+2} + \cdots + a_{n-1} \sum_{k=1}^{n} D_k x_k^{4m+4n-6}$$

$$+ \sum_{k=1}^{n} D_k x_k^{4m+4n-2} = \sum_{j=0}^{n} a_j \mu_{4m+4j-2},$$

i.e.,

$$\sum_{k=1}^{n} D_k x_k^{4m-2}\left(a_0 + a_1 x_k^4 + \cdots + a_{n-1} x_k^{4(n-1)} + x_k^{4n}\right) = \sum_{j=0}^{n} \mu_{4m+4j-2} a_j.$$

Using (2.5), the last equation becomes

$$(2.10) \qquad \sum_{k=1}^{n} D_k x_k^{4m-2} \widehat{p}(r_k) = \sum_{j=0}^{n} \mu_{4m+4j-2} a_j$$

and it is possible for each $m = 1, \ldots, \left[\dfrac{n+1}{2}\right]$.

In a similar way, by equations from (2.9), we get $\left[\dfrac{n}{2}\right]$ equations of the form

$$(2.11) \qquad \sum_{k=1}^{n} C_k r_k^m \widehat{p}(r_k) = \sum_{j=0}^{n} \mu_{4m+4j} a_j$$

and it is possible for each $m = 1, \ldots, \left[\dfrac{n}{2}\right]$.

Since $\widehat{p}_n(r_k) = 0$, $k = 1, \ldots, n$, the systems of equations (2.10) and (2.11) give a complete system of $\left[\dfrac{n+1}{2}\right] + \left[\dfrac{n}{2}\right] = n$ equations,

$$(2.12) \qquad \begin{cases} \displaystyle\sum_{j=0}^{n} \mu_{4m+4j-2} a_j = 0, & m = 1, \ldots, \left[\dfrac{n+1}{2}\right], \\[4mm] \displaystyle\sum_{j=0}^{n} \mu_{4m+4j} a_j = 0, & m = 1, \ldots, \left[\dfrac{n}{2}\right]. \end{cases}$$

Since $a_{n+1} = 1$, this system of linear equations can be represented in the following form (by taking equations from both of the systems for $m = 1$, then for $m = 2$, etc.)

$$\mu_2 a_0 \;+\; \mu_6 a_1 \;+\; \cdots + \mu_{4n-2} a_{n-1} \;=\; -\mu_{4n+2},$$

$$\mu_4 a_0 \;+\; \mu_8 a_1 \;+\; \cdots + \; \mu_{4n} a_{n-1} \;=\; -\mu_{4n+4},$$

$$\vdots$$

$$\mu_{2n} a_0 + \mu_{2n+4} a_1 + \cdots + \mu_{6n-4} a_{n-1} \;=\; -\mu_{6n},$$

The last equation is obtained from the first system in (2.12) or from the second one, if n is odd or even, respectively. This system of equations is the one given in the assertion of this theorem, i.e., (2.6).

The existence of the unique quadrature formula $\widehat{Q}_{4n+1}(f)$, with the maximal degree of precision $d_{\max} = 6n + 1$, is guaranteed if the determinant Δ_n of the matrix in (2.6) is different from zero. In order to evaluate this determinant, we use Cauchy's formula (cf. Muir [17, p. 345])

$$\det \left[\frac{1}{\alpha_k + \beta_j} \right]_{k,j=1}^{n} = \frac{\prod_{k>j=1}^{n}(\alpha_k - \alpha_j)(\beta_k - \beta_j)}{\prod_{k,j=1}^{n}(\alpha_k + \beta_j)},$$

with $\alpha_k = 2k$ and $\beta_j = 4j - 3$. Then, after some computation, we find that

$$\Delta_n = 2^{n(5n-3)/2} \prod_{j=0}^{n-1} \frac{(n+2j)!}{\binom{2j}{j}} \prod_{j=1}^{n} \frac{(4j-3)!}{(2n+4j-3)!}.$$

Thus, since $\Delta_n \neq 0$ (precisely, $\Delta_n > 0$), we conclude that the quadrature formula $\widehat{Q}_{4n+1}(f)$ uniquely exists for each $n \in \mathbb{N}$. $\qquad \square$

According to this theorem, the polynomial $\widehat{p}_n(z)$ is determined by the solution of the system of equations (2.6). Its zeros r_k, $k = 1, \ldots, n$, define nodes in the quadrature (1.3) of the maximal degree of precision.

The following theorem gives an explict form of the polynomial $\widehat{p}_n(z)$.

Theorem 2.2. *The coefficients of the polynomial $\widehat{p}_n(z)$, defined by (2.5) in Theorem 2.1, are*

$$(2.13) \qquad a_j = (-1)^{n-j} \binom{n}{j} \frac{\left(2j + \frac{3}{2}\right)_{2n-2j}}{\left(n + 2j + \frac{3}{2}\right)_{2n-2j}}, \quad j = 0, 1, \ldots, n,$$

where $(s)_j$ is the standard notation for Pochhammer's symbol

$$(s)_j = s(s+1)\cdots(s+j-1) = \frac{\Gamma(s+j)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

Proof. According to Theorem 2.1, the system of equations (2.6) has the unique solution. Therefore, it is enough to show that the coefficients from (2.13) satisfy this system (2.6). Thus, we need to prove that

$$\sum_{j=0}^{n-1} \frac{a_j}{2k+4j+1} = -\frac{1}{4n+2k+1}, \quad k = 1, \ldots, n,$$

i.e.,

$$S_n(k) := \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^{n-j}}{2k+4j+1} \cdot \frac{\left(2j + \frac{3}{2}\right)_{2n-2j}}{\left(n + 2j + \frac{3}{2}\right)_{2n-2j}} = 0, \quad k = 1, \ldots, n.$$

Since

$$\frac{\left(2j + \frac{3}{2}\right)_{2n-2j}}{\left(n + 2j + \frac{3}{2}\right)_{2n-2j}} = \frac{\Gamma\left(2n + \frac{3}{2}\right)}{\Gamma\left(3n + \frac{3}{2}\right)} \cdot \frac{\Gamma\left(n + 2j + \frac{3}{2}\right)}{\Gamma\left(2j + \frac{3}{2}\right)} = \frac{\left(2j + \frac{3}{2}\right)_n}{\left(2n + \frac{3}{2}\right)_n},$$

we have

$$S_n(k) = \frac{(-1)^n}{2\left(2n + \frac{3}{2}\right)_n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\left(2j + \frac{3}{2}\right)_n}{2j + k + \frac{1}{2}}.$$

We also note that for $1 \leq k \leq n$, $2j + k + \frac{1}{2}$ is a factor of $\left(2j + \frac{3}{2}\right)_n$, so that the last quotient is a polynomial in j of degree $n - 1$, i.e.,

$$\frac{\left(2j + \frac{3}{2}\right)_n}{2j + k + \frac{1}{2}} = \sum_{\nu=0}^{n-1} \gamma_\nu j^\nu,$$

where the coefficients γ_ν depend, in general, on n and k.

Finally, using the identity (see Gould [5, p. 2, Formula (1.13)])

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k = \begin{cases} 0, & 0 \leq k < n, \\ (-1)^n n!, & k = n, \end{cases}$$

we conclude that $S_n(k) = 0$, $1 \leq k \leq n$. $\hfill\square$

3. WEIGHT COEFFICIENTS AND ERROR ESTIMATE

Let $Q_N(f)$ be a N-point quadrature of interpolation type with simple (in general complex) nodes $z_j \in Z$,

$$Q_N(f) = \sum_{z_j \in Z} W_j f(z_j).$$

It can be obtained by an integration of the Lagrange polynomial constructed on Z, i.e.,

$$L_N(f; z) = \sum_{z_j \in Z} \frac{\omega_N(z)}{(z - z_j)\omega_N'(z_j)} f(z_j),$$

where $\omega_N(z)$ is the node polynomial. The corresponding weight coefficients can be expressed in the form

(3.14)
$$W_j = \frac{1}{\omega_N'(z_j)} \int_{-1}^{1} \frac{\omega_N(z)}{z - z_j} \mathrm{d}z.$$

In our case $N = 4n+1$ and the quadrature nodes belong to the set $Z = \{0, \pm x_k, \pm \mathrm{i}x_k, k = 1, \ldots, n\}$, so that

$$\widehat{\omega}_{4n+1}(z) = z\,\widehat{p}_n(z^4), \quad \widehat{\omega}_{4n+1}'(z) = p_n(z^4) + 4z^4\,\widehat{p}_n'(z^4).$$

The following assertion can be easily proved.

Theorem 3.3. *The weight coefficients in the quadrature formula $\widehat{Q}_{4n+1}(f)$, with the maximal degree of precision $d = 6n + 1$, are given by*

$$A_0 = \frac{1}{\widehat{p}_n(0)} \int_{-1}^{1} \widehat{p}_n(z^4)\mathrm{d}z,$$

$$A_k = \frac{1}{4r_k\widehat{p}'_n(r_k)} \int_{-1}^{1} \frac{z^2\widehat{p}_n(z^4)}{z^2 - \sqrt{r_k}}\mathrm{d}z, \quad k = 1, \ldots, n,$$

$$B_k = \frac{1}{4r_k\widehat{p}'_n(r_k)} \int_{-1}^{1} \frac{z^2\widehat{p}_n(z^4)}{z^2 + \sqrt{r_k}}\mathrm{d}z, \quad k = 1, \ldots, n,$$

where $\widehat{p}_n(z)$ is given in Theorem 3.3.

The remainder in the quadrature formula $\widehat{Q}_{4n+1}(f)$ can be expressed in the form of Taylor series at $z = 0$, where the leading term is given by

$$\frac{R_{4n+1}(z^{d_{\max}+1})}{(d_{\max} + 1)!} f^{(d_{\max}+1)}(0) = \frac{R_{4n+1}(z^{6n+2})}{(6n + 2)!} f^{(6n+2)}(0).$$

The coefficients $G_n = R_{4n+1}(z^{6n+2})$ can be computed in an exact form, viz.

$$G_1 = \frac{16}{315}, \quad G_2 = \frac{512}{165165}, \quad G_3 = \frac{4096}{22485645}, \quad G_4 = \frac{524288}{49628068875},$$

$$G_5 = \frac{4194304}{6887669463675}, \quad G_6 = \frac{134217728}{3836303358692805},$$

$$G_7 = \frac{1073741824}{535253444773400925}, \quad G_8 = \frac{549755813888}{4785728620301042601915}, \quad \text{etc.}$$

For example, the leading terms in the remainder in the quadrature formula $\widehat{Q}_{4n+1}(f)$ are $1.26 \times 10^{-6} f^{(8)}(0)$ $(n = 1)$, $3.56 \times 10^{-14} f^{(14)}(0)$ $(n = 2)$, $7.49 \times 10^{-23} f^{(20)}(0)$ $(n = 3)$, $2.62 \times 10^{-32} f^{(26)}(0)$ $(n = 4)$, etc.

4. SPECIAL CASES AND EXAMPLES

Using Theorem 2.2, we start this section by listing a few first node polynomials of the sequence $\{\widehat{p}_n(z)\}$:

$$\widehat{p}_1(z) = z - \frac{3}{7},$$

$$\widehat{p}_2(z) = z^2 - \frac{126z}{143} + \frac{15}{143},$$

$$\widehat{p}_3(z) = z^3 - \frac{429z^2}{323} + \frac{693z}{1615} - \frac{7}{323},$$

$$\widehat{p}_4(z) = z^4 - \frac{204z^3}{115} + \frac{14586z^2}{15295} - \frac{1716z}{10925} + \frac{9}{2185},$$

$$\widehat{p}_5(z) = z^5 - \frac{1995z^4}{899} + \frac{4522z^3}{2697} - \frac{92378z^2}{186093} + \frac{1001z}{20677} - \frac{77}{103385},$$

$$\widehat{p}_6(z) = z^6 - \frac{690z^5}{259} + \frac{32775z^4}{12617} - \frac{3714500z^3}{3293037} + \frac{20995z^2}{99789} - \frac{442z}{33263} + \frac{13}{99789},$$

etc. The normalized polynomials $\widehat{p}_n(z)/\widehat{p}_n(1)$ for $n \leq 5$ are displayed in Fig. 1.
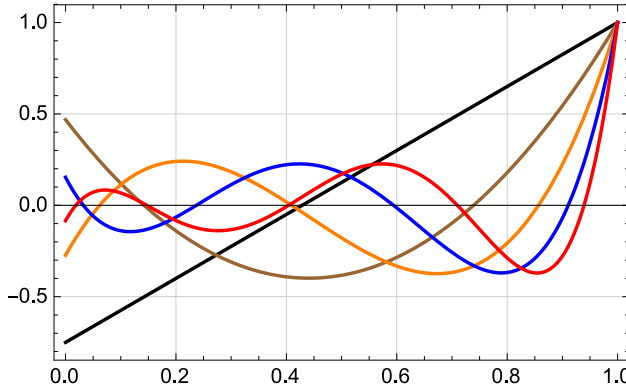
FIGURE 1. Normalized polynomials $z \mapsto \widehat{p}_n(z)/\widehat{p}_n(1)$ on $[0,1]$ for $n = 1(1)5$

Remark 4.1. Notice that for the polynomial $\widehat{p}_n(z)$ with real zeros, the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\widehat{p}'_n(z)}{\widehat{p}_n(z)}\right\} = \frac{\widehat{p}''_n(z)\widehat{p}_n(z) - \widehat{p}'_n(z)^2}{\widehat{p}_n(z)^2} = -\sum_{k=1}^{n}\frac{1}{(z - r_k)^2} < 0$$

holds, which is known as Laguerre's inequality (cf. [15, p. 100]). Therefore, the function $z \mapsto \widehat{p}'_n(z)/\widehat{p}_n(z)$ is decreasing (see Fig. 2 for the case $n = 4$).

TABLE 1. Parameters of the quadrature formula $\widehat{Q}_{4n+1}(f)$ for $n = 1(1)5$

n	x_k	A_k	B_k
1	0.	1.06666666666666667	
	0.80910671157022121	0.48792087194199111	$-2.1254205275324445(-2)$
2	0.	0.75851851851851852	
	0.61375568697566816	0.44678904212712961	$-1.3413735169030103(-2)$
	0.92724238665153225	0.18671643342768085	$6.4900035496037478(-4)$
3	0.	0.60014652014652015	
	0.96214281153608152	0.97175623744187170(-1)	$-3.2772235085563187(-5)$
	0.79992901662161506	0.22832660542564449	$3.9045344595047195(-4)$
	0.49852034774285952	0.38317748454573029	$-9.1106549996869349(-3)$
4	0.	0.50208336325983385	
	0.42334528422173443	0.33411499384824446	$-6.7950705650995491(-3)$
	0.69774446544252737	0.22326448882961959	$2.2533204936394584(-4)$
	0.87794708961316891	0.13880277686091035	$-1.8814858386783733(-5)$
	0.97687342877118281	0.59362507634352748(-1)	$2.1045710783136424(-5)$
5	0.	0.43470421061457476	
	0.37016907042014185	0.29688563312150241	$-5.4002087637467265(-3)$
	0.61942822377595288	0.20956672392426616	$1.4243077238979650(-4)$
	0.79736012613394691	0.14820497577754713	$-9.5819114497697012(-6)$
	0.91786487159093112	0.93287220829481104(-1)	$1.1646304758720715(-6)$
	0.98442790818375368	0.39969691753677868(-1)	$-1.5544143122579493(-7)$

Parameters of the generalized quadrature formula $\widehat{Q}_{4n+1}(f)$, with the maximal degree of precision $d = 6n + 1$, for $n = 1(1)5$ are presented in Table 1. Numbers in parenthesis indicate the decimal exponents.
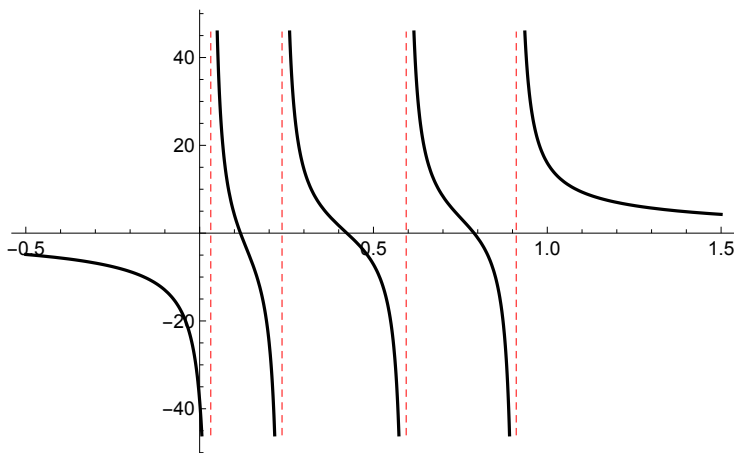
FIGURE 2. The function $z \mapsto \widehat{p}\,'_n(z)/\widehat{p}_n(z)$ for $n = 4$

Remark 4.2. In application of the quadrature formula $\widehat{Q}_{4n+1}(f)$ to integration of real functions, it is sufficient to find the values of a function only at $3n + 1$ points: 0, $\pm x_k$, $\mathrm{i}x_k$, $k = 1, \ldots, n$, regarding that the quadrature formula (1.3) can be represented in the form

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx A_0 f(0) + \sum_{k=1}^{n} \Big\{ A_k \big[f(x_k) + f(-x_k) \big] + 2B_k \mathrm{Re} f(\mathrm{i}x_k) \Big\}.$$

We note also another interesting fact. Namely, for the real functions of the form $f(x) = g(x^4)$, the previous formula can be reduced to $(n + 1)$-point quadrature formula

$$(4.15) \qquad \int_{0}^{1} f(x)\mathrm{d}x \approx K_{n+1}(f) = C_0 f(0) + \sum_{k=1}^{n} C_k f(x_k),$$

where $C_0 = \frac{1}{2} A_0$ and $C_k = A_k + B_k$, $k = 1, \ldots, n$.

Example 4.1. In order to illustrate an application of the formula (4.15) we consider the following integral

$$I(f) = \int_{0}^{1} \frac{\cos(\pi x^4)}{1 + x^8} \mathrm{d}x = 0.6708434308004106666580\ldots \, .$$

Quadrature sums $K_{n+1}(f)$ given by (4.15), as well as the relative errors

$$r_{n+1} = \left| \frac{K_{n+1}(f) - I(f)}{I(f)} \right|,$$

for $n = 1(1)10$ are presented in Table 2. In each entry in the second column of this table the first digit in error is underlined.

This formula is comparable with Gaussian one. Namely, if we consider the $(2n + 1)$-point Gauss-Legendre formula, with nodes at 0 and $\pm x_k$, $k = 1, \ldots, n$, and apply this symmetric rule to an even function, we have

$$(4.16) \qquad \int_{-1}^{1} f(x)\mathrm{d}x \approx Q_{2n+1}^{GL} = 2 \sum_{k=1}^{n} A_k f(x_k) + A_{n+1} f(0).$$

Therefore, $I \approx \frac{1}{2} Q_{2n+1}^{GL}$ (with $n + 1$ nodes) and we can compare these values with ones obtained by (4.15). The relative errors, in this case, are denoted by r_{2n+1}^{GL}.

TABLE 2. Quadrature sums $K_{n+1}(f)$ and relative errors r_{n+1}, for $n = 1(1)10$

n	$K_{n+1}(f)$	r_{n+1}	r_{2n+1}^{GL}	r_{2n+2}^{GL}
1	0.62106	7.42(−2)	2.53(−2)	9.11(−2)
2	0.67984	1.34(−2)	1.85(−2)	8.45(−3)
3	0.67024	8.90(−4)	3.91(−3)	6.53(−5)
4	0.670889	6.80(−5)	4.10(−4)	6.44(−5)
5	0.6708398	5.41(−6)	2.84(−5)	1.03(−5)
6	0.670843678	3.69(−7)	1.17(−6)	1.15(−6)
7	0.6708434139	2.52(−8)	5.36(−8)	1.03(−7)
8	0.670843431866	1.59(−9)	2.01(−8)	7.06(−9)
9	0.6708434307368	9.48(−11)	2.97(−9)	2.49(−10)
10	0.67084343080398	5.32(−12)	3.23(−10)	2.30(−11)

Also, instead of (4.16), we can take the $(2n + 2)$-point Gauss-Legendre formula on $(−1, 1)$, with nodes at $\pm x_k$, $k = 1, \ldots, n + 1$,

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx Q_{2n+2}^{GL} = 2\sum_{k=1}^{n+1} A_k f(x_k),$$

and apply it to I. Then, $I \approx \frac{1}{2}Q_{2n+2}^{GL}$ (also with $n + 1$ nodes), with the relative error r_{2n+2}^{GL}.

These relative errors in Gaussian sums are presented in the last two columns in Table 2. As we can see, fomula (4.15) has certain advantages in view of accuracy compared to the other two formulas, especially for larger value of n.

Construction of Gaussian quadratures was performed by the MATHEMATICA package `OrthogonalPolynomials` (see [3] and [13]). This package is downloadable from the web site `http://www.mi.sanu.ac.rs/˜gvm/`.

REFERENCES

[1] Acharya, M., Acharya, B. P. and Pati, S., *Numerical evaluation of integrals of analytic functions*, Internat. J. Comput. Math., **87** (2010), 2747–2751

[2] Birkhoff, G. and Young, D. M., *Numerical quadrature of analytic and harmonic functions*, J. Math. Phys., **29** (1950), 217–221

[3] Cvetković, A. S. and Milovanović, G. V., *The Mathematica Package "OrthogonalPolynomials"*, Facta Univ. Ser. Math. Inform., **19** (2004), 17–36

[4] Davis, P. J. and Rabinowitz, P., *Methods of Numerical Integration*, Academic Press, New York, 1975

[5] Gould, H. W.,*Combinatorial Identities*, Morgantown, W. Va., 1972

[6] Lether, F., *On Birkhoff-Young quadrature of analytic functions*, J. Comput. Appl. Math., **2** (1976), 81–84

[7] Lyness, J. N., *Quadrature methods based on complex function values*, Math. Comp., **23** (1969), 601–619

[8] Lyness, J. N. and Delves, L. M., *On numerical contour integration round a closed contour*, Math. Comp., **21** (1967), 561–577

[9] Lyness, J. N. and Moler, C. B., *Numerical differentiation of analytic functions*, SIAM J. Numer. Anal., **4** (1967), 202–210

[10] Mastroianni, G. and Milovanović, G. V., *Interpolation Processes – Basic Theory and Applications*, Springer Monographs in Mathematics, Berlin – Heidelberg: Springer – Verlag, 2008

[11] McGregor, M. T., *On a modified Birkhoff-Young quadrature formula for analytic functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **3** (1992), 13–16

[12] Milovanović, G. V., *Generalized quadrature formulae for analytic functions*, Appl. Math. Comput. **218** (2012), 8537–8551

[13] Milovanović, G. V. and Cvetković, A. S., *Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type*, Math. Balkanica, **26** (2012), 169–184

[14] Milovanović, G. V., Cvetković, A. S. and Stanić, M., *A generalized Birkhoff-Young-Chebyshev quadrature formula for analytic functions*, Appl. Math. Comput., **218** (2011), 944–948

[15] Milovanović, G. V., Mitrinović, D. S. and Rassias, Th. M., *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore – New Jersey – London - Hong Kong, 1994

[16] Milovanović, G. V. and Đorđević, R. Ž., *On a generalization of modified Birkhoff-Young quadrature formula*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 735 - 762 (1982), 130–134

[17] Muir, T., *The Theory of Determinants*, Vol. I, MacMillan & Co., London, 1906

[18] Scarborough, J. B., *Numerical Mathematical Analysis*, The Johns Hopkins Press, Baltimore, 1930

[19] Tošić, D. Đ., *A modification of the Birkhoff–Young quadrature formula for analytic functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 602 - 633 (1978), 73–77

[20] Tošić, D. Đ., *Numerical differenntiation of analytic functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 678 -715 (1980), 175–182

[21] Tošić, D. Đ. and ElBahi, A. A., *Optimal numerical differenntiation of real-valued analytic functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 735 - 762 (1982), 122–126

[22] Young, D. M., *An error bound for the numerical quadrature of analytic functions*, J. Math. Phys. **31** (1952), 42–44

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