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Malmquist and Müntz orthogonal systems and applications(*)

1. Introduction

The last fifty years have seen a great deal of progress in the field of orthogonal systems (orthogonal algebraic and trigonometric polynomials, orthogonal Müntz polynomials, orthogonal rational functions, etc.), as well as many their applications in mathematics, physics and other computational and applied sciences (electronics and communication, control system theory, process identification, etc.). The orthogonal polynomial systems, especially classical orthogonal polynomials (cf. Szegő [53], Chihara [11], Freud [16], Suetin [50], Geronimus [23–24], Nevai [43], Milovanović, Mitrinović, Rassias [36]), play very important role in many problems in approximation theory and numerical analysis. Some of them found very important applications in applied sciences and have become the main tool in several methods and procedures. Also, there are many non-classical orthogonal polynomials on the real line (cf. Gautschi [18] and [17] for software), as well as several classes of non-standard orthogonal polynomials (cf. Milovanović [32–33] and [35]). For example, the Szegő class on the unit circle (cf. Szegő [51–52] and [53, pp. 287–295], Geronimus [22–23], Nevai [44]), orthogonal polynomials on the semicircle and on a circular arc (see Gautschi and Milovanović [20–21], Gautschi, Landau, Milovanović [19], de Bruin [7], and also papers [29–30], [37–39]), orthogonal polynomials on the radial rays in the complex plane (see [31], [33–34] and [40]), etc.

In the recent years, the other classes of orthogonal systems, as Malmquist systems (cf. Walsh [55], Müntz systems (cf. Borwein and Erdélyi [5]), and others, have also taken a significant role in applications. A great progress in the theoretical sense for such orthogonal systems was made too. The first papers for orthogonal rational functions on the unit circle whose poles are fixed were given by M.M. Djrbashian [12–13]. These rational functions generalize the orthogonal polynomials of Szegő [53, pp. 287–295]. Recently, the paper of Djrbashian [13], which originally appeared in two parts, has been translated to English by K. Müller and A. Bultheel [42]. A survey on the theory of orthogonal systems and some open problems was written also by Djrbashian [14]. Several papers in this direction have been appeared in the last period ([8–10], [26], [46–49]).

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The orthogonal Müntz systems were considered first by Armanian mathematicians Badalyan [3] and Taslakyan [54]. Recently, it was investigated by McCarthy, Sayre and Shawyer [27] and more completely by Borwein, Erdélyi, and Zhang [6] (see also the recent book [5]).

This paper is organized as follows. In Section 2 we use the orthogonality relation for Jacobi polynomials in order to obtain rational functions (via Laplace transform), which can be applied in many technical fields. In the same section, we mention some of these important applications. Section 3 is devoted to the orthogonal Müntz systems which represent an important extension of the orthogonal polynomial systems. The Malmquist systems of orthogonal rational functions are considered in Section 4. Finally, in Section 5 we give some connections between the previous mentioned orthogonal systems using one new nonstandard inner product.

2. Applications of Orthogonal Systems

The applications of the classical orthogonal polynomials in technical fields as *electrical network synthesis, electronics and telecommunication, signal processing theory, control system theory* and *process identification* are well known. The Legendre, Laguerre, Chebyshev and Jacobi polynomials are very useful for design and construction of electrical network, transfer functions, orthogonal filters, adaptive control, telecommunication systems, etc. These applications are based on the least squares polynomial approximations. The orthogonality of these polynomials enable the construction of optimal network and optimal filters. Moreover, the Laplace transforms of the classical polynomials, or their modifications are rational functions, which can be easy factorized. This property is very convenient in constructing simple procedures for forming signal generators, adaptive controllers, and practical realizations of the transfer functions. For instance, Chebyshev orthogonal polynomials are used very much for technical applications, so there exist Chebyshev orthogonal filters. Many papers were published about Chebyshev polynomials and their applications in technics.

For designing orthogonal filters and optimal transfer functions may be used some modifications of the Legendre or Jacobi polynomials, which are orthogonal on the interval $(-1,1)$. On the other side, technical systems which are designed using orthogonal polynomials work in the real time, so we need the corresponding approximations on the interval $(0,\infty)$. For example, by substituting $x = 2e^{-at} - 1$ $(a > 0)$ in a polynomial orthogonal on $(-1,1)$, we obtain an exponential polynomial orthogonal on the $(0,\infty)$. For polynomials orthogonal on $(0,1)$ we use the substitution $x = e^{-at}$ $(a > 0)$.

Applying the Laplace transform on the exponential orthogonal polynomials we obtain orthogonal rational functions. In the sequel we will consider some applications of the classical orthogonal polynomials in technics, using their Laplace transforms and corresponding orthogonal functions.

Starting from the orthogonality relation for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with the parameters $\alpha, \beta > -1$, i.e.,

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx = \|P_n^{(\alpha,\beta)}\|^2 \delta_{nm}, \tag{2.1}$$

where

$$\|P_n^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

and putting $x+1 = 2e^{-2t}$, we obtain

$$\int_0^\infty \varphi_n(t)\varphi_m(t)\, dt = \delta_{nm},$$

where

$$\varphi_n(t) = \frac{2^{(\alpha+\beta+2)/2}}{\|P_n^{(\alpha,\beta)}\|}\, e^{-(\beta+1)t}(1-e^{-2t})^{\alpha/2} P_n^{(\alpha,\beta)}(2e^{-2t}-1).$$

Then, the Laplace transform of φ_n can be expressed in the form

$$\begin{aligned}
W_n(s) = \mathcal{L}[\varphi_n(t)] &= \int_0^\infty e^{-st}\varphi_n(t)\, dt \\
&= \frac{2^{-(s+1)/2}}{\|P_n^{(\alpha,\beta)}\|} \int_{-1}^{1} (1-x)^{\alpha/2}(1+x)^{(s+\beta-1)/2} P_n^{(\alpha,\beta)}(x)\, dx \\
&= \frac{2^{(\alpha+\beta)/2}}{\|P_n^{(\alpha,\beta)}\|} \cdot \frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{1}{2}(s+\beta+1)\right)}{\Gamma\left(\frac{1}{2}(\alpha+\beta+s+3)\right)} \times \\
&\qquad \times {}_3F_2\left(-n, \alpha+\beta+n+1, \tfrac{1}{2}\,\alpha+1; \alpha+1, \tfrac{1}{2}(\alpha+\beta+s+3); 1\right),
\end{aligned}$$

where the hypergeometric function ${}_3F_2$ is reduces to

$$\begin{aligned}
Q_n^{(\alpha,\beta)}(s) &= {}_3F_2\left(-n, \alpha+\beta+n+1, \tfrac{1}{2}\,\alpha+1; \alpha+1, \tfrac{1}{2}(\alpha+\beta+s+3); 1\right) \\
&= \sum_{k=0}^{\infty} \frac{(-n)_k(\alpha+\beta+n+1)_k \left(\frac{1}{2}\,\alpha+1\right)_k}{(\alpha+1)_k \left(\frac{1}{2}(\alpha+\beta+s+3)\right)_k k!} \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \frac{(\alpha+\beta+n+1)_k \left(\frac{1}{2}\,\alpha+1\right)_k}{(\alpha+1)_k \left(\frac{1}{2}(\alpha+\beta+s+3)\right)_k}\,.
\end{aligned}$$

In a simpler case when $\alpha = 0$, the function ${}_3F_2$ can be reduced to ${}_2F_1$. Namely,

$$\begin{aligned}
Q_n^{(0,\beta)}(s) &= {}_3F_2\left(-n, \beta+n+1, 1; 1, \tfrac{1}{2}(\beta+s+3); 1\right) \\
&= {}_2F_1\left(-n, \beta+n+1; \tfrac{1}{2}(\beta+s+3); 1\right).
\end{aligned}$$

Using a representation of the Jacobi polynomials, with parameters

$$\hat{\alpha} = \frac{s+\beta+1}{2} \qquad \text{and} \qquad \hat{\beta} = -\frac{s-\beta+1}{2}\,,$$

in terms of hypergeometric function (see Bateman and Erdélyi [2])

$${}_2F_1\left(-n, n+\hat{\alpha}+\hat{\beta}+1; \hat{\alpha}+1; \tfrac{1}{2}(1-x)\right) = \frac{P_n^{(\hat{\alpha},\hat{\beta})}(x)}{\binom{n+\hat{\alpha}}{n}}\,,$$

we find

$$Q_n^{(0,\beta)}(s) = \frac{P_n^{(\hat{\alpha},\hat{\beta})}(-1)}{\binom{n+\hat{\alpha}}{n}} = (-1)^n \frac{\binom{n+\hat{\beta}}{n}}{\binom{n+\hat{\alpha}}{n}}\,,$$

because of

$$n+\hat{\alpha}+\hat{\beta}+1 = n+\beta+1 \qquad \text{and} \qquad \hat{\alpha}+1 = \frac{s+\beta+3}{2}\,.$$

Thus, in this case we get

$$\begin{aligned} W_n(s) &= \mathcal{L}\left[\frac{2^{\beta/2+1}}{\|P_n^{(0,\beta)}\|}\, e^{-(\beta+1)t} P_n^{(0,\beta)}(2e^{-2t}-1)\right] \\ &= \frac{2^{\beta/2}}{\|P_n^{(0,\beta)}\|} \cdot \frac{\Gamma\left(\frac{1}{2}(s+\beta+1)\right)}{\Gamma\left(\frac{1}{2}(s+\beta+3)\right)} \cdot \frac{(-1)^n(n+\hat{\beta})(n-1+\hat{\beta})\cdots(1+\hat{\beta})}{(n+\hat{\alpha})(n-1+\hat{\alpha})\cdots(1+\hat{\alpha})}\,, \end{aligned}$$

i.e.,

$$W_n(s) = \sqrt{2(2n+\beta+1)}\, \frac{\prod_{i=0}^{n-1}(s-(2i+1+\beta))}{\prod_{i=0}^{n}(s+(2i+1+\beta))}\,. \tag{2.2}$$

Sometimes, we put $x+1 = 2e^{-t}$ in (2.1). Then the previous rational functions become

$$\widetilde{W}_n(s) = \sqrt{2}\, W_n(2s) = \sqrt{2n+\beta+1}\, \frac{\prod_{i=0}^{n-1}\left(s-\dfrac{2i+1+\beta}{2}\right)}{\prod_{i=0}^{n}\left(s+\dfrac{2i+1+\beta}{2}\right)}\,. \tag{2.3}$$

In a degenerating case $\beta = -1$, when the Jacobi polynomial $P_n^{(0,-1)}(x)$ can be expressed in the form

$$P_n^{(0,-1)}(x) = \frac{1+x}{2}\, P_{n-1}^{(0,1)}(x),$$

from (2.3) we obtain

$$\widetilde{W}_n(s) = \sqrt{2n}\, \frac{\prod\limits_{i=1}^{n-1}(s-i)}{\prod\limits_{i=1}^{n}(s+i)}\,. \tag{2.4}$$

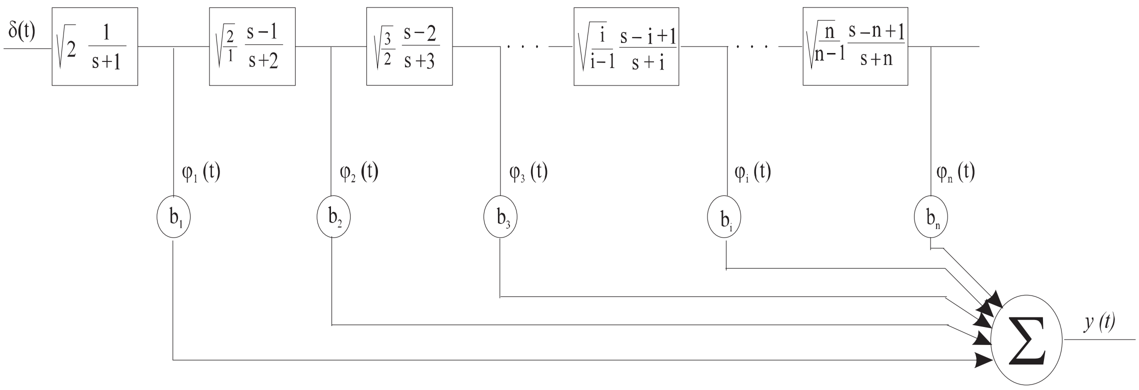

Fig. 2.1

Using (2.4) we can obtain an orthogonal filter (see [1, pp. 42–71]), which is given in Fig. 2.1. In this case the corresponding function $\widetilde{\varphi}(t)$ can be expressed in the form

$$\widetilde{\varphi}(t) = \sqrt{2n} \sum_{i=1}^{n} a_{n,i} e^{-it},$$

where

$$a_{n,i} = (-1)^i \binom{n+i-1}{i-1} \binom{n}{i}.$$

This Jacobi filter has been used in signal theory for obtaining the orthogonal signals $\phi(t) = \widetilde{\varphi}(t)$. Also, this filter is useful in the adaptive control in construction of adjustable models. The application of this filter is very often in the process identification as well. In this case, the network, obtained by Jacobi's filter, is connected with an unknown process as it is shown in Fig. 2.2.

By adjusting the parameters b_i, so that the error $e(t)$ is minimal (or zero), one can derive the model of an unknown process.

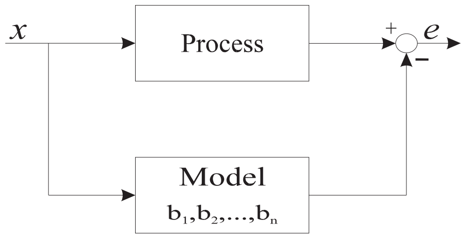

Fig. 2.2

An application of the Legendre polynomials is also given in [1]. Namely, using (2.2) with $\beta = 0$, one can obtain the following sequence of the rational functions

$$W_n(s) = \sqrt{2(2n+1)}\,\frac{\prod_{i=0}^{n-1}(s-(2i+1))}{\prod_{i=0}^{n}(s+2i+1)}.$$

These functions can be used in practical realization in the same way as the previous functions of the Jacobi type.

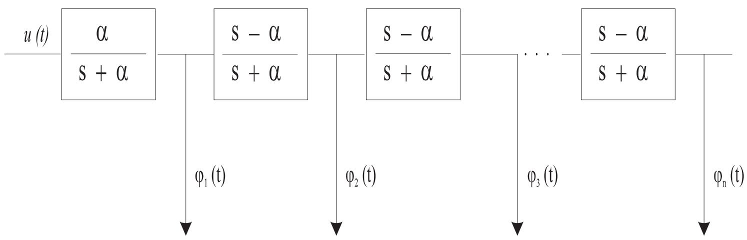

Fig. 2.3

In [15] it was also presented a method for designing of orthogonal filters, based on the classical orthogonal polynomials. In construction of some real filters has been used the Hermite polynomials. A method for process identification was derived in [4] and based on the Laguerre polynomials. It is well known that the Laplace transform of the Laguerre polynomials is given by

$$\mathcal{L}[L_n(x)] = \frac{(-1)^n}{n!}\frac{(s-1)^n}{s^{n+1}}\,.$$

In order to obtain the Laguerre orthogonal filters, we can take the modified Laguerre functions $\phi_{n,\alpha}(x)$, which Laplace transform is given by

$$\mathcal{L}[\phi_{n,\alpha}(x)] = \frac{\sqrt{2\alpha}}{s+\alpha}\left(\frac{s-\alpha}{s+\alpha}\right)^n. \tag{2.5}$$

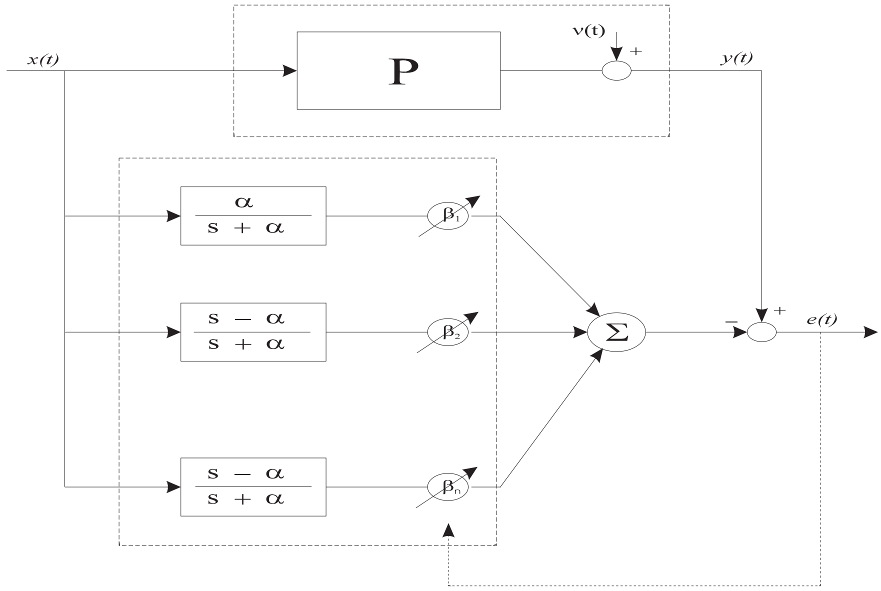

Fig. 2.4

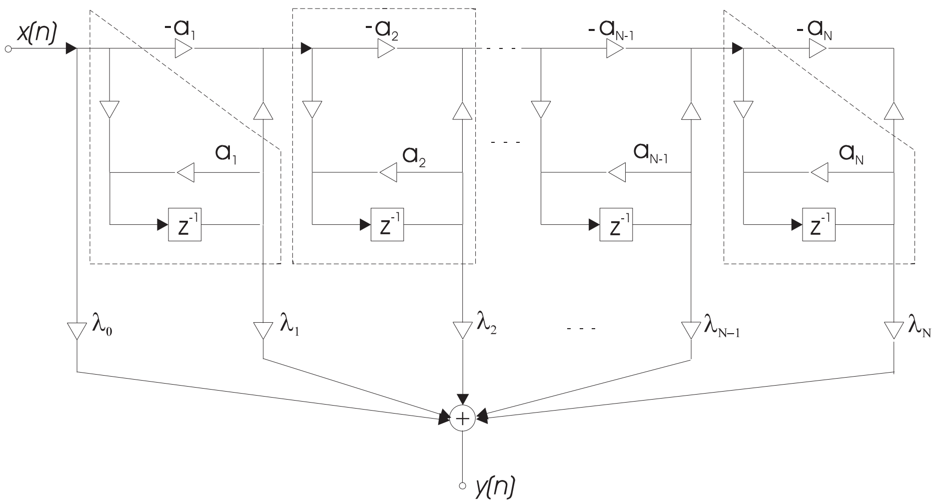

Fig. 2.5

The orthogonal filter obtained by (2.5) is displayed in Fig. 2.3.

In this way, we can obtain one base of the orthogonal functions $\phi_{n,\alpha}(x)$, useful for process identification. Such a system is presented in Fig. 2.4.

Using the complex rational functions $F_n(s)$ defined by

$$F_n(s) = \frac{\sqrt{1-\alpha_n^2}}{s-\alpha_n} \prod_{i=0}^{n-1} \frac{1-\bar{\alpha}_i s}{s-\alpha_i}$$

(see Section 4), some applications in obtaining orthogonal filters are given in [25] and [45]. Such an orthogonal filter is presented in Fig. 2.5.

3. Müntz Orthogonal Systems

Let $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ be a given sequence of complex numbers. Taking the following definition for x^λ:

$$x^\lambda = e^{\lambda \log x}, \qquad x \in (0, +\infty),\ \lambda \in \mathbb{C},$$

and the value at $x = 0$ is defined to be the limit of x^λ as $x \to 0$ from $(0, +\infty)$ whenever the limits exists, we will consider orthogonal Müntz polynomials as linear combinations of the Müntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ (see [5–6]). The set of all such polynomials we will denote by $M_n(\Lambda)$, i.e.,

$$M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\},$$

where the linear span is over the complex numbers $\mathbb{C}$ in general. The union of all $M_n(\Lambda)$ is denoted by $M(\Lambda)$.

The first considerations of orthogonal Müntz systems were made by Badalyan [3] and Taslakyan [54]. Recently, it was investigated by McCarthy, Sayre and Shawyer [27] and more completely by Borwein, Erdélyi, and Zhang [6].

Supposing that $\mathrm{Re}(\lambda_k) > -1/2$ $(k = 0, 1, \dots)$ we can give the following definition of the *Müntz-Legendre polynomials* on $(0, 1]$ (see [54], [6]):

Definition 3.1. The nth Müntz-Legendre polynomial on $(0, 1]$ is given by

$$P_n(\lambda_0, \dots, \lambda_n; x) = \frac{1}{2\pi i} \oint_\Gamma \prod_{\nu=0}^{n-1} \frac{s + \bar{\lambda}_\nu + 1}{s - \lambda_\nu} \frac{x^s\, ds}{s - \lambda_n} \qquad (n = 0, 1, \dots), \tag{3.1}$$

where the simple contour Γ surrounds all the zeros of the denominator in the integrand.

For polynomials $P_n(x) \equiv P_n(\lambda_0, \dots, \lambda_n; x)$ one can prove an orthogonality relation on $(0, 1)$:

Theorem 3.1. *Let the polynomials $P_n(x)$ be defined by* (3.1)*. Then*

$$\int_0^1 P_n(x) \overline{P_m(x)}\, dx = \delta_{n,m} / (1 + \lambda_n + \overline{\lambda}_n)$$

holds for every $n, m = 0, 1, \dots$.

Evidently, that the polynomials $P_n^*(x) = (1+\lambda_n+\overline{\lambda}_n)^{1/2}P_n(x)$ are *orthonormal*. Supposing that $\lambda_\nu \neq \lambda_\mu$ ($\nu \neq \mu$) it is easy to show that polynomials $P_n(x)$ can be expressed in the form

$$P_n(x) = \sum_{k=0}^{n} c_{n,k} x^{\lambda_k},$$

where

$$c_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1}(1+\lambda_k+\bar{\lambda}_\nu)}{\prod\limits_{\substack{\nu=0\\ \nu\neq k}}^{n}(\lambda_k-\lambda_\nu)}.$$

In a limit case when $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$, the polynomials (3.1) reduce to

$$P_n(\lambda_0,\ldots,\lambda_n;x) = x^\lambda L_n(-(1+\lambda+\bar{\lambda})\log x),$$

where $L_n(x)$ is the Laguerre polynomial orthogonal on the $(0,\infty)$ with respect to the exponential weight e^{-x}, and for which $L_n(0) = 1$.

There is also a generalized Rodrigues formula for the Müntz-Legendre polynomials (see [27]). Some recurrence relations also hold:

Theorem 3.2. *Let the polynomials $P_n(x)$ be defined by* (3.1)*. Then*

$$x(P_n'(x) - P_{n-1}'(x)) = \lambda_n P_n(x) + (1+\bar{\lambda}_{n-1})P_{n-1}(x),$$

$$xP_n'(x) = \lambda_n P_n(x) + \sum_{\nu=0}^{n-1}(1+\lambda_\nu+\overline{\lambda}_\nu)P_\nu(x),$$

$$xP_n''(x) = (\lambda_n - 1)P_n'(x) + \sum_{\nu=0}^{n-1}(1+\lambda_\nu+\overline{\lambda}_\nu)P_\nu'(x).$$

Similar to the Legendre polynomials and here also $P_n(1) = 1$ for each n.

An interesting question is connected by the zero distribution of the Müntz-Legendre polynomials. A nice proof of the following result was given in [6].

Theorem 3.3. *For real numbers $\lambda_\nu > -1/2$ ($\nu = 0,1,\ldots$) the Müntz-Legendre polynomial $P_n(x)$ has exactly n distinct zeros in $(0,1)$, and it changes sign at each of these zeros.*

Some inequalities for Müntz polynomials were also investigated in [6].

4. Malmquist Orthogonal Systems

Let $A = \{a_0, a_1, \dots\}$ be an arbitrary sequence of complex numbers in the unit circle ($|a_\nu| < 1$). The Malmquist system of rational functions (see [55], [12–14]) is defined in the following way

$$\phi_n(s) = \frac{(1-|a_n|^2)^{1/2}}{1-\bar{a}_n s} \prod_{\nu=0}^{n-1} \frac{a_\nu - s}{1-\bar{a}_\nu s} \cdot \frac{|a_\nu|}{a_\nu} \qquad (n = 0, 1, \dots), \tag{4.1}$$

where for $a_\nu = 0$ we put $|a_\nu|/a_\nu = \bar{a}_\nu/|a_\nu| = -1$. Such system of functions was intensively investigated in several papers by Djrbashian [12–14], Bultheel, González-Vera, Hendriksen, and Njåstad [8–10], Pan [46–49], etc.

In this section we want to prove some auxiliary results in order to connect this system of orthogonal functions with some Müntz system of polynomials.

Excluding the normalization constants, the system (4.1) can be represented in the form

$$W_n(s) = \frac{\prod\limits_{\nu=0}^{n-1} (s - a_\nu)}{\prod\limits_{\nu=0}^{n} (s - a_\nu^*)}, \tag{4.2}$$

where $a_\nu^* = 1/\bar{a}_\nu$. For $a_\nu = 0$ we put only s instead of $(s - a_\nu)/(s - a_\nu^*)$.

Suppose now that $a_\nu \neq a_\mu$ for $\nu \neq \mu$. Then (4.2) can be written in the form

$$W_n(s) = \sum_{k=0}^{n} \frac{A_{n,k}}{s - a_k^*}, \tag{4.3}$$

where

$$A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1} (a_k^* - a_\nu)}{\prod\limits_{\substack{\nu=0 \\ \nu\neq k}}^{n} (a_k^* - a_\nu^*)} \qquad (k = 0, 1, \dots, n). \tag{4.4}$$

The case when $a_\nu = a_\mu$ can be considered as a limiting process $a_\nu \to a_\mu$.

Alternatively, for $|a_\nu| < 1$, (4.4) can be reduced to

$$A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1} \left(\dfrac{1}{\bar{a}_k} - a_\nu\right)}{\prod\limits_{\substack{\nu=0 \\ \nu\neq k}}^{n} \left(\dfrac{1}{\bar{a}_k} - \dfrac{1}{\bar{a}_\nu}\right)} = \frac{\bar{a}_0 \cdots \bar{a}_n}{\bar{a}_k} \cdot \frac{\prod\limits_{\nu=0}^{n-1} (\bar{a}_k a_\nu - 1)}{\prod\limits_{\substack{\nu=0 \\ \nu\neq k}}^{n} (\bar{a}_k - \bar{a}_\nu)}. \tag{4.5}$$

It is well-known that system of functions (4.1) is orthonormal on the unit circle $|s| = 1$ with respect to the inner product

$$(u, v) = \frac{1}{2\pi i} \oint_{|s|=1} u(s)\overline{v(s)}\,\frac{ds}{s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta})\overline{v(e^{i\theta})}\,d\theta. \tag{4.6}$$

Namely, $(\phi_n, \phi_m) = \delta_{nm}$ $(n, m = 0, 1, \dots)$.

We note also that

$$(u, v) = \frac{1}{2\pi i} \oint_{|s|=1} u(s)\overline{v(s^*)}\,\frac{ds}{s}\,,$$

where $s^* = 1/\bar{s}$ on the unit circle $|s| = 1$.

Because of completeness we prove the following result:

Theorem 4.1. *If the system of rational functions* $\{W_n\}_{n=0}^{+\infty}$ *defined by* (4.2) *and the inner product* $(\,.\,,\,.\,)$ *by* (4.6)*, then*

$$(W_n, W_m) = \|W_n\|^2 \delta_{nm},$$

where

$$\|W_n\|^2 = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}\,.$$

Proof. It is enough to prove that $(W_n, W_m) = 0$ for $m < n$, because of the property $\overline{(u, v)} = (v, u)$. Supposing that $m < n$, we have

$$\begin{aligned}
(W_n, W_m) &= \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{\prod\limits_{\nu=0}^{n-1}(s - a_\nu)}{\prod\limits_{\nu=0}^{n}(s - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1}(\bar{s} - \bar{a}_\nu)}{\prod\limits_{\nu=0}^{m}(\bar{s} - \bar{a}_\nu^*)} \cdot \frac{ds}{s} \\
&= \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{\prod\limits_{\nu=0}^{n-1}(s - a_\nu)}{\prod\limits_{\nu=0}^{n}(s - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1}\left(\frac{1}{s} - \frac{1}{a_\nu^*}\right)}{\prod\limits_{\nu=0}^{m}\left(\frac{1}{s} - \frac{1}{a_\nu}\right)} \cdot \frac{ds}{s} \\
&= \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{\prod\limits_{\nu=0}^{n-1}(s - a_\nu)}{\prod\limits_{\nu=0}^{n}(s - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1}(s - a_\nu^*)}{\prod\limits_{\nu=0}^{m}(s - a_\nu)} \cdot \frac{(-1)a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*}\,ds.
\end{aligned}$$

Evidently, for $|a_\nu| < 1$, i.e., $|a_\nu^*| > 1$, and $m < n$, Cauchy's theorem gives $(W_n, W_m) = 0$. For $m = n$ we see that

$$\begin{aligned}\|W_n\|^2 &= -\frac{a_0 \cdots a_n}{a_0^* \cdots a_{n-1}^*} \cdot \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{ds}{(s-a_n^*)(s-a_n)} \\ &= -\frac{a_0 \cdots a_n}{(1/\bar{a}_0) \cdots (1/\bar{a}_{n-1})} \cdot \frac{1}{a_n - a_n^*} = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}. \quad \square\end{aligned}$$

Lemma 4.1. *Let $|z| \le 1$ and let F be defined by*

$$F(z) = \frac{1}{2\pi i} \oint\limits_{|s|=1} W_n(s) \overline{W_m(\bar{z}s)} \frac{ds}{s}, \tag{4.7}$$

where the system functions $W_n(s)$ is defined by (4.2) with mutually different numbers a_ν ($\nu = 0, 1, \dots$) in the unit circle $|s| = 1$. Then

$$F(z) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{A_{n,i} \bar{A}_{m,j}}{a_i^* \bar{a}_j^* - z}, \tag{4.8}$$

where the numbers $A_{n,k}$ are given in (4.4).

Proof. For $|z| \le 1$ we conclude that the function

$$\begin{aligned}F(z) &= \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{\prod\limits_{\nu=0}^{n-1} (s - a_\nu)}{\prod\limits_{\nu=0}^{n} (s - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1} \left(\frac{z}{s} - \frac{1}{a_\nu^*} \right)}{\prod\limits_{\nu=0}^{m} \left(\frac{z}{s} - \frac{1}{a_\nu} \right)} \cdot \frac{ds}{s} \\ &= \frac{(-1) a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} \cdot \frac{1}{2\pi i} \oint\limits_{|s|=1} \frac{\prod\limits_{\nu=0}^{n-1} (s - a_\nu)}{\prod\limits_{\nu=0}^{n} (s - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1} (s - a_\nu^* z)}{\prod\limits_{\nu=0}^{m} (s - a_\nu z)} \, ds\end{aligned}$$

has $(m+1)$ poles in the circle $|s| = 1$: $a_j z$ ($j = 0, 1, \dots, m$). By Cauchy's residue theorem we find that

$$F(z) = \frac{(-1) a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} \sum_{j=0}^{m} \frac{\prod\limits_{\nu=0}^{n-1} (a_j z - a_\nu)}{\prod\limits_{\nu=0}^{n} (a_j z - a_\nu^*)} \cdot \frac{\prod\limits_{\nu=0}^{m-1} (a_j z - a_\nu^* z)}{\prod\limits_{\substack{\nu=0 \\ \nu \ne j}}^{m} (a_j z - a_\nu z)}.$$

Since

$$G_{m,j} = \frac{\prod_{\nu=0}^{m-1}(a_j z - a_\nu^* z)}{\prod_{\substack{\nu=0 \\ \nu \neq j}}^{m}(a_j z - a_\nu z)} = \frac{\prod_{\nu=0}^{m-1}(a_j \bar{a}_\nu - 1)}{\prod_{\substack{\nu=0 \\ \nu \neq j}}^{m}(a_j - a_\nu)} \cdot \frac{1}{\bar{a}_0 \cdots \bar{a}_{m-1}} \,,$$

because of (4.5), we have

$$G_{m,j} = \frac{a_j}{a_0 \cdots a_m} \cdot \bar{A}_{m,j} \cdot \frac{1}{\bar{a}_0 \cdots \bar{a}_{m-1}} = \frac{a_j \bar{a}_m \bar{A}_{m,j}}{|\bar{a}_0 \cdots \bar{a}_m|^2} \,.$$

Thus,

$$\frac{a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} G_{m,j} = a_j \bar{A}_{m,j},$$

and we obtain

$$F(z) = -\sum_{j=0}^{m} \bar{A}_{m,j} \frac{\prod_{\nu=0}^{n-1}\left(z - \frac{a_\nu}{a_j}\right)}{\prod_{\nu=0}^{n}\left(z - \frac{a_\nu^*}{a_j}\right)} \,.$$

On the other hand, expanding $Q_j(z) = \prod_{\nu=0}^{n-1}(z - a_\nu/a_j) \Big/ \prod_{\nu=0}^{n}(z - a_\nu^*/a_j)$ in partial fractions, we get

$$Q_j(z) = \sum_{i=0}^{n} \frac{\prod_{\nu=0}^{n-1}\left(\frac{a_i^*}{a_j} - \frac{a_\nu}{a_j}\right)}{\prod_{\substack{\nu=0 \\ \nu \neq i}}^{n}\left(\frac{a_i^*}{a_j} - \frac{a_\nu^*}{a_j}\right)} \cdot \frac{1}{z - \frac{a_i^*}{a_j}} = \sum_{i=0}^{n} \frac{\prod_{\nu=0}^{n-1}(a_i^* - a_\nu)}{\prod_{\substack{\nu=0 \\ \nu \neq i}}^{n}(a_i^* - a_\nu^*)} \cdot \frac{1}{z - a_i^* \bar{a}_j^*} \,.$$

Because of (4.4), the right hand side of the last equality becomes

$$\sum_{i=0}^{n} \frac{A_{n,i}}{z - a_i^* \bar{a}_j^*} \,,$$

so that we obtain

$$F(z) = \sum_{j=0}^{m} \bar{A}_{m,j} \sum_{i=0}^{n} A_{n,i} \frac{1}{a_i^* \bar{a}_j^* - z} \,,$$

i.e., (4.8). □

For $z = 1$, from (4.7) we see that $F(1) = (W_n, W_m)$. Thus,

$$(W_n, W_m) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{A_{n,i} \bar{A}_{m,j}}{a_i^* \bar{a}_j^* - 1} \,. \tag{4.9}$$

5. A Connection Between Malmquist and Müntz Systems

In this section we give a connection between the Malmquist system of rational functions (4.2) and a Müntz system, which is orthogonal with respect to a new inner product.

Using the numbers a_ν^* which appear in (4.2), i.e., $A^* = \left\{a_0^*, a_1^*, \dots\right\}$, we form the Müntz system $\left\{x^{a_0^*}, x^{a_1^*}, \dots, x^{a_n^*}\right\}$. Notice that all such numbers are outside the unit circle. In order to short our notation and to be consistent to Section 3, we write $a_\nu^* = \lambda_\nu$ $(\nu = 0, 1, \dots)$.

In our consideration we need a new operation:

Definition 5.1. For $\alpha, \beta \in \mathbb{C}$ we have

$$x^\alpha \odot x^\beta = x^{\alpha\beta} \qquad (x \in (0, \infty)). \tag{5.1}$$

Using (5.1) we can introduce an external operation for the Müntz polynomials from $M(A^*) = M(\Lambda)$.

Definition 5.2. Let $P \in M_n(\Lambda)$ and $Q \in M_m(\Lambda)$, i.e.,

$$P(x) = \sum_{i=0}^{n} p_i x^{\lambda_i} \quad \text{and} \quad Q(x) = \sum_{j=0}^{m} q_j x^{\lambda_j}, \tag{5.2}$$

then

$$(P \odot Q)(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_i q_j x^{\lambda_i \lambda_j}. \tag{5.3}$$

Under restrictions that for each i and j we have

$$|\lambda_i| > 1, \qquad \mathrm{Re}(\lambda_i \bar{\lambda}_j - 1) > 0, \tag{5.4}$$

then we can introduce a new inner product for Müntz polynomials.

Definition 5.3. Let the conditions (5.4) be satisfied for the Müntz polynomials $P(x)$ and $Q(x)$ given by (5.2). Their inner product $[P, Q]$ is defined by

$$[P, Q] = \int_0^1 (P \odot \overline{Q})(x) \, \frac{dx}{x^2}, \tag{5.5}$$

where $(P \odot Q)(x)$ is determined by (5.3).

It is not clear immediately that (5.5) represents an inner product. Therefore, we prove the following result:

Theorem 5.1. *Let* $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ *be a sequence of the complex numbers such that the conditions* (5.4) *hold. Then*

(i) $[P, P] \geq 0$;

(ii) $[P, P] = 0 \iff P(x) \equiv 0$;

(iii) $[P + Q, R] = [P, R] + [Q, R]$;

(iv) $[cP, Q] = c[P, Q]$;

(v) $[P, Q] = \overline{[Q, P]}$

for each $P, Q, R \in M(\Lambda)$ *and each* $c \in \mathbb{C}$.

Proof. Let $P(x)$ and $Q(x)$ be given by (5.2). Using (5.5) and (5.3) we have

$$[P, Q] = \int_0^1 (P \odot Q)(x) \frac{dx}{x^2} = \sum_{i=0}^{n} \sum_{j=0}^{m} p_i \bar{q}_j \int_0^1 x^{\lambda_i \bar{\lambda}_j - 2} dx.$$

Because of (5.4), we find that $\int_0^1 x^{\lambda_i \bar{\lambda}_j - 2} dx = 1/(\lambda_i \bar{\lambda}_j - 1)$ for each i and j, so that we get

$$[P, Q] = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{p_i \bar{q}_j}{\lambda_i \bar{\lambda}_j - 1}. \tag{5.6}$$

In order to prove (i) and (ii) it is enough to conclude that the quadratic form

$$[P, P] = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{\lambda_i \bar{\lambda}_j - 1} p_i \bar{p}_j,$$

i.e., its matrix $H_n = \left[1/(\lambda_i \bar{\lambda}_j - 1)\right]_{i,j=0}^{n}$, is positive definite. Therefore, we use the Sylvester's necessary and sufficient conditions (cf. [28, p. 214])

$$D_k = \det H_k = \det \left[1/(\lambda_i \bar{\lambda}_j - 1)\right]_{i,j=0}^{k} > 0 \qquad (k = 0, 1, \dots, n).$$

In order to evaluate the determinants D_k, we use Cauchy's formula (see Muir [41, p. 345])

$$\det \left[\frac{1}{a_i + b_j}\right]_{i,j=0}^{k} = \frac{\prod\limits_{i>j=0}^{k} (a_i - a_j)(b_i - b_j)}{\prod\limits_{i,j=0}^{k} (a_i + b_j)}$$

with $a_i = \lambda_i$ and $b_j = -1/\bar\lambda_j$. Thus, we obtain

$$D_k = \frac{1}{\prod_{j=0}^{k} \bar\lambda_j} \det\left[\frac{1}{\lambda_i - 1/\bar\lambda_j}\right]_{i,j=0}^{k} = \frac{1}{\prod_{j=0}^{k} \bar\lambda_j} \cdot \frac{\prod_{i>j=0}^{k} (\lambda_i - \lambda_j)\Bigl(-\dfrac{1}{\bar\lambda_i} + \dfrac{1}{\bar\lambda_j}\Bigr)}{\prod_{i,j=0}^{k} \Bigl(\lambda_i - \dfrac{1}{\bar\lambda_j}\Bigr)},$$

i.e.,

$$D_k = \frac{1}{\prod_{j=0}^{k} \bar\lambda_j} \cdot \frac{\prod_{i>j=0}^{k} \dfrac{|\lambda_i - \lambda_j|^2}{\bar\lambda_i \bar\lambda_j}}{\prod_{i,j=0}^{k} \dfrac{\lambda_i \bar\lambda_j - 1}{\bar\lambda_j}}.$$

Since $D_0 = 1/(|\lambda_0|^2 - 1) > 0$ (because of (5.4)) and

$$D_k = \frac{D_{k-1}}{|\lambda_k|^2 - 1} \prod_{i=0}^{k-1} \frac{|\lambda_k - \lambda_i|^2}{|\lambda_i \bar\lambda_k - 1|^2},$$

by induction we conclude that $D_k > 0$ for all k.

The properties (iii)–(v) follow directly from (5.5) or (5.6). $\square$

Now we are ready to define the Müntz polynomials

$$Q_n(x) \equiv Q_n(\lambda_0, \lambda_1, \ldots, \lambda_n; x), \qquad n = 0, 1, \ldots, \tag{5.7}$$

orthogonal with respect to the inner product (5.5).

Consider again the Malmquist system of rational functions (4.2), i.e.,

$$W_n(s) = \frac{\prod_{\nu=0}^{n-1} (s - 1/\bar\lambda_\nu)}{\prod_{\nu=0}^{n} (s - \lambda_\nu)}, \tag{5.8}$$

where $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$ is a complex sequence such that (5.4) holds.

Definition 5.4. The nth Müntz polynomial $Q_n(x)$, associated to the Malmquist function (5.8), is defined by

$$Q_n(x) = \frac{1}{2\pi i} \oint_\Gamma W_n(s) x^s \, ds, \tag{5.9}$$

where the simple contour Γ surrounds all the points λ_ν $(\nu = 0, 1, \ldots, n)$.

Using (5.9), (4.3), (4.4) and Cauchy's residue theorem we get, a representation of (5.7) in the form

$$Q_n(x) = \sum_{k=0}^{n} A_{n,k} x^{\lambda_k}, \tag{5.10}$$

where

$$A_{n,k} = \frac{\prod_{\nu=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_\nu)}{\prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} (\lambda_k - \lambda_\nu)} \qquad (k = 0, 1, \ldots, n), \tag{5.11}$$

and where we assumed that $\lambda_i \neq \lambda_j$ $(i \neq j)$.

Finally, we prove an orthogonality relation for polynomials $Q_n(x)$. Other results in this direction will be published elsewhere.

Theorem 5.2. *Under previous conditions on the sequence Λ, the Müntz polynomials $Q_n(x)$, $n = 0, 1, \ldots$, defined by* (5.9), *are orthogonal with respect to the inner product* (5.5). *i.e.,*

$$[Q_n, Q_m] = \frac{1}{(|\lambda_n|^2 - 1)|\lambda_0 \lambda_1 \cdots \lambda_{n-1}|^2} \, \delta_{n,m}. \tag{5.12}$$

Proof. According to (5.6) and (5.10), we have

$$[Q_n, Q_m] = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{A_{n,i} \bar{A}_{m,j}}{\lambda_i \bar{\lambda}_j - 1},$$

where $A_{n,k}$ is given by (5.11).

Using Lemma 4.1 with $z = 1$, i.e., equality (4.9), we conclude that

$$[Q_n, Q_m] = (W_n, W_m),$$

where $W_n(s)$ is determined by (5.8).

Since $a_\nu = 1/\bar{\lambda}_\nu$ $(\nu = 0, 1, \ldots)$, from Theorem 4.1 it follows (5.12). $\square$

For the norm of polynomial $Q_n(x)$ we obtain

$$\|Q_n\| = \sqrt{[Q_n, Q_n]} = \frac{1}{|\lambda_0 \lambda_1 \cdots \lambda_{n-1}|} \cdot \frac{1}{\sqrt{|\lambda_n|^2 - 1}},$$

where the complex numbers satisfy the condition (5.4).

One particular result can be interesting:

Corollary 5.3. *Let $Q_n(x)$ be defined by* (5.9) *and let $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$. Then*

$$Q_n(x) = x^\lambda L_n\big(-(\lambda - 1/\bar{\lambda})\log x\big), \tag{5.13}$$

where $L_n(x)$ is the Laguerre polynomial orthogonal with respect to e^{-x} on $[0,\infty)$ and such that $L_n(0) = 1$.

Proof. For $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$, (5.9) reduces to

$$Q_n(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{(s-1/\bar{\lambda})^n}{(s-\lambda)^{n+1}}\, x^s\, ds,$$

where $\lambda \in \operatorname{int}\Gamma$. Since

$$\operatorname*{Res}_{z=\lambda}\left[\frac{(s-1/\bar{\lambda})^n}{(s-\lambda)^{n+1}}\, x^s\right] = \frac{1}{n!}\lim_{z\to\lambda}\frac{d^n}{ds^n}\left[\big(s-1/\bar{\lambda}\big)^n\, x^s\right]$$

we obtain by Cauchy's residue theorem

$$\begin{aligned} Q_n(x) &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} n(n-1)\cdots(k+1)(\lambda-1/\bar{\lambda})^k x^\lambda(\log x)^k \\ &= x^\lambda\sum_{k=0}^{n}\frac{1}{k!}\binom{n}{k}(\lambda-1/\bar{\lambda})^k(\log x)^k, \end{aligned}$$

which gives (5.13). $\square$

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