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AN APPLICATION OF LITTLE 1/q-JACOBI POLYNOMIALS TO SUMMATION OF CERTAIN SERIES

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Abstract. In this paper we consider an application of a special class of little $1/q$-Jacobi polynomials to summation of series. Beside the three-term recurrence relation for such polynomials, a zero distribution as well as the corresponding quadratures of Gaussian type are investigated. Some numerical examples are included.

1. Introduction

We are primarily interested with the linear functional of the form

$$L_a^{p,q}(f) = \sum_{k=0}^{+\infty} \frac{1}{p^k} f\left(\frac{a}{q^k}\right) = \int f\,d\mu, \tag{1.1}$$

which is a direct generalization of the q-integral which is defined by

$$\int_0^a f(x)d_{1/q}x := a(1-1/q)\sum_{k=0}^{+\infty} \frac{1}{q^k} f\left(\frac{a}{q^k}\right).$$

There is a simple connection between q-integral and $L_a^{p,q}$, namely

$$\int_0^a f(x)d_{1/q}x = a(1-1/q)L_a^{q,q}(f).$$

This paper is organized as follows. Orthogonal polynomials with respect to the linear functional $L_a^{p,q}$ are derived in Section 2. Section 3 is devoted to the problem of zero distribution of such polynomials and the corresponding quadratures of Gaussian type. A few numerical examples are considered in Section 4.

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2. Polynomials Orthogonal With Respect to $L_a^{p,q}$

It is well known that polynomials orthogonal with respect to linear functional of the form

$$L(f) = \sum_{k=0}^{+\infty} \frac{(\beta/q;1/q)_k}{(1/q;1/q)_k} (\alpha/q)^k f\left(\frac{a}{q^k}\right)$$

for $0 < \alpha < q$, $0 < \beta < q$ and $q > 1$ are little $1/q$-Jacobi polynomials (for reference see [7]). Choosing $\beta = 1$ and $\alpha = q/p$ the previous linear functional becomes our functional $L_a^{p,q}$. This means that the polynomials orthogonal with respect to $L_a^{p,q}$ are little $1/q$-Jacobi polynomials under conditions $p > 1$ and $q > 1$.

For arbitrary (possibly complex values of p and q) it can be proved that polynomials orthogonal with respect to $L_a^{p,q}$ exist under conditions

$$|pq^n| > 1 \quad (n \in \mathbb{N}_0) \quad \text{and} \quad q^n \neq 1 \quad (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *The polynomials* $\{p_k\}_{k\in\mathbb{N}_0}$*, orthonormal with respect to* $L_a^{p,q}$*, exist under conditions given in* (2.1)*, and they satisfy the following three-term recurrence relation*

$$xp_k(x) = \beta_{k+1}p_{k+1} + \alpha_k p_k(x) + \beta_k p_{k-1}(x),$$

where

$$\begin{aligned} \alpha_k &= aq^k \frac{p+q-2pq^k(1+q)+pq^{2k}(p+q)}{(pq^{2k-1}-1)(pq^{2k+1}-1)}, \quad k \geq 0, \\ \beta_0^2 &= \frac{p}{p-1}, \\ \beta_k^2 &= a^2p\; q^{2k} \frac{(q^k-1)^2(pq^{k-1}-1)^2}{(pq^{2k-2}-1)(pq^{2k-1}-1)^2(pq^{2k}-1)}, \quad k \geq 1. \end{aligned} \tag{2.2}$$

Proof. The case $p > 1$ and $q > 1$ is directly connected to the little $1/q$-Jacobi polynomials. In order to prove the rest of the theorem moments are needed. They can be calculated as

$$\mu_n = \sum_{k=0}^{+\infty} \frac{1}{p^k} \frac{a^n}{q^{nk}} = \frac{p(aq)^n}{pq^n - 1},$$

where a simple summation of the geometric series is used, which is valid under condition $|pq^n| > 1$, $n \in \mathbb{N}_0$. This set of conditions is equivalent to conditions $|p| > 1$ and $|q| \geq 1$.

From the previous it can be concluded that formulas for the moments are valid, provided

$$|p| > 1, \quad |q| \geq 1 \quad \text{and} \quad q^n \neq 1 \quad (n \in \mathbb{N}).$$

Under an elementary transformation the moments can be expressed as

$$\frac{\mu_{i+j-2}}{pa^{i+j-2}q^{i-1}} = \frac{1}{pq^{i-1} - q^{1-j}},$$

which means that values of Hankel determinants (cf. [3]) $H_n = |\mu_{i+j-2}|_{i,j=1}^n$, can be calculated using Cauchy's double alternant determinant formula (see [11]), so that the values of Hankel and modified Hankel determinant[1] can be expressed in the form

$$H_n = p^{n(n+1)/2} a^{n(n-1)} q^{n(n^2-1)/3} \frac{\prod_{j=1}^{n-1} (1 - q^j)^{2(n-j)}}{\prod_{k=0}^{2n-2} (pq^k - 1)^{n-|n-1-k|}}$$

and

$$H'_n = aqH_n \frac{q^n - 1}{q - 1} \frac{pq^{n-1} - 1}{pq^{2n-1} - 1},$$

respectively.

It is known that under condition $H_n \neq 0$, orthogonal polynomials exist (see [3]).

From these values of Hankel determinant it is easy to conclude that polynomials orthogonal with respect to $L_a^{p,q}$ exist under condition given in (2.1). Using formulas given in [10], three-term recurrence coefficients can be calculated as

$$\alpha_n = \frac{H'_{n+1}}{H_{n+1}} - \frac{H'_n}{H_n}, \quad \beta_n^2 = \frac{H_{n+1}H_{n-1}}{H_n^2}.$$

According to these formulas the recursion coefficients can be calculated to be equal to expressions given in (2.2). $\square$

It is easy to check, using values of Hankel determinants, that for a given sequence of moments

$$\mu_n = \frac{p(aq)^n}{pq^n - 1}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

positive definite case appears under condition $pq^k > 1, \ k \in \mathbb{N}_0$.

[1] The modified Hankel determinant H'_n has the same columns as Hankel determinant except for the last column which is equal to the vector $(\mu_k)_{k=n}^{2n-1}$, (cf. [3])

Remark 1. For the sequence of moments given in (2.3), using values of Hankel determinants, one more positive definite case can be encountered. Under condition $0 < pq^k < 1,\ k \in \mathbb{N}_0$ it is easy to check that $H_n > 0$.

However, using again formula for summation of geometric series, it is easy to verify that essentially unique moment functional is given by (see [3])

$$L(f) = -\sum_{k=1}^{+\infty} p^k f(aq^k).$$

This moment functional is clearly the same as $L_a^{p,q}$ (it is enough to perform substitutions $p := 1/p$, $a := aq$ and $q := 1/q$ to have an equivalence to the multiplicative factor $-p$.

3. Zero Distribution and Gaussian Quadrature Rules

Following [2], we give the next definition:

Definition 3.1. For every two uniformly bounded sequences of complex numbers $\alpha_k,\ \beta_k,\ k \in \mathbb{N}_0$, we associate infinite (possibly) complex Jacobi matrix

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & 0 & \dots \\ \beta_1 & \alpha_1 & \beta_2 & \dots \\ 0 & \beta_2 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.1}$$

This Jacobi matrix can be interpreted to be linear operator acting on the Hilbert space ℓ^2 of all complex square-summable sequences with usual scalar product $\langle u, v\rangle = \sum_{i\in\mathbb{N}_0} u_i\overline{v_i}$. The value of related operator can be defined as a result of matrix multiplication of the infinite matrix given in (3.1) with an infinite vector representing an element from ℓ^2. We refer to Jacobi matrix when we mean to refer to associated linear operator and vice versa.

Definition 3.2. For the functional $L_a^{p,q}$, the symbol $J_a^{p,q}$ represents a complex Jacobi matrix and/or a linear operator acting on ℓ^2 related with it and constructed with sequences given in (2.2).

For bounded operator, which is true for our operator $J_a^{p,q}$ (see Theorem below), it is easy to conclude that domain is the whole ℓ^2, since rows and columns are in ℓ^2 for a banded $J_a^{p,q}$.

It is known that all zeros of orthogonal polynomials lie in the closure of the numerical range of the operator $J_a^{p,q}$ (for reference see [2]).

Definition 3.3. Numerical range of operator J is defined by

$$\Theta(J) = \{\langle Jx, x\rangle \mid x \in \ell^2,\ ||x|| = 1\}.$$

Its closure is denoted by $\Gamma(J) = \overline{\Theta(J)}$.

Theorem 3.1. *Under condition $|q| > 1$, the linear operator $J_a^{p,q}$ is compact even more it is of trace class. In the case when $|q| = 1$, with $q^n \neq 1,\ n \in \mathbb{N}$, and $|p| > 1$, the linear operator $J_a^{p,q}$ is bounded, but not compact. All zeros of related orthogonal polynomials lie in the set*

$$\Gamma(J_a^{p,q}) \subset \{z||z| \leq |\beta_1| + |\alpha_0|\},$$

when $|q| > 1$. In the second case, all zeros of related orthogonal polynomials lie in the set

$$\Gamma(J_a^{p,q}) \subset \left\{ z \,\middle|\, |z| \leq |a| \frac{|p|^2 + 6|p| + 1 + 4\sqrt{|p|}(|p| - 1)}{(|p| - 1)^2} \right\}.$$

Proof. To see the first part of the statement it is enough to note the asymptotic formulas

$$|\alpha_k| \approx \frac{\max(|p|, |q|)|a|}{|p||q|^k} \to 0, \quad |\beta_k| \approx \frac{|a|}{\sqrt{|p|}|q|^{k-1}} \to 0, \quad k \to +\infty.$$

From these formulas we can also conclude

$$\begin{aligned}\sum_{k=0}^{+\infty} |\alpha_k| + |\beta_{k+1}| &\approx \sum_{k=1}^{+\infty} \frac{\max(|p|, |q|)|a|}{|p||q|^{k-1}} + \sum_{k=1}^{+\infty} \frac{|a|}{\sqrt{|p|}|q|^{k-1}} \\ &= \frac{\max(|p|, |q|) + \sqrt{|p|}}{|p|} \frac{|a||q|}{|q| - 1} < +\infty.\end{aligned}$$

In the case $|q| = 1$, with conditions $q^n \neq 1,\ n \in \mathbb{N}$, the following bounds

$$|\alpha_k| \leq |a| \frac{|p|^2 + 6|p| + 1}{(|p| - 1)^2}, \quad |\beta_k|^2 \leq \frac{4|a|^2|p|}{(|p| - 1)^2}$$

hold, which means that operator $J_a^{p,q}$ is bounded.

To prove the second part we refer to [2] and note that the upper bound for the norm of Jacobi operator $J_a^{p,q}$ can be calculated as

$$||J_a^{p,q}|| \leq \sup_{n \in \mathbb{N}_0} (|\beta_n| + |\alpha_n| + |\beta_{n+1}|).$$

It is enough to see that, under condition $|q| > 1$, $|\alpha_k|$ and $|\beta_k|$ are decreasing in k, i.e., the bound for the norm is given by

$$||J_a^{p,q}|| \leq |\beta_1| + |\alpha_0|.$$

For $|q| = 1$, with $q^n \neq 1,\ n \in \mathbb{N}$, it is enough to use the bounds for $|\alpha_k|$ and $|\beta_k|$, given before and a bound for the norm. Thus, we get

$$||J_a^{p,q}|| \leq |a| \frac{|p|^2 + 6|p| + 1 + 4\sqrt{|p|}(|p| - 1)}{(|p| - 1)^2}.$$

Applying Cauchy-Schwartz-Buniakowski inequality to the condition with which the numerical range is defined, we get

$$|(Jx, x)| \leq ||Jx||||x|| \leq ||J||,$$

so $||x|| \leq ||J||$ for every $x \in \Theta(J)$. Since $\Gamma(J) = \overline{\Theta(J)}$, the previous is true for $x \in \Gamma(J)$. It is also a classical result that in the case J is compact and self-adjoint, the largest eigenvalue has modulus exactly equal to $||J||$ (cf. [8]). It means that the bound $||J||$ is sharp in the sense that it cannot be smaller if all (real) zeros are going to be included. It is shown in the next section that the bound cannot be improved on any part of the complex plane. $\square$

Theorem 3.2. *Under condition $|q| > 1$, we have*

$$\sigma(J_a^{p,q}) = \overline{\{a/q^k \,|k \in N_0\}}.$$

Proof. Note that the Weil function

$$f(z) = \langle (zI - J_a^{p,q})^{-1} e_0, e_0 \rangle,$$

can be written in the following form

$$f(z) = \sum_{k=0}^{+\infty} \mu_k z^{-k-1} = L_a^{p,q}\left(\frac{1}{z - x}\right) = \sum_{j=0}^{+\infty} \frac{1}{p^j} \frac{1}{z - a/q^j}.$$

The previous is valid under assumption $|z| > \sup |\sigma(J_a^{p,q})|$. Since the function f is analytic in the neighborhood of infinity, using theorem of analytic continuation, the functional series on the right hand side in the previous formula can be understood to be valid in the set $\mathbb{C}\backslash\overline{\{a/q^k \mid k \in N_0\}}$. Then it can be easily seen that f has simple poles at $a/q^k,\ k \in \mathbb{N}_0$. Using a theorem from [2] it can be concluded that $J_a^{p,q}$ has spectrum supported on the set $\overline{\{a/q^k,\ k \in \mathbb{N}_0\}}$ (for details under some general conditions see [14]). $\square$

Theorem 3.3. *For polynomials orthogonal with respect to $L_a^{p,q}$ under condition $|q| > 1$, the following formulas are true*

$$(3.2)\quad \lim_{n\to+\infty} p_n(a/q^k) = 0, \quad |P_n(a/q^k)| = o(|a|^n |q|^{-(n^2-n)/4}),\ n \to +\infty$$

where P_n is the monic orthogonal polynomial ($P_n = p_n||P_n||$), and also

$$\lim_{n\to+\infty} |P_n(z)|^{1/n} = |z|,$$

for any $z \in \mathbb{C}\backslash\sigma(J_a^{p,q})$.

Proof. Since $a/q^k,\ k \in \mathbb{N}_0$, are eigenvalues of $J_a^{p,q}$, the respective eigenvectors are $(p_n(a/q^k))_{n\in\mathbb{N}_0},\ k \in \mathbb{N}_0$. These vectors are elements of ℓ^2, hence, they are square-summable, which means $\lim\limits_{n\to+\infty} p_n(a/q^k) = 0$. Using an expression for the norm $\prod\limits_{k=0}^{n} \beta_k$ and bounds for β-coefficients the second part of the statement is proved.

Using Poincare theorem (see [9]), we know there are only two possibilities for the expression $|P_n(z)|^{1/n},\ z \neq 0$, to converge. Those limits are 0 and $|z|$. It can be proved that the convergence to 0 is impossible under assumption $z \in \mathbb{C}\backslash\sigma(J_a^{p,q})$ (see [14]). □

From (3.2), it can be easily seen that the polynomial p_n has a zero near the point a/q^k. However, it can be claimed even more. Namely, we can claim that under condition $|q| > 1$, there are no spurious zeros of the orthogonal polynomials.

Lemma 3.1. *Under condition $|q| > 1$ there are no spurious zeros of related orthogonal polynomials.*

Proof. Following [16] and [2] the number ζ is spurious zero for the sequence of orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}_0}$, if there is an infinite subset $N' \subset \mathbb{N}_0$, such that for every $n \in N'$ there is a zero of p_n, denoted by x_n, such that $\lim\limits_{n\to+\infty,\ n\in N'} x_n = \zeta$.

It is easy to prove that

$$\Gamma_{\text{ess}}(J_a^{p,q}) = \{0\},$$

where $\Gamma_{\text{ess}}(J_a^{p,q}) = \bigcap\limits_{m\in\mathbb{N}_0} \Gamma((J_a^{p,q})^{(m)})$ and operator $(J_a^{p,q})^{(m)}$ is created from the sequences $\alpha_k^{(m)} = \alpha_{k+m},\ \beta_k^{(m)} = \beta_{k+m}$, i.e., from the same sequences

α and β but without m elements from the beginning, which are neglected. Using bounds for the norm of the operator $J_a^{p,q}$, it can be easily concluded that

$$||(J_a^{p,q})^{(m)}|| \leq |\beta_{m+1}| + |\alpha_m|,$$

since $J_a^{p,q}$ is of a trace class $\beta_k \to 0$ and $\alpha_k \to 0$, i.e., $||(J_a^{p,q})^{(m)}|| \to 0$. The previous means $\operatorname{diam}(\Gamma((J_a^{p,q})^{(m)})) \to 0$ and we have

$$\Gamma_{\text{ess}}(J_a^{p,q}) = \{0\}.$$

From [2], it is known that for any compact set C, which intersection with $\Gamma_{\text{ess}}(J_a^{p,q})$ is empty, there is an $n(C)$ such that there are no zeros of orthogonal polynomials with degrees $n > n(C)$ in C.

Suppose there is a spurious zero for $\{p_n\}_{n\in\mathbb{N}_0}$, and that zero is ζ. It is enough to consider the compact set $C = \{z \mid |z-\zeta| \leq \varepsilon,\ \varepsilon > 0\}$ to conclude that starting with degree $n(C)+1$ there are no zeros of polynomials p_n in C. Hence, no ζ can be spurious zero of the sequence $\{p_n\}_{n\in\mathbb{N}_0}$. $\square$

In the case $|q| = 1,\ q^n \neq 1,\ n \in \mathbb{N}$, it can be easily concluded that q has a representation of the form $q = \exp(ib\pi)$, for some irrational b. In [6] the following lemma can be found.

Lemma 3.2. *Under condition $q = \exp(ib\pi)$, with some irrational b, the set $\{q^{-k} \mid k \in \mathbb{N}_0\}$ is dense on the unit circle.*

Theorem 3.4. *Under condition $|q| = 1,\ q^n \neq 1,\ n \in \mathbb{N}$, the following is true*

$$\{z \mid |z| = |a|\} \subset \sigma(J_a^{p,q}), \quad \operatorname{diam}(\sigma(J_a^{p,q})) = 2|a|.$$

Proof. It is easy to see that the Weyl function has a representation in the form

$$f(z) = \sum_{k=0}^{+\infty} \frac{1}{p^k} \frac{1}{z - a/q^k},$$

for all $|z| > ||J_a^{p,q}||$. Since, the series on the right hand side represents an analytic function for $|z| > |a|$, it is easy to conclude that the Weyl function has a unique analytic continuation on the set $\{z \mid |z| > |a|\}$ and that analytic continuation is the series on right hand side. However, since series has singularities in the points $a/q^k,\ k \in \mathbb{N}_0$, and these points are dense on the circle $|z| = |a|$, the Weyl function cannot be continued inside the circle $|z| = |a|$. The previous means that, since on the spectrum of $J_a^{p,q}$ the Weyl function has singularities, the circle $|z| = |a|$ is a part of the spectrum of $J_a^{p,q}$. Also, since the Weyl function is analytic on the set $|z| > |a|$, the diameter of $\sigma(J_a^{p,q})$, $\mathrm{diam}(\sigma(J_a^{p,q}))$, cannot be bigger then $2|a|$. However, it cannot be smaller neither since $a/q^k,\ k \in \mathbb{N}_0$, is dense on the set $|z| = |a|$. $\square$

Since we cannot claim simplicity of the zeros of orthogonal polynomials, in order to apply the Gaussian quadrature rule the multiplicity of the zeros has to be included, i.e., our quadrature rule has the following form

$$G_n(f) = \sum_{k=0}^{M} \sum_{i=0}^{M_k} w_{i,k}^n f^{(i)}(x_k^n), \tag{3.3}$$

where $w_{i,k}^n$ are known as the weights and x_k^n are known as the nodes of the quadrature rule. In order to have the maximal degree of exactness $(2n-1)$, the nodes have to be the zeros of the polynomial p_n. Note that in the case of simple nodes this quadrature rule becomes simply the Gaussian quadrature rule

$$G_n(f) = \sum_{k=0}^{n} w_k^n f(x_k^n).$$

The following error bound, for the quadrature rule given by (3.3), can be done in the following way (see [13]):

Theorem 3.5. *Let C be a rectifiable curve such that all zeros of polynomial p_n lie in the set $\mathrm{int}(C)$. For the quadrature rule (3.3) the following error bound*

$$\left| \int f d\mu - G_n(f) \right| \le \frac{\ell(C)||f||_C \int |P_n|^2 d|\mu|}{2\pi d |P_n(z)|^2_{z\in C}}$$

holds, where $d = \mathrm{dist}(C, \mathrm{supp}(\mu))$ denotes the distance between the support of measure μ and the curve C, $\ell(C)$ is the length of C, $||f||_C$ denotes usual sup norm on the compact set C, P_n is the monic orthogonal polynomial with respect to the measure of orthogonality μ.

Under condition $|q| > 1$, the curve C appearing in the previous theorem can be constructed in the following way. First choose some $0 < r$ and construct the circle K around origin with the radius r. If $r < |a|$ there exist finite number of points $aq^{-\nu}$ with the property $|aq^{-\nu}| > r$, it is assumed there are $k+1$ such points, if $k = -1$ contour C equals K, if $k > -1$ we proceed as follows. Choose some $R > 0$ and construct the circles C_ν, with the radius R, around points $aq^{-\nu}$, $\nu \le k$. By l_ν, $\nu \le k$, (rectifiable) arcs connecting each circle C_ν with the circle K are denoted. By l_ν^+, $\nu \le k$, arc with direction from circle K to circle C_ν is denoted and, by l_ν^-, $\nu \le k$, same arc with opposite direction is denoted. The curve C, with a positive orientation, can be constructed in the following way

$$C = \begin{cases} K \bigcup \left(\bigcup\limits_{\nu=0}^{k} C_\nu \right) \bigcup \left(\bigcup\limits_{\nu=0}^{k} (l_\nu^+ \bigcup l_\nu^-) \right), & k > -1, \\ K, & k = -1. \end{cases} \tag{3.4}$$

Then we have the following theorem for the rate of convergence (see [14]):

Theorem 3.6. *Suppose a rectifiable curve C is given as in* (3.4) *and the function f is analytic in the domain* $\mathrm{int}(C)$ *and continuous on C. Then for every ρ, $\rho \ne aq^{-k}$, $k \in \mathbb{N}_0$, there exist $n_0(\rho)$, such that for $n > n_0$*

$$\left| \int f d\mu - G_n(f) \right|^{1/2n} \approx a_n \frac{2\rho}{r},$$

where $a_n > 0$, $a_n \to 1$, $n \to +\infty$.

In the case $q = \exp(ib\pi)$, for some irrational b, a strong asymptotic error bound, as it is given in the previous theorem can hardly be given. Since sequences of the three-term recurrence coefficients are not convergent series. However, we can still claim the convergence of the related quadrature rules. Suppose that a rectifiable curve C is such that it winds around $\Gamma(J_a^{p,q})$ once, i.e., such that $\mathrm{int}(C) \supset \Gamma(J_a^{p,q})$.

Theorem 3.7. *Suppose rectifiable curve C is such that* $\mathrm{int}(C) \supset \Gamma(J_a^{p,q})$ *and the function f is analytic in* $\mathrm{int}(C)$ *and continuous on curve C. Then there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ there exists a sequence $a_n > 0$ with property $\lim\limits_{n\to+\infty} a_n = 1$ and*

$$\left| \int f d\mu - G_n(f) \right|^{1/2n} \le a_n \frac{2||J_a^{p,q}||}{\min\limits_{z\in C,\ \phi\in[0,2\pi)} \left| z - ||J_a^{p,q}||e^{i\phi} \right|}.$$

Proof. Since it is known that all zeros of orthogonal polynomials p_n, $n \in \mathbb{N}_0$ are in a circle with the radius $||J_a^{p,q}||$, we can give the following bounds

$$|P_n(z)| = \prod_{k=1}^{n} |z - x_k^n| \le \prod_{k=1}^{n} 2||J_a^{p,q}|| = (2||J_a^{p,q}||)^n, \quad |z| \le ||J_a^{p,q}||,$$

and

$$\int |P_n(z)|^2 d|\mu| \leq \frac{(2||J_a^{p,q}||)^{2n}|p|}{|p|-1}.$$

For $z \in C$ we have

$$\max_{z\in C} \frac{1}{|P_n(z)|} \leq \frac{1}{\min\limits_{z\in C,\ x_k^n\in\Gamma(J_a^{p,q})} \prod\limits_{k=1}^{n} |z-x_k^n|} \leq \frac{1}{\min\limits_{z\in C,\ \phi\in[0,\pi)} |z-||J_a^{p,q}||e^{i\phi}\,|^n}.$$

Using Theorem 3.5 we can choose

$$a_n = \left(\frac{\ell(C)||f||_C|p|}{2\pi d(|p|-1)}\right)^{1/2n},$$

the property $\lim\limits_{n\to+\infty} a_n = 1$ is obvious, and the statement of this theorem is proved. □

From this theorem the convergence of the quadrature rules can be claimed choosing C to be circle with the center at the origin and with radius greater then $3||J_a^{p,q}||$. Thus, we have

Theorem 3.8. *If a rectifiable curve C in the previous theorem is chosen to be $C = \{z \,|\, |z| = 3||J_a^{p,q}|| + \varepsilon, \varepsilon > 0\}$, then for some function f which is analytic in the* int(C) *and continuous on C, the quadrature rule of the form* (3.3) *is converging with the geometric speed* (*at least*).

Proof. Using Theorem 3.7, for every n sufficiently large, the quadrature error is bounded, i.e.,

$$\left|\int f d\mu - G_n(f)\right|^{1/2n} \leq a_n \frac{2||J_a^{p,q}||}{2||J_a^{p,q}|| + \varepsilon} < 1,$$

where $\varepsilon > 0$ and $a_n \to 1$. The convergence speed is directly verified. □

4. Numerical Examples

At first we consider a positive definite case. We take $p = q = 2,\ a = 1$, in (1.1), and

$$f(x) = \frac{1}{x - 3^{-20}}.$$

Quadrature rules G_n can be constructed using QR-algorithm (cf. [4], [5]), especially it is stable if modification given in [1] is used. The Gaussian

n	G_n	r. err.
10	6.605719151641912	0.54
20	11.60759673863332	0.20
30	18.19943971672188	0.26
40	14.46129284956265	m.p.

Table 4.1: Gaussian approximations G_n and relative errors for the functional $L_1^{2,2}(1/(. - 3^{-20}))$

approximations $G_n(f)$ and the corresponding relative errors (r. err.) are given in Table 4.1 (m.p. stands for the machine precision $\approx 10^{-16}$ in double precision).

Choosing $a = 1,\ p = q = 2i,\ i = \sqrt{-1}$, when the quadrature rule is applied for integration of the same function f, we get the same behavior. Table 4.2 displays the corresponding results for this case.

n	G_n	r.err.
10	$9.577889690630307 + 4.0332355954016054i$	0.66
20	$19.57775686236893 + 9.0341888460741501i$	0.30
30	$29.71613796045620 + 14.066446517362046i$	0.07
40	$27.52570186363645 + 13.689702133806741i$	m.p.

Table 4.2: Gaussian approximations G_n and relative errors for the functional $L_1^{2i,2i}(1/(. - 3^{-20}))$

The behavior of relative errors can be fully understood using Theorem 3.6. While zeros of orthogonal polynomials have modulus grater then 3^{-20}, the convergence of our quadrature rules is bad since ρ/r is greater than 1. In this case it can be rather said that quadrature rules are diverging. When some zeros drop below 3^{-20}, ρ can also be improved and hence ρ/r is smaller then 1, having as a consequence a very fast convergence. For example, the smallest modulus of zeros for $n = 40$ is of order 10^{-13} and for $n = 30$ is of order 10^{-10}. Note that $3^{-20} \approx 3 \times 10^{-10}$.

For $|q| = 1,\ q^n \neq 1,\ n \in \mathbb{N},\ |p| > 1$, for many examples we have run, we were unable to encounter a case when zeros of orthogonal polynomials have higher multiplicity than 1. A construction of zeros can be done using QR-algorithm. For a stable construction of weights in Gaussian quadrature rule we need a modified algorithm given in [12].

It turns out again that all zeros are attracted to the supporting set of the functional $L_a^{p,q}$. Some finite number of zeros that have modulus bigger then a are not spurious, it turns out they are falling down to the boundary set of the unit circle.

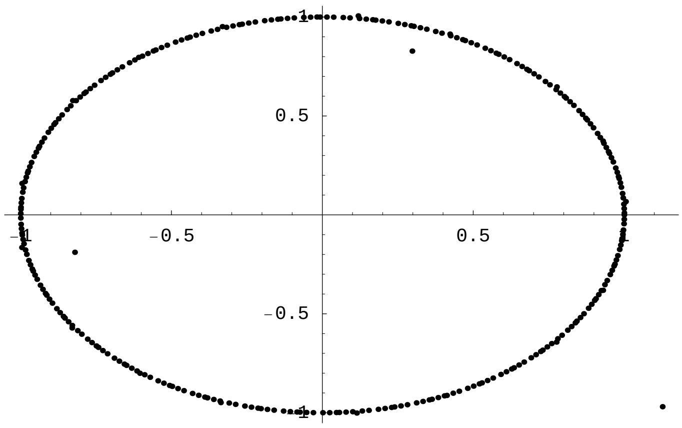

FIG. 4.1: Distribution of zeros for $n = 300,\ p = -1.1,\ q = \exp(i\sqrt{2}\pi),\ a = 1$

For example, we give results for the convergence of the sum

$$\sum_{k=0}^{+\infty} \left(-\frac{10}{11}\right)^k \exp(\exp(-ik\sqrt{2}\pi)).$$

The machine precision is achieved with only 10 nodes in the corresponding Gaussian quadrature rule. Distribution of zeros for $n = 300,\ p = -1.1,\ q = \exp(i\sqrt{2}\pi),\ a = 1$ is given in Figure 4.1.

Completely different situation occurs when instead of $\sqrt{2}$ (algebraic number) some transcendental number is introduces, for example π. In Figure 4.2 this case is presented with $n = 300,\ p = -1.1,\ q = \exp(i\pi^2),\ a = 1$.

However, the Gaussian quadrature rule is equally successful as with an algebraic number. For example, the sum of the series

$$\sum_{k=0}^{+\infty} \left(-\frac{10}{11}\right)^k \exp(\exp(-ik\pi^2))$$

can be achieved with the machine precision with only 10 nodes in the Gaussian quadrature rule.

Especially an interesting case is when $|p|$ is large. In this case, as it can be seen zeros are contained almost in the closure of the unit circle according

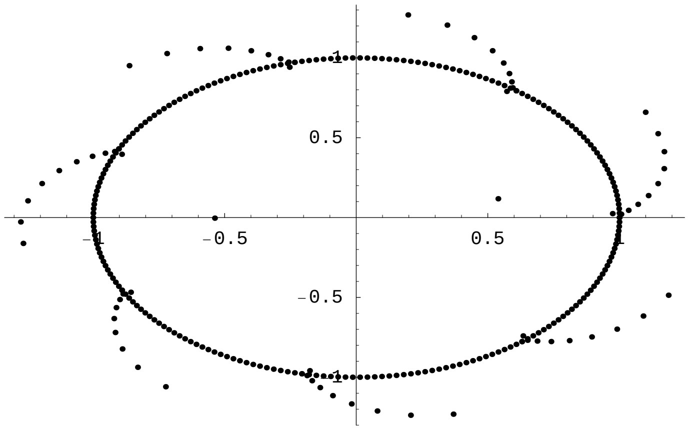

FIG. 4.2: Distribution of zeros for $n = 300,\ p = -1.1,\ q = \exp(i\pi^2),\ a = 1$

to the bound from Theorem 3.1. The supporting set of the functional $L_a^{p,q}$ is a subset of the unit circle. This example shows that for this case the bound $||J||$ for the set of zeros is sharp and that in a general case it cannot be improved to much. In Figure 4.3 is presented the case with $n = 300$, $p = 1000$, $q = \exp(i\pi^2)$, $a = 1$. As we can see the all zeros are attracted to the supporting set of the functional $L_1^{1000,\exp(i\pi^2)}$.

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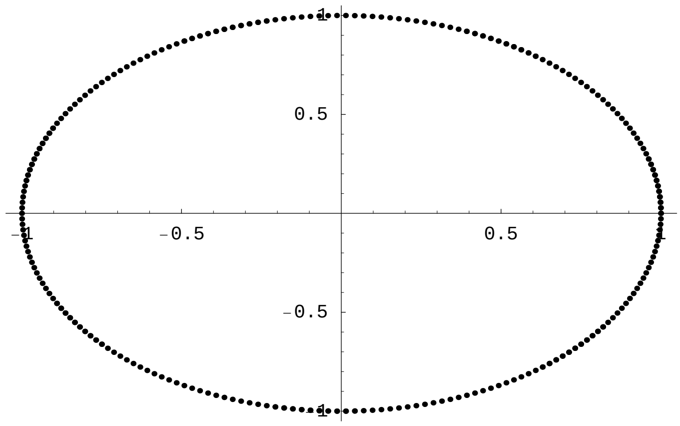

FIG. 4.3: Distribution of zeros for $n = 300,\ p = 1000,\ q = \exp(i\pi^2),\ a = 1$

REFERENCES

1. M. AHUES and F. TISSEUR: *A New Deflation Criterion for the QR Algorithm.* LAPACK Working Note 122, March 1997 (Technical Report).
2. B. BECKERMANN: *Complex Jacobi matrices*. J. Comput. Appl. Math. **127** (2001), 17–65.
3. T. S. CHIHARA: *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York, 1962.
4. W. GAUTSCHI: *Algorithm* 726*: ORTHPOL – A package of routines for generating orthogonal polynomials and Gauss-type quadrature rules*. ACM Trans. Math. Software **10** (1994), 21–62.
5. G.H. GOLUB and J.H. WELSCH: *Calculation of Gauss quadrature rule*. Math. Comput. **23** (1986), 221–230.
6. G. H. HARDY and E. M. WRIGHT: *An Introduction to the Theory of Numbers.* Fifth edition. The Clarendon Press, Oxford University Press, New York, 1979.
7. R. KOEKOEK and R. P. SWARTTOUW: *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue*. Report 98–17, TU Delft, 1998.
8. P. D. LAX: *Functional Analysis*. Wiley-Interscience [John Wiley & Sons], New York, 2002.
9. A. MATE and P. NEVAI: *A generalization of poincare theorem for recurrence equations*. J. Approx. Theory **63** (1990), 92–97.
10. G. V. MILOVANOVIĆ: *Complex orthogonality on the semicircle with respect to Gegenbauer weight: Theory and applications.* In: Topics in Mathematical

Analysis (Th.M. Rassias, ed.), pp. 695–722, Ser. Pure Math., 11, World Sci. Publishing, Teaneck, NJ, 1989.

11. D. S. Mitrinović and J. D. Kečkić: *The Cauchy Method of Residues. Theory and Applications*. Mathematics and its Applications (East European Series) **9**, D. Reidel Publishing Co., Dordrecht, 1984.
12. G .V. Milovanović and A. S. Cvetković: *Note on a construction of weights in Gauss-type quadrature rule*. Facta Univer. Ser. Math. Inform. **15** (2000), 69–83.
13. G .V. Milovanović and A. S. Cvetković: *Complex Jacobi matrices and quadrature rules*. (to appear in FILOMAT).
14. G .V. Milovanović and A. S. Cvetković: *Convergence of Gaussian Quadrature Rules for Approximation of Certain Series*. (in manuscript).
15. L. Pasquini: *Accurate computation of the zeroes of the generalized Bessel polynomials*. Numer. Math. **86** (2000), 507–538.
16. H. Stahl: *Spurious poles in Pade approximation*. J. Comput. Appl. Math. **99** (1998), 511–527.

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