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MULTIPLE ORTHOGONAL POLYNOMIALS ON THE SEMICIRCLE*

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Abstract. In this paper multiple orthogonal polynomials on the semicircle, investigated by Milovanović and Stanić in [Math. Balkanica (N. S.) **18** (2004), 373–387] (complex polynomials orthogonal with respect to the complex-valued inner products $[f,g]_m = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_m(e^{i\theta})\,d\theta$, for $m = 1,2,\dots,r$) are considered. These polynomials satisfy a linear recurrence relation of order $r+1$. Under suitable assumption on the weight functions w_m, $m = 1,2,\dots,r$, we express multiple orthogonal polynomials on the semicircle in terms of the type II multiple orthogonal (real) polynomials with respect to the weight function $w_m(x)$, $m = 1,2,\dots,r$. Specially, we consider the case $r = 2$ and express coefficients of corresponding recurrence relations in terms of coefficients of recurrence relation for the type II multiple orthogonal (real) polynomials. In particular, we obtain these type of polynomials associated with Jacobi weight functions.

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy r ($\in \mathbb{N}$) orthogonality conditions (see [1], [2], [12]–[14], [15], [16]).

Let $r \geq 1$ be an integer and let $w_1, w_2, \dots, w_r$ be r weight functions on the real line so that the support of each w_i is a subset of an interval E_i. Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be a vector of r nonnegative integers, which is called a *multi-index* with length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

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Type II multiple orthogonal polynomial is monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ such that it satisfies the following orthogonality conditions:

$$\begin{cases} \displaystyle\int_{E_1} P_{\vec{n}}(x)\,x^k w_1(x)\,dx & = & 0, & k=0,1,\ldots,n_1-1, \\ \displaystyle\int_{E_2} P_{\vec{n}}(x)\,x^k w_2(x)\,dx & = & 0, & k=0,1,\ldots,n_2-1, \\ & \vdots & & \\ \displaystyle\int_{E_r} P_{\vec{n}}(x)\,x^k w_r(x)\,dx & = & 0, & k=0,1,\ldots,n_r-1. \end{cases}$$

If the polynomial $P_{\vec{n}}(x)$ is unique, then we say that $\vec{n}$ is *normal index* and if all indices are normal then we have a *complete system.*

For $r=1$ we have the ordinary orthogonal polynomials.

For each of the weight functions w_k, $k=1,2,\ldots,r$,

$$(f,g)_k = \int_{E_k} f(x)g(x)w_k(x)dx \tag{1.1}$$

denotes the corresponding inner product of f and g.

Type II multiple orthogonal polynomials with nearly diagonal multi-index always satisfy a recurrence relation of order $r+1$. Let $n\in\mathbb{N}$ and write it as $n=kr+j$, with $0\le j<r$. The *nearly diagonal multi-index* $\vec{s}(n)$ corresponding to n is given by

$$\vec{s}(n) = (\underbrace{k+1,k+1,\ldots,k+1}_{j\ \text{times}},\underbrace{k,k,\ldots,k}_{r-j\ \text{times}}).$$

Denote the corresponding type II multiple orthogonal polynomials by $P_n(x)=P_{\vec{s}(n)}(x)$.

The following recurrence relation

$$xP_k(x) = P_{k+1}(x) + \sum_{i=0}^{r} a_{k,r-i}P_{k-i}(x)\,, \quad k\ge 0, \tag{1.2}$$

holds with initial conditions $P_0(x)=1$ and $P_i(x)=0$ for $i=-1,-2,\ldots,-r$ (see [15]).

For some classical weight functions (Jacobi, Laguerre, Hermite) one can find explicit formulas for the recurrence coefficients (see [15], [16],[3]).

In [12] an effective numerical method for construction of the type II multiple orthogonal polynomials has been presented. The recurrence coefficients have been computed using the discretized Stieltjes-Gautschi procedure [5].[1] At first, we express the recurrence coefficients in terms of the inner products (1.1), and then we use the corresponding Gaussian formulas to discretize these inner products.

In this paper we repeat some basic results on polynomials orthogonal on the semicircle and multiple orthogonal polynomials on the semicircle. Multiple orthogonal polynomials on the semicircle are considered in Section 2.. We express them in terms of the type II multiple orthogonal (real) polynomials and obtain a linear recurrence relation of order $r+1$. Specially, we consider case $r=2$ and give formulas for the coefficients appearing in the representation of multiple orthogonal polynomials on the semicircle over the type II (real) multiple orthogonal polynomials, and using them we express the recurrence coefficients for polynomials orthogonal on the semicircle. Finally, these coefficients for the multiple orthogonal polynomials on the semicircle associated to the two Jacobi weights are analyzed.

2. Multiple Orthogonal Polynomials on the Semicircle

Multiple orthogonal polynomials on the semicircle are a generalization of orthogonal polynomials on the semicircle in the sense that they satisfy r ($\in \mathbb{N}$) orthogonality conditions (see [13]). Polynomials orthogonal on the semicircle have been introduced by Gautschi and Milovanović in [7].

Let w be a weight function which is positive and integrable on the open interval $(-1,1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc

$$D_+ = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\ z > 0\}.$$

Consider the following two inner products,

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx, \tag{2.1}$$

$$[f,g] = \int_{\Gamma} f(z)g(z)w(z)(iz)^{-1}\,dz = \int_{0}^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta, \tag{2.2}$$

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[1] A similar procedure was used in a numerical construction of orthogonal polynomials on the radial rays in the complex plane (see [8]).

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist, possibly as appropriately defined improper integrals.

The inner product (2.1) is positive definite and therefore generates a unique set of real orthogonal polynomials $\{p_k\}$ (p_k is monic polynomial of degree k). This inner product (2.2) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

A system of complex polynomials $\{\pi_k\}$ (π_k is monic of degree k) is called *orthogonal on the semicircle* if $[\pi_k, \pi_\ell] = 0$ for $k \neq \ell$ and $[\pi_k, \pi_\ell] \neq 0$ for $k = \ell$, $k, \ell = 0, 1, 2, \ldots$.

Gautschi, Landau and Milovanović in [6] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$\text{Re}\,[1,1] = \text{Re}\int_0^\pi w(e^{i\theta})\,d\theta \neq 0. \tag{2.3}$$

Let C_ε, $\varepsilon > 0$, denotes the boundary of D_+ with small circular parts of radius ε and centers at ± 1 spared out. Let $c_{\varepsilon,\pm 1}$ are the circular parts of C_ε with centers at ± 1 and radii ε. We assume that w is such that

$$\lim_{\varepsilon \downarrow 0} \int_{c_{\varepsilon,\pm 1}} g(z)w(z)\,dz = 0, \quad \text{for all } \ g \in \mathcal{P}. \tag{2.4}$$

It is easy to prove that the following equations hold

$$0 = \int_\Gamma g(z)w(z)\,dz + \int_{-1}^1 g(x)w(x)\,dx, \quad g \in \mathcal{P}. \tag{2.5}$$

It is well known that the real (monic) polynomials $\{p_k(z)\}$, orthogonal with respect to the inner product (2.1), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^1 \frac{p_k(z) - p_k(x)}{z - x} w(x)\,dx, \quad k = 0, 1, 2, \ldots,$$

satisfy a three-term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1}, \quad k = 0, 1, 2, \ldots,$$

whit initial conditions $y_{-1} = 0$, $y_0 = 1$ for $\{p_k\}$, and $y_{-1} = -1$, $y_0 = 0$ for $\{q_k\}$.

Definition 3.1. For a positive integer r, a set $W = \{w_1, \ldots, w_r\}$ is *admissible set of weight functions* if for the set W there exist a unique system of the (real) type II multiple orthogonal polynomials and for all w_j, $j = 1, \dots, r$, there exist a unique system of (monic, complex) orthogonal polynomials relative to the inner product (2.2).

Let $r \geq 1$ be an integer and let $W = \{w_1, w_2, \dots, w_r\}$ be admissible set of weight functions. Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be the multi-index with length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$. Multiple orthogonal polynomial on the semicircle is monic polynomial $\Pi_{\vec{n}}(z)$ of degree $|\vec{n}|$ such that it satisfies the following orthogonality conditions:

$$\left\{\begin{array}{rcll} \displaystyle\int_\Gamma \Pi_{\vec{n}}\,(z)\, z^k\, w_1(z)(iz)^{-1}\,dz & = & 0, & k = 0, 1, \dots, n_1 - 1, \\ \displaystyle\int_\Gamma \Pi_{\vec{n}}\,(z)\, z^k\, w_2(z)(iz)^{-1}\,dz & = & 0, & k = 0, 1, \dots, n_2 - 1, \\ & \vdots & & \\ \displaystyle\int_\Gamma \Pi_{\vec{n}}\,(z)\, z^k\, w_r(z)(iz)^{-1}\,dz & = & 0, & k = 0, 1, \dots, n_r - 1. \end{array}\right. \tag{2.6}$$

For $r = 1$ we have the ordinary orthogonal polynomials on the semicircle.

Let denote for $m = 1, 2, \dots, r$

$$[f, g]_m = \int_\Gamma f(z)g(z)w_m(z)(iz)^{-1}\,dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_m(e^{i\theta})\,d\theta \tag{2.7}$$

corresponding complex inner products.

For any polynomial g the following equations hold

$$0 = \int_\Gamma g(z)w_m(z)\,dz + \int_{-1}^1 g(x)w_m(x)\,dx \tag{2.8}$$

and

$$\int_\Gamma \frac{g(z)w_m(z)}{iz}\,dz = \pi g(0)w_m(0) + i \,⨍_{-1}^1 \frac{g(x)w_m(x)}{x}\,dx \tag{2.9}$$

for $m = 1, 2, \dots, r$.

We consider only the nearly diagonal multi-indices.

The corresponding type II multiple orthogonal polynomials (real) $\{P_n\}$ satisfy recurrence relation (1.2).

It is easy to see that for $m = 1, 2, \dots, r$ associated polynomials of the second kind

$$Q_n^{(m)}(z) = \int_{-1}^1 \frac{P_n(z) - P_n(x)}{z - x}\,w_m(x)\,dx, \quad n = 0, 1, \dots, \tag{2.10}$$

satisfy the same recurrence relation (but with different initial conditions).

The multiple orthogonal polynomials on the semicircle satisfy the following recurrence relation of order $r+1$:

$$z\Pi_k(z) = \Pi_{k+1}(z) + \sum_{i=0}^{r} \alpha_{k,r-i}\Pi_{k-i}(z)\,, \quad k \ge 1,$$

with initial conditions $\Pi_0(z) = 1$, and $\Pi_{-1}(z) = \Pi_{-2}(z) = \cdots = \Pi_{-r}(z) = 0$ (see [13]).

Let denote moments for the inner products given in (2.7) with $\mu_k^{(m)}$, $m = 1, 2, \ldots, r$, $k \in \mathbb{N}_0$, i.e.,

$$\mu_k^{(m)} = [z^k, 1]_m = \int_\Gamma z^k w_m(z)(iz)^{-1}\, dz, \quad m = 1, 2, \ldots, r, \ \ k \in \mathbb{N}_0.$$

For zero moments we have

$$\mu_0^{(m)} = \int_\Gamma \frac{w_m(z)}{iz}\, dz = \pi w_m(0) + i \fint_{-1}^{1} \frac{w_m(x)}{x}\, dx, \quad m = 1, 2, \ldots, r\,.$$

Denote also

$$(2.11) \quad D_n = \begin{bmatrix} Q_{n-1}^{(1)}(0) - i\mu_0^{(1)} P_{n-1}(0) & \cdots & Q_{n-r}^{(1)}(0) - i\mu_0^{(1)} P_{n-r}(0) \\ Q_{n-1}^{(2)}(0) - i\mu_0^{(2)} P_{n-1}(0) & \cdots & Q_{n-r}^{(2)}(0) - i\mu_0^{(2)} P_{n-r}(0) \\ \vdots & & \vdots \\ Q_{n-1}^{(r)}(0) - i\mu_0^{(r)} P_{n-1}(0) & \cdots & Q_{n-r}^{(r)}(0) - i\mu_0^{(r)} P_{n-r}(0) \end{bmatrix}.$$

Using equations (2.8), (2.9) for appropriately chosen polynomials g and orthogonality conditions (2.6), one can prove existence and uniqueness of multiple orthogonal polynomials on the semicircle with additional conditions that all matrices D_n are regular.

Theorem 2.1. *Let r be positive integer and $W = \{w_1, \ldots, w_r\}$ be admissible set of weight functions. Assume in additional that all matrices D_n in* (2.11) *are regular. Denoting by $\{P_k\}$ the (real) type II multiple orthogonal polynomials, relative to the set W, we have the following representation*

$$(2.12) \quad \Pi_k(z) = P_k(z) + \theta_{k,1}P_{k-1}(z) + \theta_{k,2}P_{k-2}(z) + \cdots + \theta_{k,r}P_{k-r}(z).$$

The coefficients $\theta_{k,j}$, $j = 1, 2, \ldots, r$, are solution of the following system of linear equations

$$(2.13) \quad \sum_{j=1}^{r} \theta_{k,j} \left(Q_{k-j}^{(m)}(0) - i\mu_0^{(m)} P_{k-j}(0) \right) = i\mu_0^{(m)} P_k(0) - Q_k^{(m)}(0),$$

$m = 1, 2, \ldots, r.$

Proof. Assume first that the orthogonal polynomials $\{\Pi_k\}$ exist. Putting

$$g(z) = \frac{1}{i}\Pi_k(z)z^{\ell_m - 1}, \quad 1 \leqslant \ell_m < k_m$$

(for $k \in \mathbb{N}$, $(k_1, \ldots, k_r)$ is the corresponding nearly diagonal multi–index) in (2.8) for $m = 1, 2, \ldots, r$, we find

$$\begin{aligned} 0 &= \int_\Gamma \Pi_k(z) z^{\ell_m} (iz)^{-1} w_m(z)\, dz - i\int_{-1}^{1} \Pi_k(x) x^{\ell_m - 1} w(x)\, dx \\ &= \Big[\Pi_k, z^{\ell_m}\Big]_m - i\Big(\Pi_k, x^{\ell_m - 1}\Big)_m , \end{aligned}$$

so we have the representation (2.12).

To determine the constants $\theta_{k,j}$, $j = 1, 2, \ldots, r$, we put

$$\begin{aligned} g(z) &= \frac{\Pi_k(z) - \Pi_k(0)}{iz} \\ &= \frac{1}{i}\Bigg[\frac{P_k(z) - P_k(0)}{z} + \theta_{k,1}\frac{P_{k-1}(z) - P_{k-1}(0)}{z} \\ &\qquad + \cdots + \theta_{k,r}\frac{P_{k-r}(z) - P_{k-r}(0)}{z}\Bigg] \end{aligned}$$

in (2.8) for $m = 1, 2, \ldots, r$, and use the first expression for g to evaluate the first integral, and the second one to evaluate the second integral in (2.8). This gives the system of equations (2.13) for $k \geq r$. From (2.8) and (2.9), putting for the polynomial g $\Pi_0, \Pi_1, \ldots, \Pi_{r-1}$ successively, using (2.12), (1.2) and (2.10), we obtain $\theta_{k,j}$ for $k < r$, i.e., a system of equations of the same form as in the case $k \geq r$. The system of equations (2.13) has a regular matrix, so it has the unique solution.

Conversely, defining Π_k with (2.12), where $\theta_{k,j}$, $j = 1, \ldots, r$ is a solution of the system of equations (2.13), it is easy to see that

$$[\Pi_k, z^{\ell_m}]_m = 0, \quad 0 \leq \ell_m < k_m,$$

for $m = 1, \ldots, r$. □

2.1. Case $r = 2$

Let $W = \{w_1, w_2\}$ be admissible set of weight functions.

The type II (real) multiple orthogonal polynomials satisfy the following recurrence relations

$$P_{k+1}(x) = (x - b_k)P_k(x) - c_k P_{k-1}(x) - d_k P_{k-2}(x), \quad k \geq 0, \tag{2.14}$$

with initial conditions $P_0(x) = 1$, $P_{-1}(x) = P_{-2} = 0$.

Multiple orthogonal polynomials on the semicircle satisfy the following recurrence relations

$$\Pi_{k+1}(z) = (z - \beta_k)\Pi_k(z) - \gamma_k \Pi_{k-1}(z) - \delta_k \Pi_{k-2}(z), \quad k \geq 0, \tag{2.15}$$

with initial conditions $\Pi_0(z) = 1$, $\Pi_{-1}(z) = \Pi_{-2}(z) = 0$.

Using theorem 2.1 we have for $k \geq 2$ the following equation

$$\Pi_k(z) = P_k(z) + \theta_{k,1} P_{k-1}(z) + \theta_{k,2} P_{k-2}(z), \tag{2.16}$$

where $\theta_{k,1}$ and $\theta_{k,2}$ are solution of the following system of linear equations

$$\begin{aligned} \theta_{k,1}\left(Q_{k-1}^{(1)}(0) - i\mu_0^{(1)} P_{k-1}(0)\right) + \theta_{k,2}\left(Q_{k-2}^{(1)}(0) - i\mu_0^{(1)} P_{k-2}(0)\right) \\ = i\mu_0^{(1)} P_k(0) - Q_k^{(1)}(0), \\ \theta_{k,1}\left(Q_{k-1}^{(2)}(0) - i\mu_0^{(2)} P_{k-1}(0)\right) + \theta_{k,2}\left(Q_{k-2}^{(2)}(0) - i\mu_0^{(2)} P_{k-2}(0)\right) \\ = i\mu_0^{(2)} P_k(0) - Q_k^{(2)}(0). \end{aligned}$$

At first, we will find relations between $\theta_{k,1}$, $\theta_{k,2}$ and recurrence coefficients b_k, c_k, d_k, and then we will express the recurrence coefficients β_k, γ_k and δ_k as functions of b_k, c_k, d_k, $\theta_{k,1}$ and $\theta_{k,2}$.

If we denote

$$R_k^{(j)} = Q_k^{(j)}(0) - i\mu_0^{(j)} P_k(0), \quad j = 1, 2, \tag{2.17}$$

the previous system of equations can be written in form

$$\theta_{k,1} R_{k-1}^{(j)} + \theta_{k,2} R_{k-2}^{(j)} = -R_k^{(j)}, \quad j = 1, 2,$$

and the solution is

$$\theta_{k,1} = \frac{R_{k-2}^{(1)} R_k^{(2)} - R_k^{(1)} R_{k-2}^{(2)}}{R_{k-1}^{(1)} R_{k-2}^{(2)} - R_{k-2}^{(1)} R_{k-1}^{(2)}}, \quad \theta_{k,2} = \frac{R_k^{(1)} R_{k-1}^{(2)} - R_{k-1}^{(1)} R_k^{(2)}}{R_{k-1}^{(1)} R_{k-2}^{(2)} - R_{k-2}^{(1)} R_{k-1}^{(2)}}. \tag{2.18}$$

According to (2.14), for $k \geq 3$ we have

$$P_k(0) = -b_{k-1} P_{k-1}(0) - c_{k-1} P_{k-2}(0) - d_{k-1} P_{k-3}(0),$$

and

$$Q_k^{(j)}(0) = -b_{k-1} Q_{k-1}^{(j)}(0) - c_{k-1} Q_{k-2}^{(j)}(0) - d_{k-1} Q_{k-3}^{(j)}(0), \quad j = 1, 2.$$

Then

$$\begin{aligned}
R_k^{(j)} &= -b_{k-1}Q_{k-1}^{(j)}(0) - c_{k-1}Q_{k-2}^{(j)}(0) - d_{k-1}Q_{k-3}^{(j)}(0) \\
&\quad -i\mu_0^{(j)}\left(-b_{k-1}P_{k-1}(0) - c_{k-1}P_{k-2}(0) - d_{k-1}P_{k-3}(0)\right) \\
&= -b_{k-1}R_{k-1}^{(j)} - c_{k-1}R_{k-2}^{(j)} - d_{k-1}R_{k-3}^{(j)}.
\end{aligned}$$

Now we apply some elementary transformations to obtain

$$\begin{aligned}
\theta_{k,1} &= \frac{\left(-b_{k-1}R_{k-1}^{(2)} - c_{k-1}R_{k-2}^{(2)} - d_{k-1}R_{k-3}^{(2)}\right)R_{k-2}^{(1)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}} \\
&\qquad - \frac{\left(-b_{k-1}R_{k-1}^{(1)} - c_{k-1}R_{k-2}^{(1)} - d_{k-1}R_{k-3}^{(1)}\right)R_{k-2}^{(2)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}} \\
&= b_{k-1} + d_{k-1}\frac{R_{k-3}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-3}^{(2)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}},
\end{aligned}$$

i.e.,

$$\theta_{k,1} = b_{k-1} - \frac{d_{k-1}}{\theta_{k-1,2}}, \quad k \geq 3, \tag{2.19}$$

and

$$\begin{aligned}
\theta_{k,2} &= \frac{\left(-b_{k-1}R_{k-1}^{(1)} - c_{k-1}R_{k-2}^{(1)} - d_{k-1}R_{k-3}^{(1)}\right)R_{k-1}^{(2)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}} \\
&\qquad - \frac{\left(-b_{k-1}R_{k-1}^{(2)} - c_{k-1}R_{k-2}^{(2)} - d_{k-1}R_{k-3}^{(2)}\right)R_{k-1}^{(1)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}} \\
&= c_{k-1} + d_{k-1}\frac{R_{k-1}^{(1)}R_{k-3}^{(2)} - R_{k-3}^{(1)}R_{k-1}^{(2)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}} \\
&= c_{k-1} - d_{k-1}\frac{R_{k-3}^{(1)}R_{k-1}^{(2)} - R_{k-1}^{(1)}R_{k-3}^{(2)}}{R_{k-2}^{(1)}R_{k-3}^{(2)} - R_{k-3}^{(1)}R_{k-2}^{(2)}} \cdot \frac{R_{k-2}^{(1)}R_{k-3}^{(2)} - R_{k-3}^{(1)}R_{k-2}^{(2)}}{R_{k-1}^{(1)}R_{k-2}^{(2)} - R_{k-2}^{(1)}R_{k-1}^{(2)}},
\end{aligned}$$

i.e.,

$$\theta_{k,2} = c_{k-1} - d_{k-1}\frac{\theta_{k-1,1}}{\theta_{k-1,2}}, \quad k \geq 3. \tag{2.20}$$

Using (2.18) we can calculate $\theta_{2,1}$ and $\theta_{2,2}$ directly. For this purpose we need $R_0^{(j)}$, $R_1^{(j)}$ and $R_2^{(j)}$, $j=1,2$. From (2.14) (for $k=0,1$) we get

$$P_0(0)=1,\quad P_1(0)=-b_0,\quad P_2(0)=b_0b_1-c_1,$$

and from (2.10), using (2.8), we obtain

$$\begin{aligned}
Q_0^{(j)}(0) &= \int_{-1}^{1}\frac{P_0(0)-P_0(x)}{-x}w_j(x)\,dx=0,\\
Q_1^{(j)}(0) &= \int_{-1}^{1}\frac{P_1(0)-P_1(x)}{-x}w_j(x)\,dx=\int_{-1}^{1}w_j(x)\,dx\\
&= -\int_\Gamma w_j(z)\,dz=-i\int_\Gamma zw_j(z)(iz)^{-1}\,dz=-i\mu_1^{(j)},\\
Q_2^{(j)}(0) &= \int_{-1}^{1}\frac{P_2(0)-P_2(x)}{-x}w_j(x)\,dx=\int_{-1}^{1}(x-(b_0+b_1))w_j(x)\,dx\\
&= -i\int_\Gamma(z^2-(b_0+b_1)z)w_j(z)(iz)^{-1}\,dz=-i\mu_2^{(j)}+i(b_0+b_1)\mu_1^{(j)},
\end{aligned}$$

$j=1,2$. Substituting these expressions for $P_k(0)$ and $Q_k^{(j)}(0)$, $k=0,1,2$, $j=1,2$, in (2.17) we get

$$\begin{aligned}
&R_0^{(j)}=-\mu_0^{(j)},\qquad R_1^{(j)}=-i\mu_1^{(j)}+i\mu_0^{(j)}b_0,\\
&R_2^{(j)}=-i\mu_2^{(j)}+i\mu_1^{(j)}(b_0+b_1)+i\mu_0^{(j)}(c_1-b_0b_1),\quad j=1,2.
\end{aligned}$$

Finally, from (2.18) we obtain

$$\begin{aligned}
\theta_{2,1}&=b_0+b_1-\frac{\mu_0^{(1)}\mu_2^{(2)}-\mu_2^{(1)}\mu_0^{(2)}}{\mu_0^{(1)}\mu_1^{(2)}-\mu_1^{(1)}\mu_0^{(2)}}\,,\\
\theta_{2,2}&=c_1+b_0^2-b_0\frac{\mu_0^{(1)}\mu_2^{(2)}-\mu_2^{(1)}\mu_0^{(2)}}{\mu_0^{(1)}\mu_1^{(2)}-\mu_1^{(1)}\mu_0^{(2)}}+\frac{\mu_1^{(1)}\mu_2^{(2)}-\mu_2^{(1)}\mu_1^{(2)}}{\mu_0^{(1)}\mu_1^{(2)}-\mu_1^{(1)}\mu_0^{(2)}}\,.
\end{aligned}$$

For $k=1$ we have $P_1(x)=x-b_0$ and

$$\Pi_1(z)=P_1(z)+\theta_{1,1}P_0(z)=z-b_0+\theta_{1,1}.$$

Using the orthogonality condition

$$\begin{aligned}
0=[\Pi_1,1]_1 &= \int_\Gamma \Pi_1(z)w_1(z)(iz)^{-1}\,dz\\
&= \int_\Gamma zw_1(z)(iz)^{-1}\,dz+(\theta_{1,1}-b_0)\int_\Gamma w_1(z)(iz)^{-1}\,dz,
\end{aligned}$$

we obtain

$$\theta_{1,1} = b_0 - \frac{\mu_1^{(1)}}{\mu_0^{(1)}}.$$

Now, we are ready to obtain formulas for the coefficients β_k, γ_k and δ_k in recurrence relation (2.15). Using (2.16) in (2.15) for $k \geq 4$ we get

$$\begin{aligned} P_{k+1}(z) + \theta_{k+1,1}P_k(z) + \theta_{k+1,2}P_{k-1}(z) \\ = (z-\beta_k)(P_k(z) + \theta_{k,1}P_{k-1}(z) + \theta_{k,2}P_{k-2}(z)) \\ -\gamma_k(P_{k-1}(z) + \theta_{k-1,1}P_{k-2}(z) + \theta_{k-1,2}P_{k-3}(z)) \\ -\delta_k(P_{k-2}(z) + \theta_{k-2,1}P_{k-3}(z) + \theta_{k-2,2}P_{k-4}(z)) \end{aligned}$$

and substituting here for $zP_k(z)$, $zP_{k-1}(z)$ and $zP_{k-2}(z)$ the expressions obtained from the recurrence relation (2.14) yields

$$\begin{aligned} (\theta_{k+1,1} - b_k - \theta_{k,1} + \beta_k)P_k(z) \\ +(\theta_{k+1,2} - c_k - b_{k-1}\theta_{k,1} - \theta_{k,2} + \beta_k\theta_{k,1} + \gamma_k)P_{k-1}(z) \\ +(\delta_k + \gamma_k\theta_{k-1,1} + \beta_k\theta_{k,2} - b_{k-2}\theta_{k,2} - c_{k-1}\theta_{k,1} - d_k)P_{k-2}(z) \\ +(\delta_k\theta_{k-2,1} + \gamma_k\theta_{k-1,2} - c_{k-2}\theta_{k,2} - d_{k-1}\theta_{k,1})P_{k-3}(z) \\ +(\delta_k\theta_{k-2,2} - d_{k-2}\theta_{k,2})P_{k-4}(z) \equiv 0. \end{aligned}$$

By the linear independence of the polynomials $\{P_k\}$ we conclude that

$$\theta_{k+1,1} - b_k - \theta_{k,1} + \beta_k = 0, \tag{2.21}$$

$$\theta_{k+1,2} - c_k - b_{k-1}\theta_{k,1} - \theta_{k,2} + \beta_k\theta_{k,1} + \gamma_k = 0, \tag{2.22}$$

$$\delta_k + \gamma_k\theta_{k-1,1} + \beta_k\theta_{k,2} - b_{k-2}\theta_{k,2} - c_{k-1}\theta_{k,1} - d_k = 0, \tag{2.23}$$

$$\delta_k\theta_{k-2,1} + \gamma_k\theta_{k-1,2} - c_{k-2}\theta_{k,2} - d_{k-1}\theta_{k,1} = 0, \tag{2.24}$$

$$\delta_k\theta_{k-2,2} - d_{k-2}\theta_{k,2} = 0. \tag{2.25}$$

Using (2.21) and (2.19), we get for $k \geq 4$

$$\beta_k = \theta_{k,1} + \frac{d_k}{\theta_{k,2}}\,; \tag{2.26}$$

from (2.22), (2.20), (2.26) and (2.19) we get

$$\gamma_k = \theta_{k,2} + d_{k-1}\frac{\theta_{k,1}}{\theta_{k-1,2}}; \tag{2.27}$$

and, finally, from (2.25) we get

$$\delta_k = d_{k-2}\frac{\theta_{k,2}}{\theta_{k-2,2}}\,. \tag{2.28}$$

Substituting β_k, γ_k and δ_k given by (2.26), (2.27) and (2.28) in (2.23) and (2.24), using (2.19) and (2.20) it is easy to see that equations (2.23) and (2.24) are satisfied.

For $k = 0$ using the same procedure, instead of equations (2.21)–(2.25) we have only one equation $\theta_{1,1} - b_0 + \beta_0 = 0$, and easily obtain

$$\beta_0 = b_0 - \theta_{1,1}.$$

Using the same procedure (with $k = 1, 2, 3$) we get:

1° For $k = 1$

$$\beta_1 = b_1 + \theta_{1,1} - \theta_{2,1}, \quad \gamma_1 = c_1 + \theta_{1,1} b_0 - \theta_{2,2} - \beta_1 \theta_{1,1};$$

2° For $k = 2$ that (2.26) holds also for $k = 2$, and

$$\begin{aligned} \gamma_2 &= \theta_{2,2} + \theta_{2,1}(b_1 - \theta_{2,1}), \\ \delta_2 &= d_2 - \gamma_2 \theta_{1,1} - \beta_2 \theta_{2,2} + c_1 \theta_{2,1} + b_0 \theta_{2,2}; \end{aligned}$$

3° For $k = 3$ that (2.26) and (2.27) hold also for $k = 3$, and

$$\delta_3 = \theta_{3,2}(b_1 - \theta_{2,1}).$$

2.2. Jacobi weight functions

In this subsection the multiple orthogonal polynomials on the semicircle associated with an AT system consisting of two Jacobi weight functions on $[-1, 1]$ with different singularities at -1 and the same singularity at 1 are considered.

The weight functions are

$$w_m(x) = (1 - x)^{\alpha}(1 + x)^{\beta_m}, \quad m = 1, 2,$$

where $\alpha, \beta_m > -1$, $m = 1, 2$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$.

The recursion coefficients b_n, c_n, d_n in (2.14) (see [16]) for Jacobi weights satisfy[2]

$$\lim_{n\to+\infty} b_n = -\frac{1}{9}, \quad \lim_{n\to+\infty} c_n = 3\left(\frac{8}{27}\right)^2, \quad \lim_{n\to+\infty} d_n = \left(\frac{8}{27}\right)^3.$$

Based on the numerous numerical experiments we can state the following conjecture:

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[2] Notice that in [16] the recurrence coefficients for the type II multiple orthogonal polynomials (real) associated with an AT system consisting of Jacobi weights on $[0, 1]$ with different singularities at 0 and the same singularity at 1 have been given.

Conjecture 2.1. *The sequences* $\{\theta_{k,1}\}_{k=1}^{+\infty}$ *and* $\{\theta_{k,2}\}_{k=2}^{+\infty}$ *are convergent, with*

$$\begin{aligned}\theta_1 &= \lim_{k\to+\infty}\theta_{k,1}\\ &= -\frac{2}{27}+\frac{7\left(1-i\sqrt{3}\right)}{18\left(-13+16\sqrt{2}\right)^{1/3}}-\frac{1}{18}\left(-13+16\sqrt{2}\right)^{1/3}\left(1+i\sqrt{3}\right)\\ &\cong -0.009454178427325359-0.5213314224171121\,i\end{aligned}$$

and

$$\begin{aligned}\theta_2 &= \lim_{k\to+\infty}\theta_{k,2}\\ &= \frac{8}{729}\Big(8+\big(2(2+\sqrt{2})\big)^{1/3}\left(-3-3i\sqrt{3}\right)+3i\left(4-2\sqrt{2}\right)^{1/3}\left(i+\sqrt{3}\right)\Big)\\ &\cong -0.009373049182708634-0.04806819302729593\,i.\end{aligned}$$

Namely, θ_1 and θ_2 are the solutions, lying in the IV quadrant, of the equations

$$c(b-\theta_1)-\theta_1(b-\theta_1)^2=d,\quad \theta_2^3-c\theta_2^2+bd\theta_2=d^2,$$

respectively, where $b=\lim_{n\to+\infty}b_n$, $c=\lim_{n\to+\infty}c_n$, $d=\lim_{n\to+\infty}d_n$.

In Table 2.1 numerical values for $\theta_{k,1}$ and $\theta_{k,2}$ (for some values of $k\le 70$) in case of AT system consisting of two Jacobi weight functions:

$$w_1(x)=(1-x^2)^{-1/2},\ w_2(x)=(1-x)^{-1/2}$$

are given. Numbers in parentheses denote decimal exponents.

Theorem 2.2. *The sequences of recurrence coefficients* β_n, γ_n *and* δ_n *in* (2.15) *are convergent and*

$$\lim_{n\to+\infty}\beta_n=\lim_{n\to+\infty}b_n,\quad \lim_{n\to+\infty}\gamma_n=\lim_{n\to+\infty}c_n,\quad \lim_{n\to+\infty}\delta_n=\lim_{n\to+\infty}d_n.$$

Proof. If we take $\lim_{k\to+\infty}$ in (2.26) and (2.19), according to previous conjecture, we immediately get the first assertion, i.e., $\lim_{k\to+\infty}\beta_k=\lim_{k\to+\infty}b_k$. In similar way, from (2.27) and (2.20) one can obtain the second assertion, and finally, the third assertion follows from (2.28). □

Table 2.1: Numerical values for $\theta_{k,1}$ and $\theta_{k,2}$ for two Jacobi weight functions with $\alpha = -1/2$, $\beta_1 = -1/2$, $\beta_2 = 0$

k	$\theta_{k,1}$	$\theta_{k,2}$
2	$-0.788673023(-2) - 0.5184874803\,i$	$-0.184874803(-1) - 0.9211326977(-1)i$
3	$-0.956026150(-2) - 0.5217911640\,i$	$-0.993955584(-2) - 0.5018361000(-1)i$
4	$-0.964558273(-2) - 0.5217583278\,i$	$-0.963141412(-2) - 0.4893449913(-1)i$
5	$-0.961680868(-2) - 0.5216419087\,i$	$-0.952284796(-2) - 0.4853785880(-1)i$
10	$-0.951057947(-2) - 0.5214183834\,i$	$-0.940492376(-2) - 0.4815528993(-1)i$
15	$-0.948111084(-2) - 0.5213706104\,i$	$-0.938656887(-2) - 0.4810346785(-1)i$
20	$-0.946980988(-2) - 0.5213535513\,i$	$-0.938048469(-2) - 0.4808715112(-1)i$
25	$-0.946435934(-2) - 0.5213456057\,i$	$-0.937774509(-2) - 0.4808000184(-1)i$
30	$-0.946132802(-2) - 0.5213412783\,i$	$-0.937628173(-2) - 0.4807624754(-1)i$
35	$-0.945947210(-2) - 0.5213386657\,i$	$-0.937540935(-2) - 0.4807403541(-1)i$
40	$-0.945825454(-2) - 0.5213369688\,i$	$-0.937484779(-2) - 0.4807262350(-1)i$
45	$-0.945741313(-2) - 0.5213358051\,i$	$-0.937446518(-2) - 0.4807166773(-1)i$
50	$-0.945680757(-2) - 0.5213349724\,i$	$-0.937419284(-2) - 0.4807099088(-1)i$
55	$-0.945635734(-2) - 0.5213343563\,i$	$-0.937399213(-2) - 0.4807049411(-1)i$
60	$-0.945601354(-2) - 0.5213338877\,i$	$-0.937383997(-2) - 0.4807011879(-1)i$
61	$-0.945595460(-2) - 0.5213338075\,i$	$-0.937381398(-2) - 0.4807005481(-1)i$
62	$-0.945589845(-2) - 0.5213337312\,i$	$-0.937378926(-2) - 0.4806999397(-1)i$
63	$-0.945584493(-2) - 0.5213336584\,i$	$-0.937376571(-2) - 0.4806993606(-1)i$
64	$-0.945579386(-2) - 0.5213335891\,i$	$-0.937374328(-2) - 0.4806988090(-1)i$
65	$-0.945574510(-2) - 0.5213335230\,i$	$-0.937372187(-2) - 0.4806982832(-1)i$
66	$-0.945569852(-2) - 0.5213334598\,i$	$-0.937370145(-2) - 0.4806977815(-1)i$
67	$-0.945565398(-2) - 0.5213333994\,i$	$-0.937368194(-2) - 0.4806973026(-1)i$
68	$-0.945561138(-2) - 0.5213333417\,i$	$-0.937366329(-2) - 0.4806968451(-1)i$
69	$-0.945557059(-2) - 0.5213332864\,i$	$-0.937364545(-2) - 0.4806964077(-1)i$
70	$-0.945553152(-2) - 0.5213332335\,i$	$-0.937362839(-2) - 0.4806959893(-1)i$

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