FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 21 (2006), 13–21

REMARK ON ORTHOGONAL POLYNOMIALS INDUCED BY THE MODIFIED CHEBYSHEV MEASURE OF THE SECOND KIND*

G. V. Milovanović, A. S. Cvetković, M. M. Matejić

Abstract. In this note we introduce a system of polynomials $\{\widehat{P}_k\}$ orthogonal with respect to the modified Chebyshev measure of the second kind,

$$\mathrm{d}\widehat{\lambda}(t) = \frac{t + \frac{1}{2}c + \frac{1}{c}}{t + \frac{1}{2}c + \frac{1}{2c}} \sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1,1],$$

where c is a positive real number, and determine the coefficients in the corresponding three-term recurrence relation for these polynomials in an analytical form.

1. Introduction

In this note we investigate polynomials orthogonal with respect to the moment functional

$$\mathcal{L}(P) = \int_{-1}^{1} P(t) \frac{t + \frac{1}{2}c + \frac{1}{c}}{t + \frac{1}{2}c + \frac{1}{2c}} \sqrt{1-t^2}\,\mathrm{d}t, \quad P \in \mathcal{P}, \tag{1.1}$$

where $c \in \mathbb{R}\backslash\{0\}$. The special case $c = 1$ has been considered in [4]. To make it more clear, Figure 1 displays graph of the rational part of the weight for $c = \sqrt{2}$. As c tends to 1, the singularity of the rational part tends

Received January 18, 2006
2000 *Mathematics Subject Classification.* 33C45, 33C47.
*The authors were supported in part by the Serbian Ministry of Science and Environmental Protection (Project: Orthogonal Systems and Applications, grant number #144004) and the Swiss National Science Foundation (SCOPES Joint Research Project No. IB7320–111079 "New Methods for Quadrature").

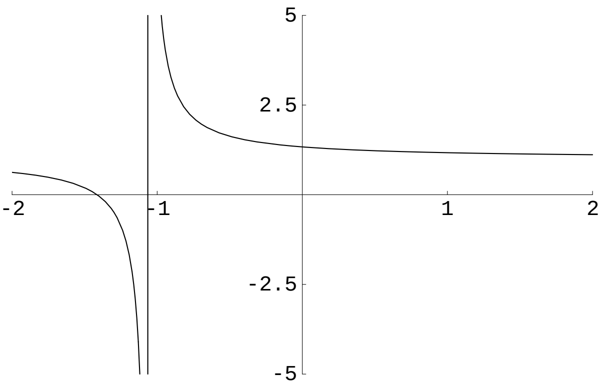

FIG. 1: Graph of the rational part of the weight in (1.1)

to -1; when c passes 1, the singularity of the rational part goes back to $-\infty$. Completely symmetric situation appears for $c < 0$. Namely, we can substitute $c := -c$, and after substitution $t := -t$, we get the same linear functional. Therefore, in the rest of this paper we only consider the case $c > 0$. Zero of the rational part is always bigger in modulus than the singularity.

We consider the modified measure

$$\mathrm{d}\widehat{\lambda}(t) = \frac{t-\gamma}{t-\delta}\sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1,1],$$

where $\gamma = -\frac{1}{2}c - \frac{1}{c}$ and $\delta = -\frac{1}{2}c - \frac{1}{2c}$. We pose the problem of determining the recurrence coefficients $\widehat{\alpha}_k = \alpha_k(\mathrm{d}\widehat{\lambda})$, $\widehat{\beta}_k = \beta_k(\mathrm{d}\widehat{\lambda})$, from those of the Chebyshev measure of the second kind, for which $\alpha_k = 0$, $\beta_k = 1/4$ for all $k \in \mathbb{N}$ and $\alpha_0 = 0$, $\beta_0 = \pi/2$.

The existence of $\{\widehat{P}_k\}$ is granted, since $\mathrm{d}\widehat{\lambda}(t)$ is a positive measure on $[-1,1]$ having finite moments of all orders

$$\mathcal{L}(t^k) = \int_{-1}^{1} t^k \frac{t+\frac{1}{2}c+\frac{1}{c}}{t+\frac{1}{2}c+\frac{1}{2c}}\sqrt{1-t^2}\,\mathrm{d}t, \quad k \in \mathbb{N}_0.$$

The problem is solved in two steps. First, we consider the modification of the Chebyshev measure of the second kind by the linear divisor, for which we are using Algorithm 1 (see the next section). We get the coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$. Then, we apply Algorithm 2 (see Section 3), which modifies by linear factor, for computing the coefficients of the three-term recurrence relation for the measure $\mathrm{d}\widehat{\lambda}(t)$, we finally get $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ for $k \in \mathbb{N}_0$.

Similar measures, e.g. with the weight function $(1-t^2)(1-k^2t^2)^{-1/2}$, $k^2<1$, were studied in [9]. There is also a great number on results for the so-called Szegő-Bernstein weight functions given by

$$w_1(t)=\frac{\rho(t)}{\sqrt{1-t^2}},\quad w_2(t)=\rho(t)\sqrt{1-t^2},\quad w_3(t)=\rho(t)\sqrt{\frac{1-t}{1+t}},$$

where ρ is a polynomial positive on the interval $(-1,1)$ (see [10], [2]). Similar weight function

$$w(t)=\frac{\sqrt{1-t^2}}{1-\mu t^2},\quad \mu\le 1,$$

has also been studied in [6]. For the Chebyshev measure of the first kind the same modification has been studied in [7]. Finally, in [1] one may find similar results even when the supporting set has two disjoint components.

2. Linear Divisors

To begin with, we consider a linear divisor

$$\mathrm{d}\widetilde{\lambda}(t)=\frac{1}{t-\delta}\sqrt{1-t^2}\ \mathrm{d}t,\quad t\in[-1,1],\quad \delta\in\mathbb{R}\setminus[-1,1],$$

where $\delta=-\frac{1}{2}c-\frac{1}{2c}$.

In order to be able to apply the modification (see Algorithm 1 given below), we must have the value of the Cauchy integral

$$\rho_0(\delta)=\int_{-1}^{1}\frac{1}{\delta-t}\sqrt{1-t^2}\ \mathrm{d}t,\quad \delta\in\mathbb{R}\setminus[-1,1].$$

Lemma 2.1. *The value of the Cauchy integral is*

$$\rho_0(\delta)=\int_{-1}^{1}\frac{1}{\delta-t}\sqrt{1-t^2}\ \mathrm{d}t=(\sqrt{\delta^2-1}+\delta)\pi.$$

Proof. Using the second Euler's substitution $\sqrt{1-t^2}=1+mt$, we get

$$t=-\frac{2m}{1+m^2}\ ,\quad \mathrm{d}t=\frac{2(m^2-1)}{(1+m^2)^2}\mathrm{d}m\ .$$

Now, it follows

$$\int\frac{1}{\delta-t}\sqrt{1-t^2}\ \mathrm{d}t=-2\int\frac{(m^2-1)^2}{(\delta m^2+2m+\delta)(1+m^2)^2}\ \mathrm{d}m,$$

which can be solved as an integral of the rational function. Now, we get

$$\frac{(m^2-1)^2}{(\delta m^2+2m+\delta)(1+m^2)^2}=\frac{\delta}{1+m^2}-\frac{2m}{(1+m^2)^2}+\frac{1-\delta^2}{\delta m^2+2m+\delta}.$$

The rest of the proof is now obvious. □

Before proving the next theorem, Algorithm 1 is presented and is going to be used to prove Theorem 1. Both Algorithms 1 and 2 can be found, for example, in [5, p. 123–129].

Algorithm 1 (Modification by a linear divisor)

Initialization:

$$\widetilde{\alpha}_0=\delta-\frac{\beta_0}{\rho_0(\delta)},\quad \widetilde{\beta}_0=-\rho_0(\delta),\quad q_0=-\frac{\beta_0}{\rho_0(\delta)}. \tag{2.1}$$

Continuation: For $k=1,2,\dots,n-1$ do

$$e_{k-1}=\alpha_{k-1}-\delta-q_{k-1}, \tag{2.2}$$

$$\widetilde{\beta}_k=q_{k-1}e_{k-1},$$

$$q_k=\beta_k/e_{k-1}, \tag{2.3}$$

$$\widetilde{\alpha}_k=q_k+e_{k-1}+\delta. \tag{2.4}$$

Theorem 2.1. *The coefficients of the three-term recurrence relation for the measure*

$$\mathrm{d}\widetilde{\lambda}(t)=\frac{1}{t+\frac{c}{2}+\frac{1}{2c}}\sqrt{1-t^2}\;\mathrm{d}t,\quad t\in[-1,1],$$

are

$$\widetilde{\alpha}_0=-\frac{1}{2c},\qquad \widetilde{\alpha}_k=0\quad for\quad k\ge 1$$

and

$$\widetilde{\beta}_0=\frac{\pi}{c},\qquad \widetilde{\beta}_k=\frac{1}{4}\quad for\quad k\ge 1.$$

Proof. The coefficients $\widetilde{\alpha}_0$ and $\widetilde{\beta}_0$ are computed directly from (2.1). Also, it is useful to compute the coefficients $\widetilde{\alpha}_1$ and $\widetilde{\beta}_1$ as the basis of mathematical induction. Using Algorithm 1, for $k=1$ we get

$$\widetilde{\alpha}_1=0,\qquad \widetilde{\beta}_1=\frac{1}{4}.$$

The rest of the proof follows using an induction argument. Thus, let the statement be true for k and we need to prove it for $k+1$. Combing (2.2) and (2.4) we get

$$\widetilde{\alpha}_k = 0 = q_k + e_{k-1} + \delta = q_k - \delta - q_{k-1} + \delta,$$

wherefrom it follows

$$q_k = q_{k-1}. \tag{2.5}$$

Now, from (2.4) we get

$$\widetilde{\alpha}_{k+1} = q_{k+1} + e_k + \delta = q_{k+1} - \delta - q_k + \delta = q_{k+1} - q_k.$$

Using (2.3), it follows

$$q_{k+1} = \frac{1}{4e_k},$$

and from (2.5)

$$e_k = e_{k-1}. \tag{2.6}$$

Finally, we get

$$\widetilde{\alpha}_k = \frac{1}{4e_{k-1}} - q_k = q_k - q_k = 0.$$

From (2.5) and (2.6), it follows

$$\widetilde{\beta}_{k+1} = q_k e_k = q_{k-1} e_{k-1} = \frac{1}{4}.$$

This completes the proof. □

3. Linear Factors

Let us consider a linear factor

$$\mathrm{d}\widehat{\lambda}(t) = (t-\gamma)\mathrm{d}\widetilde{\lambda}(t), \quad t \in [-1,1], \quad \gamma \in \mathbb{R} \setminus [-1,1],$$

where $\gamma = -\frac{1}{2}c - \frac{1}{c}$.

Before presenting Algorithm 2, we have to stress that in this algorithm we use already computed coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ to get the coefficients of the three-term recurrence relation for the measure $\mathrm{d}\widehat{\lambda}(t)$, $\widehat{\alpha}_k$ and $\widehat{\beta}_k$, for $k \in \mathbb{N}_0$.

ALGORITHM 2 (Modification by a linear factor)

Initialization:

$$e_{-1} = 0.$$

Continuation: For $k = 0, 1, \ldots, n-1$ do

$$q_k = \widetilde{\alpha}_k - e_{k-1} - \gamma, \tag{3.1}$$

$$\widehat{\beta}_k = (\widetilde{\alpha}_0 - \gamma)\widetilde{\beta}_0 \quad \text{if} \quad k = 0,$$

$$\widehat{\beta}_k = q_k e_{k-1} \quad \text{if} \quad k > 0, \tag{3.2}$$

$$e_k = \widetilde{\beta}_{k+1}/q_k, \tag{3.3}$$

$$\widehat{\alpha}_k = \gamma + q_k + e_k. \tag{3.4}$$

Theorem 3.1. *The coefficients of the three-term recurrence relation for the measure*

$$\mathrm{d}\widehat{\lambda}(t) = \frac{t + \frac{c}{2} + \frac{1}{c}}{t + \frac{c}{2} + \frac{1}{2c}} \sqrt{1 - t^2}\ \mathrm{d}t, \quad t \in [-1, 1],$$

are

$$\widehat{\alpha}_k = -\frac{Apq^k}{(1 + pq^k)(1 + pq^{k+1})}, \quad k \in \mathbb{N}_0, \tag{3.5}$$

and

$$\widehat{\beta}_k = \frac{(1 + pq^{k-1})(1 + pq^{k+1})}{4(1 + pq^k)^2}, \quad k \in \mathbb{N}, \tag{3.6}$$

where

$$A = \frac{c^4 + 4}{c(2 + c^2 + \sqrt{c^4 + 4})}, \quad p = \frac{\sqrt{c^4 + 4} - c^2}{\sqrt{c^4 + 4} + c^2}, \quad q = \frac{2 + c^2 - \sqrt{c^4 + 4}}{2 + c^2 + \sqrt{c^4 + 4}}.$$

Proof. First, we prove that

$$e_k = \frac{2 + c^2 - \sqrt{c^4 + 4}}{4c} \frac{1 + pq^k}{1 + pq^{k+1}}, \qquad k \in \mathbb{N}_0, \tag{3.7}$$

wherefrom the rest of the statement of this theorem follows directly. The proof is given by induction. For $k = 0$ from (3.7) we have $e_0 = c/(2+2c^2)$ that we can also obtain from Algorithm 2, putting $k = 0$. So, let the statement be true for $k - 1$. From (3.1) and (3.3) it follows

$$e_k = \frac{1/4}{q_k} = -\frac{1}{4(e_{k-1} + \gamma)}. \tag{3.8}$$

Using elementary calculus we get

$$-4\Big(\frac{2+c^2-\sqrt{c^4+4}}{4c}\cdot\frac{1+pq^{k-1}}{1+pq^k}-\frac{c}{2}-\frac{1}{c}\Big)=\frac{4c}{2+c^2-\sqrt{c^4+4}}\cdot\frac{1+pq^{k+1}}{1+pq^k}.$$

The term on the right side of the previous equation is $1/e_k$, which is exactly stated in (3.8).

From (3.2) and (3.3) it follows

$$\widehat{\beta}_k = q_k e_{k-1} = \frac{\widetilde{\beta}_{k+1}}{e_k} e_{k-1} = \frac{1}{4}\frac{e_{k-1}}{e_k},$$

which is exactly (3.6).

Now, (3.5) is a direct consequence of (3.4) and (3.7). □

4. Explicit Expression for Polynomials $\widehat{P}_n$

Using the following two theorems we give explicit expression for the polynomial system $\{\widehat{P}_n\}$.

Theorem 4.1. *The polynomial system orthogonal with respect to the measure* $\mathrm{d}\widetilde{\lambda}(t)$ *is given by*

$$\widetilde{P}_k(x) = U_k(x) - \widetilde{\alpha}_0 U_{k-1}(x),$$

where $\widetilde{\alpha}_0 = -1/(2c)$ *and* U_n *is the Chebyshev polynomial of the second kind, defined by* (see [8])

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad n \in \mathbb{N}_0.$$

Proof. The statement is true for $k=1$ and $k=2$. Indeed, we have

$$\widetilde{P}_1(x) = (x-\widetilde{\alpha}_0)\widetilde{P}_0(x) = x-\widetilde{\alpha}_0 = U_1(x) - \widetilde{\alpha}_0 U_0(x)$$

and

$$\begin{aligned}\widetilde{P}_2(x) &= (x-\widetilde{\alpha}_1)\widetilde{P}_1(x) - \widetilde{\beta}_1\widetilde{P}_0(x) = x(U_1(x)-\widetilde{\alpha}_0U_0(x)) - \beta_1 U_0\\ &= xU_1(x) - \beta_1 U_0(x) - \widetilde{\alpha}_0 x U_0(x) = U_2(x) - \widetilde{\alpha}_0 U_1(x).\end{aligned}$$

Let the statement be true for $k-1$ and k, and we need to prove it for $k+1$. Then, we have

$$\begin{aligned}
\widetilde{P}_{k+1}(x) &= (x-\widetilde{\alpha}_k)\widetilde{P}_k(x) - \widetilde{\beta}_k\widetilde{P}_{k-1}(x) \\
&= (x-\widetilde{\alpha}_k)(U_k(x)-\widetilde{\alpha}_0U_{k-1}(x)) - \widetilde{\beta}_k(U_{k-1}(x)-\widetilde{\alpha}_0U_{k-2}(x)) \\
&= (xU_k(x)-\widetilde{\beta}_kU_{k-1}(x)) - \widetilde{\alpha}_0(xU_{k-1}(x)-\beta_kU_{k-2}(x)) \\
&= U_{k+1}(x)-\widetilde{\alpha}_0U_k(x). \quad \square
\end{aligned}$$

Finally, we can express directly our polynomial system $\{\widehat{P}_n\}$, using Chebyshev polynomials of the second kind $\{U_n\}$.

Theorem 4.2. *Let* $\mathrm{d}\widetilde{\lambda}(t)$ *be quasi-definite and* $\gamma = -\frac{1}{2}c - \frac{1}{c}$ *be such that* $\widetilde{P}_k(\gamma) \neq 0$ *for* $k \in \mathbb{N}$. *Let* $\mathrm{d}\widehat{\lambda}(t) = (t-\gamma)\mathrm{d}\widetilde{\lambda}(t)$. *Then* $\mathrm{d}\widehat{\lambda}(t)$ *is also quasi-definite and polynomials* $\{\widehat{P}_n\}$ *are the monic formal orthogonal polynomials with respect to* $\mathrm{d}\widehat{\lambda}(t)$, *and can be expressed in the form*

$$\begin{aligned}
\widehat{P}_n(t,\gamma) &= \frac{\widetilde{P}_{n+1}(t) - \dfrac{\widetilde{P}_{n+1}(\gamma)}{\widetilde{P}_n(\gamma)}\widetilde{P}_n(t)}{t-\gamma} \\
&= \frac{U_{n+1}(t)-\widetilde{\alpha}_0U_n(t) - \dfrac{U_{n+1}(\gamma)-\widetilde{\alpha}_0U_n(\gamma)}{U_n(\gamma)-\widetilde{\alpha}_0U_{n-1}(\gamma)}(U_n(t)-\widetilde{\alpha}_0U_{n-1}(t))}{t-\gamma}.
\end{aligned}$$

Proof. The proof of this theorem is a consequence of Theorem 1.55 from [5, p. 38]. $\square$

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Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
grade@junis.ni.ac.yu (G.V. Milovanović)
aca@elfak.ni.ac.yu (A.S. Cvetković)
maki@elfak.ni.ac.yu (M.M. Matejić)