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EXTREMAL PROBLEMS FOR NONNEGATIVE POLYNOMIALS IN L^r NORM WITH GENERALIZED LAGUERRE WEIGHT

A. Guessab, G. V. Milovanović and O. Arino

Dedicated to Professor D. S. Mitrinović on the occasion of his 80th birthday

Abstract. Let W_n be the set of all algebraic polynomials of exact degree n, whose coefficients are nonnegative. For the norm in $L^r[0,+\infty)$ with generalized Laguerre weight $w(t) = t^\alpha e^{-t}$ ($\alpha > -1$), the extremal problem

$$C_{n,r}^{(m)}(\alpha) = \sup_{P\in W_n} \left(\frac{\|P^{(m)}\|_r}{\|P\|_r}\right)^r$$

is solved when $r \in \mathbb{N}$, which completes the results of Milovanović [6] for $r = 2$ and $m = 1$, and Guessab and Milovanović [2] for $r = 3$ and $m = 1$.

1. Introduction

E. Hille, G. Szegö and J. D. Tamarkin [3] extended the well-known result of A. A. Markov [5] to L^r-norm ($r \geq 1$) on $(-1, 1)$ by proving the following result:

Theorem 1.1. *Let $r > 1$ and let $P(t)$ be an arbitrary rational polynomial of degree n. Then*

$$\left(\int_{-1}^{1} |P'(t)|^r\,dt\right)^{1/r} \leq An^2 \left(\int_{-1}^{1} |P(t)|^r\,dt\right)^{1/r}, \tag{1.1}$$

where A is a positive constant which depends only on r, but not on P or on n.

This result was obtained also by N. Bari [1] using very different methods.

The factor n^2 in (1.1) cannot be replaced by any function tending to infinity more slowly. Namely, for each n exist polynomials $P(t)$ of degree n such that the left side of (1.1) is $\leq Bn^2$, where B is a constant of the same nature as A.

Under only a little restriction on the zeros of $P(z)$, M. A. Malik [4] found the following improvements of Theorem 1.1:

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Theorem 1.2. *Let $r > 1$ and let $P(t)$ be an arbitrary rational polynomial of degree n. Let $P(z)$ have no zeros in the two circular regions*

$$|z \pm a| < 1 - a \qquad (0 \le a < 1),$$

then

$$\left(\int_{-1}^{1} |P'(t)|^r \, dt\right)^{1/r} \le B n^{1+1/r} \left(\int_{-1}^{1} |P(t)|^r \, dt\right)^{1/r},$$

where B is a positive constant which depends only on r and a, but not on P or on n.

Instead of the set of all algebraic polynomials of degree n, in this paper we consider a restricted polynomial set. Namely, let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i.e.,

$$W_n = \Big\{ P \;\Big|\; P(t) = \sum_{k=0}^{n} a_k t^k, \;\; a_k \ge 0 \; (k = 0, 1, \ldots, n-1), \; a_n > 0 \Big\}.$$

We denote by $W_n^{(m-1)}$ the subset of W_n for which $a_0 = \cdots = a_{m-1} = 0$ (i.e., $P(0) = \cdots = P^{(m-1)}(0) = 0$).

Let $w(t) = t^\alpha e^{-t}$ $(\alpha > -1)$ be a weight function on $[0, +\infty)$. For $P \in W_n$, we define $\|P\|_r = \left(\int_0^\infty w(t) P(t)^r \, dt\right)^{1/r}$, $r \ge 1$, and consider the following extremal problem:

Determine the best constant in the inequality

$$\|P^{(m)}\|_r^r \le C_{n,r}^{(m)} \|P\|_r^r, \qquad P \in W_n, \tag{1.2}$$

i.e.,

$$C_{n,r}^{(m)}(\alpha) = \sup_{P \in W_n} \left(\frac{\|P^{(m)}\|_r}{\|P\|_r} \right)^r. \tag{1.3}$$

The case $r = 2$ and $m = 1$ has been recently investigated by A. K. Varma [10], [11] and G. V. Milovanović [6]. In [6] the following result was proved:

Theorem 1.3. *The best constant $C_{n,2}^{(1)}(\alpha)$ defined in* (1.3) *is*

$$C_{n,2}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n), \\[2ex] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & (\alpha_n \le \alpha < +\infty), \end{cases}$$

where

$$\alpha_n = \tfrac{1}{2}(n+1)^{-1}\big((17n^2 + 2n + 1)^{1/2} - 3n + 1\big).$$

When $r = 3$ and $m = 1$, A. Guessab and G. V. Milovanović [2] obtained the following result:

Theorem 1.4. *The best constant* $C_{n,3}^{(1)}(\alpha)$ *defined in* (1.3) *is*

$$C_{n,3}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{(3+\alpha)(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n), \\ \dfrac{n^3}{(3n+\alpha)(3n+\alpha-1)(3n+\alpha-2)} & (\alpha_n \le \alpha < +\infty), \end{cases}$$

where α_n *is the unique positive root of the equation*

$$(n^2+n+1)\alpha^3 + 3(2n^2+2n-1)\alpha^2 + (11n^2-16n+2)\alpha - 3n(7n-2) = 0.$$

An extremal problem for higher derivatives of nonnegative polynomials with respect to the same weight was investigated in [9]. A similar problem for Freud's weight function has been dealt with G. V. Milovanović and R. Ž. Djordjević [8]. A survey about extremal problems of Markov's type for algebraic polynomials is given in [7].

In this paper we consider the extremal problem (1.3) for $r \in \mathbb{N}$ and $m \le n$. Firstly, we note that the supremum in our extremal problem (1.3) is attained for some $P \in W_n^{(m-1)}$. Indeed,

$$\sup_{P\in W_n} \frac{\|P^{(m)}\|_r}{\|P\|_r} = \sup_{\substack{P\in W_n^{(m-1)} \\ a_0,\dots,a_{m-1}\ge 0}} \frac{\|P^{(m)}\|_r}{\|P+Q_{m-1}\|_r} = \sup_{P\in W_n^{(m-1)}} \frac{\|P^{(m)}\|_r}{\|P\|_r},$$

where $Q_{m-1}(t) = \sum_{k=0}^{m-1} a_k t^k$ $(a_k \ge 0)$.

An auxiliary result, necessary in solving problem (1.3), is presented in §2. The central issue of the paper, the determination of the best constant in (1.2), is given in §3. Some corollaries and special cases of importance are included.

2. An auxiliary result

Lemma 2.1. *Let* $r \in \mathbb{N}$ *and* $m \le n$. *If* $P(t) = \sum_{k=m}^{n} a_k t^k \in W_n^{(m-1)}$ *then the equality*

$$\|P^{(m)}\|_r^r = \sum_{k_1,\dots,k_r=m}^{n} H_{r,\alpha}^{(m)}(k_1,\dots,k_r)\Big(\prod_{i=1}^{r} a_{k_i}\Big)\Gamma(\alpha+k_1+\cdots+k_r+1) \tag{2.1}$$

holds, where

$$H_{r,\alpha}^{(m)}(k_1,\dots,k_r) = \frac{\prod_{i=1}^{r} k_i(k_i-1)\cdots(k_i-m+1)}{\prod_{i=0}^{mr-1}(\alpha+k_1+\cdots+k_r-i)}, \tag{2.2}$$

and Γ *is the gamma function.*

Proof. Let $P \in W_n^{(m-1)}$, i.e.,

$$P(t) = \sum_{k=m}^{n} a_k t^k$$

with $a_k \geq 0$ $(k = m, \dots, n-1)$ and $a_n > 0$. Then

$$\big(P^{(m)}(t)\big)^r = \sum_{k_1,\dots,k_r=m}^{n} B_m(k_1,\dots,k_r) t^{k_1+\cdots+k_r-mr},$$

where

$$B_m(k_1,\dots,k_r) = \prod_{i=1}^{r} a_{k_i} k_i(k_i-1)\cdots(k_i-m+1). \tag{2.3}$$

Thus, we have

$$\|P^{(m)}\|_r^r = \sum_{k_1,\dots,k_r=m}^{n} B_m(k_1,\dots,k_r)\Gamma(\alpha+k_1+\cdots+k_r-mr+1). \tag{2.4}$$

Since

$$\Gamma(\alpha+k_1+\cdots+k_r-mr+1) = \frac{\Gamma(\alpha+k_1+\cdots+k_r+1)}{\prod\limits_{i=0}^{mr-1}(\alpha+k_1+\cdots+k_r-i)}, \tag{2.5}$$

from (2.3), (2.4) and (2.5) we obtain (2.1). $\square$

3. Main result

In this section we give the results related to problem (1.3).

Theorem 3.1. *The best constant* $C_{n,r}^{(m)}(\alpha)$ *defined in* (1.3) *is*

$$C_{n,r}^{(m)}(\alpha) = \begin{cases} \dfrac{(m!)^r}{\prod\limits_{i=1}^{mr}(i+\alpha)} & (-1 < \alpha \leq \alpha_{n,r,m}), \\[2ex] \dfrac{\prod\limits_{i=0}^{m-1}(n-i)^r}{\prod\limits_{i=0}^{mr-1}(rn+\alpha-i)} & (\alpha_{n,r,m} \leq \alpha < +\infty), \end{cases} \tag{3.1}$$

where $\alpha_{n,r,m}$ is the unique positive root of the equation

$$(3.2)\qquad (m!)^r \prod_{i=0}^{mr-1} (rn+\alpha-i) = \prod_{i=1}^{mr}(i+\alpha) \prod_{i=0}^{m-1} (n-i)^r.$$

Proof. Let $P \in W_n^{(m-1)}$. Then from Lemma 1 we have

$$(3.3)\qquad \|P^{(m)}\|_r^r = \sum_{k_1,\dots,k_r=m}^{n} H_{r,\alpha}^{(m)}(k_1,\dots,k_r)\Big(\prod_{i=1}^{r} a_{k_i}\Big)\Gamma(\alpha+k_1+\cdots+k_r+1),$$

where $H_{r,\alpha}^{(m)}$ is given by (2.2). Since

$$\prod_{i=1}^{r} k_i(k_i-1)\cdots(k_i-m+1) \le \frac{1}{r^{mr}}\Big(\sum_{i=1}^{r} k_i\Big)^r \Big(\sum_{i=1}^{r}(k_i-1)\Big)^r \cdots \Big(\sum_{i=1}^{r}(k_i-m+1)\Big)^r,$$

we find

$$(3.4)\qquad H_{r,\alpha}^{(m)}(k_1,\dots,k_r) \le \frac{1}{r^{mr}} \prod_{j=0}^{m-1} K_{r,\alpha}^{(m,j)}(k_1,\dots,k_r),$$

where

$$(3.5)\qquad K_{r,\alpha}^{(m,j)}(k_1,\dots,k_r) = \frac{\Big(\big(\sum\limits_{\nu=1}^{r} k_\nu\big)-jr\Big)^r}{\prod\limits_{i=0}^{r-1}\Big(\alpha+\big(\sum\limits_{\nu=1}^{r} k_\nu\big)-jr-i\Big)}.$$

From (3.4) and (3.5) it folllows that

$$\|P^{(m)}\|_r^r \le \frac{1}{r^{mr}}\Big(\prod_{j=0}^{m-1} \max_{k_1,\dots,k_r} K_{r,\alpha}^{(m,j)}(k_1,\dots,k_r)\Big)\|P\|_r^r$$

and so we have

$$C_{n,r}^{(m)}(\alpha) \le \frac{1}{r^{mr}}\Big(\prod_{j=0}^{m-1} \max_{k_1,\dots,k_r} K_{r,\alpha}^{(m,j)}(k_1,\dots,k_r)\Big) = L,$$

where

$$L = \begin{cases} \dfrac{(m!)^r}{\prod_{i=1}^{mr}(i+\alpha)} & \text{if} \quad -1 < \alpha \le \alpha_{n,r,m}, \\ \dfrac{\prod_{i=0}^{m-1}(n-i)^r}{\prod_{i=0}^{mr-1}(rn+\alpha-i)} & \text{if} \quad \alpha_{n,r,m} \le \alpha < +\infty, \end{cases}$$

and $\alpha_{n,r,m}$ is the unique positive root of the equation (3.2).

In order to prove that $C_{n,r}^{(m)}(\alpha) = L$, we consider $P(t) = t^n + \lambda t$ $(\lambda \le 0)$ and set

$$\Phi(\lambda) = \|P^{(m)}\|_r^r \,/\, \|P\|_r^r.$$

Since

$$\Phi(0) = \frac{1}{r^{mr}} \left(\prod_{j=0}^{m-1} K_{r,\alpha}^{(m,j)}(n, \dots, n) \right)$$

and

$$\lim_{\lambda \to +\infty} \Phi(\lambda) = \frac{1}{r^{mr}} \left(\prod_{j=0}^{m-1} K_{r,\alpha}^{(m,j)}(m, \dots, m) \right),$$

we see that $\tilde{P}(t) = t^n$ is an extremal polynomial for $\alpha \ge \alpha_{n,r,m}$. When $-1 < \alpha \le \alpha_{n,r,m}$, there exists a sequence of polynomials, for example $p_k(t) = t^n + kt^m$, $k = 1, 2, \dots$, for which

$$\lim_{k \to +\infty} \|p_k^{(m)}\|_r^r \,/\, \|p_k\|_r^r = C_{n,r}^{(m)}(\alpha). \quad \square$$

Remark. The statement of our theorem holds if W_n is the set of all algebraic polynomials $P\,(\not\equiv 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients. In this case, for $-1 < \alpha \le \alpha_{n,r,m}$, we can see that $\tilde{P}(t) = \lambda t^m$ $(\lambda > 0)$ is an extremal polynomial.

In the limit case $n \to +\infty$, the equation (3.2) reduces to

$$\prod_{i=1}^{mr}(\alpha_{\infty,r,m} + i) = r^{mr}(m!)^r.$$

The simplest case is $r = 1$ and $m = 1$. Then we have

$$C_{n,1}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{\alpha+1} & (-1 < \alpha \le 0), \\ \dfrac{n}{\alpha+n} & (\alpha \ge 0). \end{cases}$$

The case $\alpha = 1$, $r = 2$ and $m = 1$ was considered by A. K. Varma [10]. Then

$$C_{n,2}^{(1)}(1) = \frac{n}{2(2n+1)}.$$

Corollary 3.2. *Let* $\alpha > -1, m = 1$ *and* $r = 2$. *Theorem* 3.1 *reduces to Theorem* 1.3.

Corollary 3.3. *Let* $\alpha > -1$, $m = 1$ *and* $r = 3$. *Theorem* 3.1 *reduces to Theorem* 1.4.

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Departement de Mathématiques
Avenue de l'Université
F–64000 Pau, France

Faculty of Electronic Engineering
Department of Mathematics, P.O. Box 73
18000 Niš, Yugoslavia

Departement de Mathématiques
Avenue de l'Université
F–64000 Pau, France

EXTREMALNI PROBLEMI ZA NENEGATIVNE POLINOME U L^r NORMI SA GENERALISANOM LAGUERREOVOM TEŽINOM

A. Guessab, G. V. Milovanović i O. Arino

Neka je W_n skup svih algebarskih polinoma stepena n sa nennegativnim koeficijentima. Za normu u $L^r[0,+\infty)$ sa generalisanom Laguerre-ovom težinom $w(t)=t^\alpha e^{-t}$ $(\alpha>-1)$, rešen je ekstremalni problem

$$C_{n,r}^{(m)}(\alpha)=\sup_{P\in W_n}\left(\frac{\|P^{(m)}\|_r}{\|P\|_r}\right)^r$$

kada $r\in\mathbb{N}$, što kompletira prethodne rezultate Milovanovića [6] za $r=2$ i $m=1$, i Guessab-a i Milovanovića za $r=3$ i $m=1$.