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ON SOME PROPERTIES OF HUMBERT'S POLYNOMIALS, II

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Abstract. In our previous paper [5], we defined and considered a class of Humbert's polynomials, which generalizes the well-known class of Gegenbauer's polynomials. Our interest here is in further investigation of this class of polynomials including a distribution of zeros. An conjecture about that is stated.

1. Introduction

In [5] we considered the polynomials $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by the generating function

$$G_m^{\lambda}(x,t) = (1-2xt+t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n,$$

where $m \in \mathbb{N}$ and $\lambda > -1/2$. Note that

$$p_{n,1}^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x-1)^n \qquad \text{(Horadam polynomials [3])},$$

$$p_{n,2}^{\lambda}(x) = C_n^{\lambda}(x) \qquad \text{(Gegenbauer polynomials [1])},$$

$$p_{n,3}^{\lambda}(x) = p_{n+1}^{\lambda}(x) \qquad \text{(Horadam–Pethe polynomials [4])},$$

where $(\lambda)_0 = 1,\ (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1),\ \lambda = 1,2,\dots$. The explicit form of the polynomials $p_{n,m}^{\lambda}(x)$ is

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n-mk)!}(2x)^{n-mk}. \tag{1.1}$$

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In this note we introduce a class of polynomials $\{Q_N^{(m,q,\lambda)}(t)\}_{N=0}^{\infty}$ which satisfy an $(m+1)$-term recurrence relation (Section 2). Some special cases are considered in Section 3, and certain numerical investigations regarding the distribution of zeros of such polynomials are given in Section 4.

2. Polynomials $Q_N^{(m,q,\lambda)}(t)$

Let $n = mN + q$, where $N = [n/m]$ and $0 \le q \le m-1$. Starting from (1.1), we have

$$\begin{aligned} p_{n,m}^{\lambda}(x) &= \sum_{k=0}^{N} (-1)^k \frac{(\lambda)_{mN+q-(m-1)k}}{k!(mN+q-mk)!} (2x)^{mN+q-mk} \\ &= (2x)^q Q_N^{(m,q,\lambda)}(t), \end{aligned}$$

where $t = (2x)^m$ and

$$Q_N^{(m,q,\lambda)}(t) = \sum_{k=0}^{N} (-1)^k \frac{(\lambda)_{mN+q-(m-1)k}}{k!(mN+q-mk)!} t^{N-k}. \tag{2.1}$$

The polynomials $Q_N^{(m,q,\lambda)}(t)$ depend of three parameters: $\lambda > -1/2$, $m \in \mathbb{N}$, and $q \in \{0, 1, \dots, m-1\}$.

Using the recurrence relation for the polynomials $p_{n,m}^{\lambda}(x)$ ([5])

$$np_{n,m}^{\lambda}(x) = (\lambda + n - 1)2xp_{n-1,m}^{\lambda}(x) - (n + m(\lambda - 1))p_{n-m,m}^{\lambda}(x),$$

where $n \ge m \ge 1$, we obtain:

Theorem 2.1. *The polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ satisfy the following recurrence relations:*

For $1 \le q \le m-1$,

$$\begin{aligned} (mN+q)Q_N^{(m,q,\lambda)}(t) &= (\lambda + mN + q - 1)Q_N^{(m,q-1,\lambda)}(t) \\ &\quad - (mN + q + m(\lambda - 1))Q_{N-1}^{(m,q,\lambda)}(t), \end{aligned} \tag{2.2}$$

and, for $q = 0$,

$$\begin{aligned} mNQ_N^{(m,0,\lambda)}(t) &= (\lambda + mN - 1)tQ_{N-1}^{(m,m-1,\lambda)}(t) \\ &\quad - m(N + \lambda - 1)Q_{N-1}^{(m,0,\lambda)}(t). \end{aligned} \tag{2.3}$$

Since $Dp_{n+k,m}^{\lambda}(x) = 2^k(\lambda)_k p_{n,m}^{\lambda+k}(x)$ (see [5, Theorem 1]), where D is the differentiation operator, we can prove the following results:

Theorem 2.2. *The polynomials* $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ *satisfy the following recurrence relations:*

For $0 \le q \le m-2$,

$$(q+1)Q_N^{(m,q+1,\lambda)}(t) + mtDQ_N^{(m,q+1,\lambda)}(t) = \lambda Q_N^{(m,q,\lambda+1)}(t),$$

and, for $q = m-1$,

$$mDQ_{N+1}^{(m,0,\lambda)}(t) = \lambda Q_N^{(m,m-1,\lambda+1)}(t).$$

Theorem 2.3. *The polynomials* $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2;\ m \ge 3)$ *satisfy the following recurrence relations:*

For $0 \le q \le m-3$,

$$\begin{aligned}(q+1)(q+2)Q_N^{(m,q+2,\lambda)}(t) &+ m(m+q+1)tDQ_N^{(m,q+2,\lambda)}(t)\\ &+ m^2t^2D^2Q_N^{(m,q+2,\lambda)}(t) = (\lambda)_2 Q_N^{(m,q,\lambda+2)}(t),\end{aligned}$$

and

$$m(m-1)DQ_{N+1}^{(m,0,\lambda)}(t) + m^2tD^2Q_{N+1}^{(m,0,\lambda)}(t) = (\lambda)_2 Q_N^{(m,m-2,\lambda+2)}(t),$$

$$m^2DQ_{N+1}^{(m,1,\lambda)}(t) + m^2tQ_{N+1}^{(m,1,\lambda)}(t) = (\lambda)_2 Q_N^{(m,m-1,\lambda+2)}(t),$$

for $q = m-2$ *and* $q = m-1$, *respectively.*

It is interesting to find a recurrence relation for the polynomials $t \mapsto Q_N^{(m,q,\lambda)}(t)$, where the parameters m, q, λ are fixed.

At first we prove the following lemmas:

Lemma 2.4. *The polynomials* $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ *satisfy a recurrence relation of the form*

$$\sum_{i=0}^{q+1} A_{i,N,q}^{(q+1)} Q_{N+1-i}^{(m,q,\lambda)}(t) = tQ_N^{(m,m-1,\lambda)}(t) \qquad (q = 0, 1, \dots, m-1), \tag{2.4}$$

where

$$A_{0,N,0}^{(1)} = \frac{m(N+1)}{\lambda + m(N+1) - 1}, \qquad A_{1,N,0}^{(1)} = \frac{m(N+\lambda)}{\lambda + m(N+1) - 1}.$$

For $q = 1, \dots, m-1$, the coefficients $A_{i,N,q}^{(q+1)}$ can be obtained by using the following procedure

$$
\begin{aligned}
&A_{0,N,q}^{(q+1)} = \frac{A_{0,N,q-1}^{(q)}}{\alpha_{N+1}^{(q)}}, \qquad A_{q+1,N,q}^{(q+1)} = \frac{A_{q,N,q-1}^{(q)}}{\alpha_{N+1-q}^{(q)}}\beta_{N+1-q}^{(q)},\\
&A_{i,N,q}^{(q+1)} = \frac{A_{i,N,q-1}^{(q)}}{\alpha_{N+1-i}^{(q)}} + \frac{A_{i-1,N,q-1}^{(q)}}{\alpha_{N+2-i}^{(q)}}\beta_{N+2-i}^{(q)} \qquad (i = 1, \dots, q),
\end{aligned}
\tag{2.5}
$$

where

$$
\alpha_k^{(q)} = \frac{\lambda + mk + q - 1}{mk + q}, \qquad \beta_k^{(q)} = \frac{m(k + \lambda - 1) + q}{mk + q}. \tag{2.6}
$$

Proof. Using (2.3), for $N := N + 1$, we obtain

$$
\frac{m(N+1)}{\lambda + m(N+1) - 1} Q_{N+1}^{(m,0,\lambda)}(t) + \frac{m(N+\lambda)}{\lambda + m(N+1) - 1} Q_N^{(m,0,\lambda)}(t) = tQ_N^{(m,m-1,\lambda)},
$$

which represents (2.4) for $q = 0$.

Suppose now that (2.4) holds for some q, i.e.,

$$
\sum_{i=0}^{q} A_{i,N,q-1}^{(q)} Q_{N+1-i}^{(m,q-1,\lambda)}(t) = tQ_N^{(m,m-1,\lambda)}(t).
$$

Using (2.2), i.e.,

$$
Q_k^{(m,q-1,\lambda)}(t) = \frac{1}{\alpha_k^{(q)}} \left(Q_k^{(m,q,\lambda)}(t) + \beta_k^{(q)} Q_{k-1}^{(m,q,\lambda)} \right),
$$

where $\alpha_k^{(q)}$ and $\beta_k^{(q)}$ are given by (2.6), we obtain (2.4), where the coefficients $A_{i,N,q}^{(q+1)}$ are expressed by (2.5). □

Lemma 2.5. *The polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ satisfy a recurrence relation of the form*

$$
\sum_{i=0}^{s} A_{i,N,q}^{(s)} Q_{N+1-i}^{(m,q,\lambda)}(t) = B_{N,q}^{(s)} tQ_N^{(m,m-s+q,\lambda)}(t), \tag{2.7}
$$

where s is an integer such that $q+1 \le s \le m$. The coefficients $B_{N,q}^{(s)}$ and $A_{i,N,q}^{(s)}$ $(i = 0, 1, \ldots, s)$ depend on the parameters m and λ. Precisely, they can be obtained

$$B_{N,q}^{(s+1)} = B_{N,q}^{(s)} \alpha_N^{(m-s+q)}, \quad A_{0,N,q}^{(s+1)} = A_{0,N,q}^{(s)},$$

$$\text{(2.8)} \qquad A_{i,N,q}^{(s+1)} = A_{i,N,q}^{(s)} + \frac{B_{N,q}^{(s)}}{B_{N-1,q}^{(s)}} \beta_N^{(m-s+q)} A_{i-1,N-1,q}^{(s)}, \quad (i = 1, \ldots, s),$$

$$A_{s+1,N,q}^{(s+1)} = \frac{B_{N,q}^{(s)}}{B_{N-1,q}^{(s)}} \beta_N^{(m-s+q)} A_{s,N-1,q}^{(s)},$$

with starting values given by Lemma 2.4 and $B_{N,q}^{(q+1)} = 1$.

Proof. The proof of the relation (2.7) can be given by induction.

For $s = q+1$, the relation (2.7) is equivalent to (2.4).

Using (2.2), i.e.,

$$Q_N^{(m,q,\lambda)}(t) = \alpha_N^{(q)} Q_N^{(m,q-1,\lambda)}(t) - \beta_N^{(q)} Q_{N-1}^{(m,q,\lambda)}(t),$$

for $q := m - s + q$, we have

$$\sum_{i=0}^{s} A_{i,N,q}^{(s)} Q_{N+1-i}^{(m,q,\lambda)}(t) = B_{N,q}^{(s)} \alpha_N^{(m-s+q)} t Q_N^{(m,m-s+q-1,\lambda)}(t) - \beta_N^{(m-s+q)} t Q_{N-1}^{(m,m-s+q,\lambda)}(t).$$

Applying again (2.7), for $N := N-1$, we eliminate the second term on the right side in the last equality. Thus, we obtain

$$\sum_{i=0}^{s+1} A_{i,N,q}^{(s+1)} Q_{N+1-i}^{(m,q,\lambda)}(t) = B_{N,q}^{(s+1)} t Q_N^{(m,m-(s+1)+q,\lambda)}(t),$$

where the coefficients $A_{i,N,q}^{(s+1)}$ and $B_{N,q}^{(s+1)}$ are given recursively by (2.8). □

Theorem 2.6. *The polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ satisfy the $(m+1)$-term recurrence relation*

$$\text{(2.9)} \qquad \sum_{i=0}^{m} A_{i,N,q} Q_{N+1-i}^{(m,q,\lambda)}(t) = B_{N,q} t Q_N^{(m,q,\lambda)}(t),$$

where the coefficients $B_{N,q}$ and $A_{i,N,q}$ $(i = 0, 1, \dots, m)$ depend on the parameters m and λ. Furthermore,

$$B_{N,q} = B_{N,q}^{(m)}, \qquad A_{i,N,q} = A_{i,N,q}^{(m)} \quad (i = 0, 1, \dots, m),$$

where $B_{N,q}^{(m)}$ and $A_{i,N,q}^{(m)}$ are given by Lemma 2.5.

Proof. For $s = m$, (2.7) reduces to (2.9). □

3. Special Cases

In this section we consider two Chebyshev cases: $\lambda = 1$ and $\lambda = 0$.

Case $\lambda = 1$. This is the simplest case. The recurence relations (2.2) and (2.3) reduce to

$$Q_N^{(m,q,1)}(t) = Q_N^{(m,q-1,1)}(t) - Q_{N-1}^{(m,q,1)}(t) \qquad (1 \le q \le m-1)$$

and

$$Q_N^{(m,0,1)}(t) = tQ_{N-1}^{(m,m-1,1)}(t) - Q_{N-1}^{(m,0,1)}(t),$$

respectively. Then we have the following corollary of Theorem 2.4.

Corollary 3.1. *The polynomials $Q_N^{(m,q,1)}(t)$ $(0 \le q \le m-1)$ satisfy the following recurrence relation*

$$\sum_{i=0}^{m} \binom{m}{i} Q_{N+1-i}^{(m,q,1)}(t) = tQ_N^{(m,q,1)}(t).$$

Case $\lambda = 0$. In this case we introduce the polynomials $Q_N^{(m,q,0)}(t)$ in the following way

$$Q_N^{(m,q,0)}(t) = \lim_{\lambda \to 0} \frac{Q_N^{(m,q,\lambda)}(t)}{\lambda}.$$

Then we have:

Corollary 3.2. *The polynomials $Q_N^{(m,q,0)}(t)$ $(0 \le q \le m-1)$ satisfy the following recurrence relation*

$$\sum_{i=0}^{m} \bigl(m(N+1-i) + q\bigr) \binom{m}{i} Q_{N+1-i}^{(m,q,0)}(t) = (mN - q)tQ_N^{(m,q,0)}(t).$$

4. Distribution of Zeros

According to the explicit representation of polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$, given by (2.1), we have:

Proposition 4.1. *The polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ have no negative real zeros.*

Numerical experiments for $N \leq 15$ and $m \leq 8$ suggested us to state the following conjecture:

Conjecture 4.2. *The all zeros of $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ are real, simple, and they lie in $(0, 2^m)$.*

Remark. According to the equality $p_{n,m}^{\lambda}(x) = (2x)^q Q_N^{(m,q,\lambda)}(t)$, where $t = (2x)^m$, $n = mN + q$, $N = [n/m]$ and $q \in \{0, 1, \dots, m-1\}$, we conclude that each zero τ_k $(k = 1, \dots, N)$ of the polynomial $Q_N^{(m,q,\lambda)}(t)$ generates m zeros $\xi_{k,\nu}$ $(\nu = 1, \dots, m)$ of the initial polynomial $p_{n,m}^{\lambda}(x)$, where

$$\xi_{k,\nu} = \frac{1}{2} \sqrt[m]{\tau_k} e^{i(\nu-1)\pi/m} \qquad (\nu = 1, \dots, m),$$

where $i = \sqrt{-1}$.

Example. Zeros of the polynomial $Q_5^{(6,1,1/2)}(t)$ are:

$$\tau_1 \approx 0.0252818422, \quad \tau_2 \approx 0.9626748835, \quad \tau_3 \approx 4.2829477870,$$

$$\tau_4 \approx 9.3779321447, \quad \tau_5 \approx 13.7192997676.$$

We see that $\tau_k \in (0, 2^6)$ $(k = 1, \dots, 5)$.

Thus, the zeros of the polynomial $p_{31,6}^{1/2}$ are:

$$\xi_{1,\nu} \approx 0.2708765796\, e^{i(\nu-1)\pi/6}, \quad \xi_{2,\nu} \approx 0.4968400664\, e^{i(\nu-1)\pi/6},$$

$$\xi_{3,\nu} \approx 0.6371775529\, e^{i(\nu-1)\pi/6}, \quad \xi_{4,\nu} \approx 0.7260856472\, e^{i(\nu-1)\pi/6},$$

$$\xi_{5,\nu} \approx 0.7736157957\, e^{i(\nu-1)\pi/6},$$

where $\nu = 1, \dots, 6$, and a simple zero in origin $\xi_0 = 0$.

REFERENCES

1. L. Gegenbauer: *Zur Theorie der Functionen $C_n^{\nu}(x)$.* Denkschriften der Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse **48** (1884), 293–316.

2. A. Horadam: *Polynomials associated with Chebyshev polynomials of the first kind.* Fibonacci Quart. **15** (1977), 255–257.
3. A. Horadam: *Gegenbauer polynomials revisited.* Fibonacci Quart. **23** (1985), 294–299, 307.
4. A. Horadam and S. Pethe: *Polynomials associated with Gegenbauer polynomials.* Fibonacci Quart. **19** (1981), 393–398.
5. G. V. Milovanović and G. Djordjević: *On some properties of Humbert's polynomials.* Fibonacci Quart. **25** (1987), 356–360.

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O NEKIM OSOBINAMA HUMBERTOVIH POLINOMA, II

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U našem prethodnom radu [5] definisana je i razmatrana klasa Humbertovih polinoma koji generališu dobro poznatu klasu Gegenbauerovih polinoma. Predmet ovog rada su dalja istraživanja ove klase polinoma uključujući i distribuciju nula o čemu je postavljenja i jedna hipoteza.