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SOME CONSIDERATION ABOUT GEGENBAUER AND HUMBERT POLYNOMIALS*

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Abstract. Expanding the Gegenbauer polynomial $C_n^\lambda(x)$ in powers of λ, coefficients of that expansion are the polynomials $g_{j,n}(x)$, $j = 1, \ldots, n$. In this paper we give some explicit expressions for $g_{j,n}(x)$, as well as a linear nonhomogenous differential equation of the second order, satisfying by these polynomials. Also, we consider the similar problems for the Humbert polynomials $p_{n,m}^\lambda(x)$.

1. Introduction

Investigating the Gegenbauer polynomials, S. Wrigge [3] used an unusual approach. Namely, the Gegenbauer polynomial $C_n^\lambda(x)$ is considered as a function of the parameter $\lambda > -1/2$, i.e., it is presented as

$$C_n^\lambda(x) = \sum_{j=1}^{n} g_{j,n}(x)\lambda^j, \tag{1.1}$$

where $g_{j,n}(x)$, $j = 1, 2, \ldots, n$ are polynomials of degree n. Thus,

$$\begin{aligned} C_1^\lambda(x) &= 2x\lambda, \\ C_2^\lambda(x) &= 2x^2\lambda^2 + (2x^2 - 1)\lambda, \\ C_3^\lambda(x) &= \frac{4}{3}x^3\lambda^3 + (4x^3 - 2x)\lambda^2 + \Big(\frac{8}{3}x^3 - 2x\Big)\lambda, \quad \text{etc.} \end{aligned}$$

The polynomials $g_{j,n}(x)$, $j = 1, 2, \ldots, n$ can be expressed in the form

$$g_{j,n}(x) = (-1)^{n-j} \sum_{k=0}^{M_n(j)} \frac{S_{n-k}^{(j)}}{k!(n-2k)!}(2x)^{n-2k}, \tag{1.2}$$

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where $M_n(j) = \min([n/2], n-j)$ and $S_n^{(j)}$ are the Stirling's numbers of the first kind defined by

$$x^{(n)} = x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} S_n^{(j)} x^j.$$

From (1.2) immediately follows

$$g_{1,n}(x) = \frac{2}{n} T_n(x), \tag{1.3}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind. Starting from the generating function for the polynomials $C_n^{\lambda}(x)$ it can be proved that

$$g_{2,n}(x) = 2\sum_{j=1}^{n-1} \frac{1}{j} T_j(x) \frac{1}{n-j} T_{n-j}(x). \tag{1.4}$$

For the polynomials $g_{j,n}(x)$, beside (1.1), Wrigge [3] found the generating functions

$$(-1)^j \frac{\log^j(1-2xt+t^2)}{j!} = \sum_{n=j}^{+\infty} g_{j,n}(x) t^n. \tag{1.5}$$

As a generalization of the Gegenbauer polynomials we mention the class of the Humbert's polynomials, which are defined by the generating function

$$(1-2xt+t^m)^{-\lambda} = \sum_{k=0}^{+\infty} p_{n,m}^{\lambda}(x) t^n,$$

where $m \in \mathbb{N}$ and $\lambda > -1/2$. The special cases of the Humbert polynomials are the Gegenbauer polynomials $C_n^{\lambda}(x)$ for $m = 2$, and the Horadam-Pethe polynomials $p_{n+1}^{\lambda}(x)$ for $m = 3$ (see [1–2]).

In this paper we give some explicit expressions for $g_{j,n}(x)$, which are different from (1.2), as well as a differential equation satisfying by these polynomials. Furthermore, we consider the analogous problems with Humbert polynomials $p_{n,m}^{\lambda}(x)$ given by

$$p_{n,m}^{\lambda}(x) = \sum_{j=1}^{n} h_{j,n}^{m}(x) \lambda^j.$$

2. Some Representations of $g_{i,n}(x)$

Starting from the generating function (1.5) for the polynomials $g_{j,n}(x)$ and the generating function Chebyshev's polynomials (see [3]),

$$-\frac{1}{2}\log(1-2xt+t^2)=\sum_{n=1}^{+\infty}\frac{T_n(x)}{n}t^n,$$

we find

$$\log^j(1-2xt+t^2)=(-1)^j2^j\sum_{n=j}^{+\infty}\sum_{\substack{i_1+\cdots+i_j=n\\ i_k\ge 1}}\left(\frac{T_{i_1}(x)}{i_1}\cdots\frac{T_{i_j}(x)}{i_j}\right)t^n,$$

i.e.,

$$g_{j,n}(x)=\frac{2^j}{j!}\sum_{\substack{i_1+\cdots+i_j=n\\ i_k\ge 1}}\left(\frac{T_{i_1}(x)}{i_1}\cdots\frac{T_{i_j}(x)}{i_j}\right). \tag{2.1}$$

It is easy to see that (2.1) reduces to (1.3) and (1.4) for $j=1$ and $j=2$, respectively.

Theorem 2.1. *Let* $\sigma_i^{(k)}$, $1\le i\le n-1$, *be the elementary symmetric functions of* $1,\dots,k-1,k+1,\dots,n$ *and* $\sigma_0^{(k)}\equiv 1$. *Then*

$$g_{j,n}(x)=\frac{(-1)^j}{n!}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\sigma_{n-j}^{(k)}C_n^k(x). \tag{2.2}$$

Proof. Starting with the Lagrange interpolation formula for $\lambda\mapsto C_n^\lambda(x)/\lambda$, with nodes $\lambda=k$, $k=1,2,\dots,n$, we have

$$\frac{C_n^\lambda(x)}{\lambda}=\sum_{k=1}^{n}\frac{C_n^k(x)}{k}\,\frac{\omega(\lambda)}{(\lambda-k)\omega'(k)}\,,$$

where $\omega(\lambda)=(\lambda-1)(\lambda-2)\cdots(\lambda-n)$ and $\omega'(k)=(-1)^{n-k}\binom{n}{k}^{-1}n!/k$.

Since

$$\frac{\omega(\lambda)}{\lambda-k}=\lambda^{n-1}-\sigma_1^{(k)}\lambda^{n-2}+\cdots+(-1)^{n-1}\sigma_{n-1}^{(k)},$$

where $\sigma_i^{(k)}$, $1\le i\le n-1$, are the elementary symmetric functions of $1,\dots,k-1,k+1,\dots,n$. Then

$$C_n^\lambda(x)=\frac{1}{n!}\sum_{j=1}^{n}(-1)^{n-j}\lambda^j\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}\sigma_{n-j}^{(k)}C_n^k(x),$$

wherefrom we obtain (2.2). □

Using (1.3) and (2.2), for $j=1$, we can prove the following result:

Corollary 2.2. *We have*

$$T_n(x) = -\frac{n}{2}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{C_n^k(x)}{k}\,.$$

Since $D^m T_n(x) = 2^{m-1}(m-1)!nC_{n-m}^m(x)$, $n \ge m$, formula (2.2) can be represented in terms of Chebyshev polynomials and their derivatives,

$$g_{j,n}(x) = \frac{(-1)^j}{n!}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{\sigma_{n-j}^{(k)}}{2^{k-1}(k-1)!(n+k)}\,T_{n+k}^{(k)}(x)\,.$$

A similar formula in terms of Legendre polynomials can be also given. Taking the nodes $\lambda = k+1/2$, $k = 0, 1, \ldots, n$, in the Lagrange interpolation formula for $\lambda \mapsto C_n^\lambda(x)$ we find

$$C_n^\lambda(x) = \frac{(-1)^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}C_n^{k+1/2}(x)\frac{\omega(\lambda)}{\lambda-k-1/2}\,, \tag{2.3}$$

where

$$\omega(\lambda) = \Big(\lambda-\frac{1}{2}\Big)\Big(\lambda-\frac{3}{2}\Big)\cdots\Big(\lambda-n-\frac{1}{2}\Big).$$

Let

$$\frac{\omega(\lambda)}{\lambda-k-1/2} = \sum_{j=0}^{n}(-1)^{n-j}F_{n-j}^{(k)}\lambda^j, \qquad F_0^{(k)} = 1. \tag{2.4}$$

Since

$$C_n^{k+1/2}(x) = \frac{(2k)!!}{(2k)!}P_{n+k}^{(k)}(x)\,,$$

where $P_m(x) = C_m^{1/2}(x)$ is the Legendre polynomial, then from (2.3) follows

$$C_n^\lambda(x) = \frac{1}{n!}\sum_{j=1}^{n}\sum_{k=0}^{n}(-1)^{k-j}\binom{n}{k}\frac{(2k)!!}{(2k)!}F_{n-j}^{(k)}P_{n+k}^{(k)}(x)\lambda^j,$$

wherefrom we obtain the following representation:

Theorem 2.3. *If the numbers $F_0^{(k)}, F_1^{(k)}, \ldots, F_n^{(k)}$ are defined as in* (2.4), *then*

$$g_{j,n}(x) = \sum_{k=0}^{n}(-1)^{k-j}\frac{2^k}{(n-k)!(2k)!}F_{n-j}^{(k)}P_{n+k}^{(k)}(x).$$

We note that $g_{0,n}(x) = 0$. Therefore,

$$\sum_{k=0}^{n}(-1)^{k-j}\frac{2^k}{(n-k)!(2k+1)!}P_{n+k}^{(k)}(x) = 0.$$

3. Difference and Differential Relations

Starting from the well-known recurrence relations for the Gegenbauer polynomials $C_n^\lambda(x)$ and using (1.1), Wrigge [3] proved the following relations:

$$\begin{aligned} Dg_{j,n-1}(x) &= xDg_{j,n}(x) - ng_{j,n}(x), \\ Dg_{k,n}(x) &= 2\sum_{j=k-1}^{n-1}\binom{j}{k-1}g_{j,n-1}(x), \\ (n+1)g_{j,n+1}(x) &= 2nxg_{j,n}(x) + 2xg_{j-1,n}(x) \\ &\quad - (n-1)g_{j,n-1}(x) - 2g_{j-1,n-1}(x). \end{aligned} \tag{3.1}$$

Since

$$C_n^\lambda(1) = \binom{n+2\lambda-1}{n} = \frac{2^n}{n!}\prod_{i=0}^{n-1}\Big(\lambda+\frac{i}{2}\Big),$$

we can conclude that

$$g_{1,n}(1) = \frac{2}{n} \quad \text{and} \quad g_{2,n}(1) = \frac{4}{n}\sum_{k=1}^{n-1}\frac{1}{k}\,. \tag{3.2}$$

Using the Gebenbauer differential equation and expansion (1.1), as well as the first equality in (3.1) it can be proved that the polynomial $g_{j,n}(x)$ satisfies the nonhomogenous differential equation

$$(1-x^2)y'' - xy + n^2y = 2Dg_{j-1,n-1}(x). \tag{3.3}$$

For $j=1$ this equation becomes the Chebyshev differential equation with the general solution

$$y = c_{1,n}T_n(x) + c_{2,n}\sqrt{1-x^2}\,S_{n-1}(x),$$

where $c_{1,n}$ and $c_{2,n}$ are constants, and $S_m(x)$ is the Chebyshev polynomial of the second kind. The polynomial solution for which $y(1) = 2/n$ is given by $y = g_{1,n}(x) = 2T_n(x)/n$.

For $j=2$, equation (3.3) takes the form

$$(1-x^2)y'' - xy' + n^2y = 4S_{n-2}(x),$$

wherefrom, we obtain

$$y = d_{1,n}T_n(x) + +d_{2,n}\sqrt{1-x^2}\,S_{n-2}(x) + \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k}T_{|n-2k|}(x),$$

where $d_{1,n}$ and $d_{2,n}$ are arbitrary constants. It is easy to see that

$$y = g_{2,n}(x) = \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k}\big(T_n(x) + T_{|n-2k|}(x)\big)$$

is a polynomial solution satisfying (3.2).

4. Humbert Polynomials

The polynomials $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ were defined by the generating function (see [1])

$$G_m^{\lambda}(x,t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x) t^n,$$

where $m \in \mathbb{N}$ and $\lambda > -1/2$. They can be expressed in the following explicit form

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n-mk)!} (2x)^{n-mk}. \tag{4.1}$$

Following Wrigge [3] we consider $p_{n,m}^{\lambda}(x)$ in the form

$$p_{n,m}^{\lambda}(x) = \sum_{j=1}^{n} h_{j,n}(x) \lambda^j. \tag{4.2}$$

Starting from (4.1) and (4.2), we can obtain an explicit expression for the polynomials $h_{j,n}(x)$, $j = 1, \ldots, n$. Namely,

$$h_{j,n}(x) = (-1)^{n-j} \sum_{k=0}^{M_n(j)} (-1)^{mk} \frac{S_{n-(m-1)k}^{(j)}}{k!(n-mk)!} (2x)^{n-mk}, \tag{4.3}$$

where $M_n(j) = \min([n/m], [(n-j)/(m-1)])$.

Expanding $\lambda \mapsto G_m^{\lambda}(x,t)$ in powers of λ and using (4.2) we get

$$\frac{(-1)^j}{j!} \log^j (1 - 2xt + t^m) = \sum_{n=j}^{+\infty} h_{j,n}(x) t^n. \tag{4.4}$$

If we define

$$p_{n,m}^{0}(x) = \lim_{\lambda \to 0} \frac{p_{n,m}^{\lambda}(x)}{\lambda},$$

then using (4.2) we have that $p_{n,m}^0(x) = h_{1,n}(x)$. Therefore, (4.4) for $j = 1$ becomes

$$-\log(1 - 2xt + t^m) = \sum_{n=1}^{+\infty} p_{n,m}^0(x) t^n. \tag{4.5}$$

Now, combining (4.4) and (4.5) we find another expression for $h_{j,n}(x)$. Namely,

$$h_{j,n}(x) = \frac{1}{j!} \sum_{\substack{i_1 + \cdots + i_j = n \\ i_k \geq 1}} p_{i_1,m}^0(x) \cdots p_{i_j,m}^0(x).$$

Using the similar method as in Section 2, we can prove some representations of $h_{j,n}(x)$ which are different from (4.3).

Theorem 4.1. *Let* $\sigma_i^{(k)}$, $1 \le i \le n-1$, *be the elementary symmetric functions of* $1, \dots, k-1, k+1, \dots, n$ *and* $\sigma_0^{(k)} \equiv 1$. *Then*

$$h_{j,n}(x) = \frac{(-1)^j}{n!} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \sigma_{n-j}^{(k)} p_{n,m}^{k}(x).$$

Theorem 4.2. *If the numbers* $F_0^{(k)}, F_1^{(k)}, \dots, F_n^{(k)}$ *are defined as in* (2.4), *then*

$$h_{j,n}(x) = \frac{(-1)^j}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{n-j}^{(k)} p_{n,m}^{k+1/2}(x).$$

Several recurrence relations for the polynomials $p_{n,m}^{\lambda}(x)$ can be found in [1–2].

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NEKA RAZMATRANJA OKO GEGENBAUEROVIH I HUMBERTOVIH POLINOMA

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Razvijajući Gegenbauerove polinome $C_n^{\lambda}(x)$ po stepenima od λ dobijamo polinome $g_{j,n}(x)$, $j = 1, \dots, n$, kao koeficijente tog razvoja. U radu dajemo neke eksplicitne izraze za $g_{j,n}(x)$, kao i jednu nehomogenu linearnu diferencijalnu jednačinu drugog reda koju ovi polinomi zadovoljavaju. Takođe, razmatramo i slične probleme za Humbertove polinome $p_{n,m}^{\lambda}(x)$.