FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 9 (1994), 1–5

EXPLICIT FORMULAS FOR NUMBERS OF CARLITZ AND TOSCANO

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Abstract. In this paper we give some explicit exspressions for numbers treated by Carlitz [1]. Also, we consider the similar problems for the generalized Stirling numbers of the second kind, introduced by Toscano [2].

1. Introduction

In [1] L. Carlitz introduced arrays of numbers $a_{n,t}$ and $b_{n,t}$ in the following way

$$(1.1) \qquad (x^{\lambda+\mu}D^{\mu})^n = \sum_{t=1}^{\mu(n-1)+1} a_{n,t}(x^{t+\mu-1+n\lambda}D^{t+\mu-1})$$

and

$$(1.2) \qquad (x^{\mu}D^{\lambda+\mu})^n = \sum_{t=1}^{\mu(n-1)+1} b_{n,t}(x^{t+\mu-1}D^{t+\mu-1+n\lambda}).$$

As a generalization of the Stirling numbers of the second kind, L. Toscano (see [2] and [3]) definied the numbers $K_{n,k,p}$ with the recurrence relation ($p \in \mathbb{R}^+$):

$$K_{n,1,p} = K_{n,n,p} = 1, \qquad K_{n,k,p} = K_{n-1,k-1,p} + k^p K_{n-1,k,p}.$$

In this paper we give explicit expressions for the numbers $a_{n,t}$, $b_{n,t}$, and $K_{n,k,p}$. As a consequence of that, we also give two combinatorial identities.

Received November 17, 1993.
1991 *Mathematics Subject Classification.* Primary 11B73, 11B75.
This work was supported in part by the Science Fund of Serbia under grant 0401F.

2. Representation of $a_{n,t}$ and $b_{n,t}$

For numbers $a \in \mathbb{R}$ and $k, i_s \in \mathbb{N}_0$ $(s = 0, 1, \ldots, n-1)$ we introduce the notations

$$a^{(k)} = a(a-1)\cdots(a-k+1) \quad (k > 0), \quad a^{(0)} = 1, \quad a^{(k)} = 0 \quad (k > a)$$

and

$$S_n = \sum_{s=0}^{n-1} i_s \, .$$

Theorem 2.1. *Let $n, t, \mu \in \mathbb{N}$. The explicit representation of the numbers $a_{n,t}$ defined in* (1.1) *is given by*

$$a_{n,t} = \sum_{i_1+\cdots+i_{n-1}=(n-1)\mu-t+1} \prod_{k=1}^{n-1} \binom{\mu}{i_k} \big(k(\lambda+\mu) - S_k\big)^{(i_k)}, \tag{2.1}$$

where $0 = i_0 \le i_1 \le i_2 \le \cdots \le i_{n-1} \le \mu$.

Proof. We start with the following expansion (see [4])

$$(x^r D^s)^n = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n-1} \le s} \left(\prod_{k=1}^{n-1} \binom{s}{i_k} (kr - S_k)^{(i_k)} \right) x^{nr-S_n} D^{ns-S_n},$$

where $i_0 = 0$.

If we introduce supstitution $ns - S_n = t + s - 1$ $(i_0 = 0)$, then $t \in \{1, 2, \ldots, (n-1)s + 1\}$, and

$$(x^r D^s)^n = \sum_{t=1}^{(n-1)s+1} Q_t,$$

where

$$Q_t = \sum_{i_1+\cdots+i_{n-1}=(n-1)s-t+1} \left(\prod_{k=1}^{n-1} \binom{s}{i_k} (kr - S_k)^{(i_k)} \right) x^{(r-s)n+s+t-1} D^{t+s-1}$$

and $0 = i_0 \le i_1 \le \cdots \le i_{n-1} \le s$.

For $\mu = s$ and $\lambda = r - s$, we have

$$(x^{\lambda+\mu} D^{\mu})^n = \sum_{t=1}^{\mu(n-1)+1} R_t, \tag{2.2}$$

where

$$R_t = \sum_{i_1+\cdots+i_{n-1}=(n-1)\mu+1-t} \left(\prod_{k=1}^{n-1} \binom{\mu}{i_k} (k(\lambda+\mu) - S_k)^{(i_k)} \right) x^{t+\mu-1+n\lambda} D^{t+\mu-1}.$$

So, from (1.1) and (2.2) we obtain the exspression (2.1). □

Similarly, we can formulate the following result:

Theorem 2.2. *The explicit representation for the numbers $b_{n,t}$ defined by* (1.2) *is given by*

$$b_{n,t} = \sum_{i_1+\cdots+i_{n-1}=(n-1)\mu-t+1} \prod_{k=1}^{n-1} \binom{\lambda+\mu}{i_k} (\mu k - S_k)^{(i_k)},$$

where $0 = i_0 \le i_1 \le \cdots \le i_{n-1} \le \lambda + \mu$.

The Carlitz's explicit exspression for $a_{n,t}$ is

$$(2.3)\quad a_{n+1,t} = \frac{1}{(t-1)!} \sum_{j=1}^{t} \frac{(-1)^{t-j}}{(t-j)!(j-1)!} (j+\lambda+\mu-1)^{(\mu)} \cdots (j+n\lambda+\mu-1)^{(\mu)},$$

and therefore, if we identify the right sides of (2.1) and (2.3) (substitute n with $n+1$), we get the following

Corollary 2.3. *If n, t, λ, μ are natural numbers and $0 = i_0 \le i_1 \le \cdots \le i_{n-1} \le \mu$, then the identity*

$$\sum_{i_1+\cdots+i_n=n\mu-t+1} \prod_{k=1}^{n} \binom{\mu}{i_k} (k(\lambda+\mu) - S_k)^{(i_k)} = \sum_{j=1}^{t} \frac{(-1)^{t-1}}{(t-j)!(j-1)!} (j+\lambda+\mu-1)^{(\mu)} \cdots (j+n\lambda+\mu-1)^{(\mu)}$$

is valid.

3. Representation of $K_{n,k,p}$

L. Toscano [2–3] introduced the numbers $K_{n,k,2}$ in the following way

$$(3.1)\qquad (xD)^{2n} = \sum_{i=1}^{n} K_{n,i,2} x^i D^{2i-1} x^{i-1}$$

The explicit exspression for $K_{n,k,2}$ is

$$K_{n,k,2} = \frac{2(-1)^k}{(2k)!} \sum_{i=1}^{k} (-1)^i \binom{2k}{k-i} i^{2n}.$$

But, Toscano did not give any explicit representation for the generalized numbers $K_{n,k,p}$.

Theorem 3.1. *The explicit representation for the numbers* $K_{n,k,p}$ (*with* $K_{n,1,p} = K_{n,n,p} = 1$) *is given by*

$$K_{n,k,p} = \sum \prod_{s=1}^{n-1} \binom{s - S_s}{i_s}^p, \tag{3.2}$$

where the sumation on the right is over all $i_1, \ldots, i_{n-1} \in \{0, 1\}$ *such that* $i_1 + i_2 + \cdots + i_{n-1} = n - k$ *and* $i_0 = 0$.

Proof. For $i_{n-1} \in \{0, 1\}$ we get from (3.2) that

$$\begin{aligned} K_{n,k,p} = & \sum_{i_1+\cdots+i_{n-2}=n-k} \prod_{s=1}^{n-2} \binom{s - S_s}{i_s}^p \\ & + \sum_{i_1+\cdots+i_{n-2}=n-k-1} (n-1-i_1-\cdots-i_{n-2})^p \prod_{s=1}^{n-2} \binom{s - S_s}{i_s}^p, \end{aligned}$$

i.e., $K_{n,k,p} = K_{n-1,k-1,p} + k^p K_{n-1,k,p}$, because $n - 1 - i_1 - \cdots - i_{n-2} = k$ and $n - k = (n - 1) - (k - 1)$. This completes the proof. □

For $p = 2$ we have

$$K_{n,k,2} = \sum \prod_{s=1}^{n-1} \binom{s - S_s}{i_s}^2, \tag{3.3}$$

where the sumation on the right is over all $i_1, \ldots, i_{n-1} \in \{0, 1\}$ such that $i_1 + i_2 + \cdots + i_{n-1} = n - k$ and $i_0 = 0$.

Notice that for bigger values of k ($k > n/2$) this formula is simpler than the Toscano expression (3.1). For example, if we take $k = n - 1$ then from (3.1) we find

$$K_{n,n-1,2} = \frac{2(-1)^{n-1}}{(2n-2)!} \sum_{i=1}^{n-1} (-1)^i \binom{2n-2}{n-i-1} i^{2n},$$

while representation (3.3) gives

$$K_{n,n-1,2} = \sum_{i_1+\cdots+i_{n-1}=1} \prod_{s=1}^{n-1} \binom{s-S_s}{i_s}^2 = 1^2 + 2^2 + \cdots + (n-1)^2,$$

i.e., $K_{n,n-1,2} = n(n-1)(2n-1)/6$.

From (3.1) and (3.2) we have:

Corollary 3.2. *The identity*

$$\sum_{i_1+\cdots+i_{n-1}=n-k} \prod_{s=1}^{n-1} \binom{s-S_s}{i_s}^2 = \frac{2(-1)^k}{(2k)!} \sum_{i=1}^{k} (-1)^i \binom{2k}{k-i} i^{2n}$$

is valid (*with* $i_1, i_2, \ldots, i_{n-1} \in \{0,1\}$).

Remark 3.1. Many authors have studied "new" numbers related to numbers $a_{n,t}, b_{n,t}$ and $K_{n,k,p}$. In a forthcoming paper we will analyse several results of other authors which are consequences of fundamental results of L. Carlitz and L. Toscano.

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EXPLICITNE FORMULE ZA CARLITZ-OVE I TOSCANO-OVE BROJEVE

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U radu se izvode eksplicitni izrazi za brojeve koje je definisao L. Carlitz [1]. Takođe, razmatraju se slični problemi za generalisane Stirlingove brojeve druge vrste koje je uveo L. Toscano [2].