BIT 27 (1987), 389-402

THE NUMERICAL EVALUATION OF SINGULAR INTEGRALS WITH COTH-KERNEL *

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Abstract.

Singular integrals with hyperbolic cotangent kernel present their own numerical problems because of the poles of the kernel located in the complex plane. We write such integrals as ordinary Cauchy principal value integrals involving an appropriate (nonclassical) weight function and apply quadrature methods of Gaussian and interpolatory type. The most accurate one is based on Gauss-Christoffel quadrature relative to the weight function in question. Its error is studied both by real- and complex-variable techniques. Numerical examples are given to illustrate the theory.

AMS Categories: Primary 65D30.

1. Introduction.

Cauchy principal value integrals of the form

$$(1.1)\qquad (I_{[\alpha,\beta]}\phi)(\xi) = \fint_{\alpha}^{\beta} \phi(\tau)\coth\frac{\tau-\xi}{2}\,d\tau, \qquad \alpha < \xi < \beta,$$

where $[\alpha, \beta]$ is a finite or infinite interval, arise in problems of fluid mechanics in connection with conformal mapping (see, e.g., [5, Ch. 4, §17]). The hyperbolic cotangent kernel in (1.1) does not only have a pole on the segment $[\alpha, \beta]$, as the more customary Cauchy kernel, but has infinitely many additional poles in the complex plane. This makes the numerical evaluation of (1.1) more difficult, particularly if the complex poles nearest to the real line are close to the segment $[\alpha, \beta]$, relative to its length, i.e., $2\pi \ll \beta-\alpha$. In such cases, when $[\alpha, \beta]$ is finite, we propose to use Gauss-Christoffel quadrature relative to a weight function that depends both on ξ and $[\alpha, \beta]$. Recently developed methods for generating orthogonal polynomials for nonclassical weight functions thus find a

*) The work of the first author was supported in part by the National Science Foundation under grant DCR-8320561.

Received May 1986.

useful application here. The results obtainable in this way are extremely accurate, though they require some effort. Cheaper, but correspondingly less accurate results can be obtained by Gauss-Legendre quadrature.

In Section 2 we develop the two methods based on Gauss-Christoffel and Gauss-Legendre quadrature and illustrate them on simple examples in Section 3. The remainder term of the more accurate of these quadrature methods is analyzed in Section 4, both for real-valued and holomorphic functions. An alternative, less accurate, but more stable procedure is discussed in Section 5. The case of an infinite interval $[\alpha, \beta]$ is considered in Section 6.

2. Two quadrature rules of Gaussian type.

We first assume that $[\alpha, \beta]$ is a finite interval. By a linear change of variables the integral (1.1) can then be brought to the form

$$(I_a f)(x) = \fint_{-1}^{1} f(t)\coth(a(t-x))dt, \tag{2.1}$$

where

$$a = \tfrac{1}{4}(\beta-\alpha), \qquad x = \frac{2\xi-(\alpha+\beta)}{\beta-\alpha}, \qquad -1 < x < 1, \tag{2.2}$$

and

$$f(t) = \tfrac{1}{2}(\beta-\alpha)\varphi(\tfrac{1}{2}(\alpha+\beta)+\tfrac{1}{2}(\beta-\alpha)t), \qquad -1 \leq t \leq 1. \tag{2.3}$$

It suffices, therefore, to consider the integral (2.1). Dividing and multiplying the integrand by $a(t-x)$, we can write (2.1) as a conventional Cauchy principal value integral,

$$(I_a f)(x) = \frac{1}{a}\fint_{-1}^{1} \frac{f(t)}{t-x}w(t)dt, \qquad -1 < x < 1, \tag{2.4}$$

involving the weight function

$$w(t) = w(t; a, x) = \omega(a(t-x)), \qquad \omega(u) = u\coth u. \tag{2.5}$$

Following standard procedure (cf., e.g., [2, §3.2.1]), we decompose (2.4) into a singular and regular part,

$$(I_a f)(x) = \frac{1}{a}f(x)\fint_{-1}^{1} \frac{w(t)}{t-x}dt + \frac{1}{a}\int_{-1}^{1} \frac{f(t)-f(x)}{t-x}w(t)dt,$$

and note that the former can be evaluated in closed form. The result is

$$(2.6)\qquad (I_a f)(x) = \frac{1}{a}\left\{f(x)\ln\frac{\sinh a(1-x)}{\sinh a(1+x)} + \int_{-1}^{1} g(t)w(t)dt\right\}, \quad -1 < x < 1,$$

where

$$(2.7)\qquad g(t) = g(t;x) = \begin{cases} \dfrac{f(t)-f(x)}{t-x} & \text{if } t \neq x, \\ f'(x) & \text{if } t = x. \end{cases}$$

We assume here that $f \in C^1[-1,1]$. Note that the function $\omega(u)$ in (2.5) is meromorphic, with poles at $\pm i\pi$, $\pm 2i\pi$, $\pm 3i\pi, \ldots$, and $\omega(u) \geq 1$ for real u.

The integral in (2.6) may now be approximated either by Gauss-Christoffel quadrature relative to the weight function w,

$$(2.8)\qquad \int_{-1}^{1} g(t)w(t)dt = \sum_{\nu=1}^{n} \lambda_\nu^C g(\tau_\nu^C) + R_n^C(g),$$

or by Gauss-Legendre quadrature,

$$(2.9)\qquad \int_{-1}^{1} g(t)w(t)dt = \sum_{\nu=1}^{n} \lambda_\nu^L g(\tau_\nu^L) w(\tau_\nu^L) + R_n^L(gw).$$

The nodes τ_ν and weights λ_ν required in these quadratures can be computed by standard techniques (see, e.g., [2, §5.1]) in terms of eigenvalues and eigenvectors of the Jacobi matrix

$$(2.10)\qquad J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are the coefficients in the recurrence relation

$$(2.11)\qquad \begin{aligned} &\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \qquad k = 0, 1, 2, \ldots, \\ &\pi_{-1}(t) = 0, \qquad \pi_0(t) = 1, \end{aligned}$$

for the respective (monic) orthogonal polynomials. For (2.9), the orthogonal polynomials are the Legendre polynomials and the Jacobi matrix (2.10) is explicitly known. In the case of (2.8), the Jacobi matrix must be generated numerically. This can be done, for example, by the discretized Stieltjes procedure based on Fejér quadrature (cf. [3, §2.2]). When a is not too large, this works quite well. For $a = 1$, for example, N-point Fejér quadrature with $N = 160$ yields the first 40 coefficients (i.e., α_k and β_k for $k = 0, 1, 2, \dots, 39$) accurate to 26 decimal places, regardless of the value of x. The procedure must work a little harder when a is large; to get the same accuracy when $x = 0$ and $a = 2, 4, 8, 16$, for example, requires N to be about 160, 200, 360 and 440, respectively.

We note that replacing x by $-x$, with a held fixed, changes α_k to $-\alpha_k$ and leaves β_k the same. This follows readily from the invariance of the weight function (2.5) under the substitution $t \to -t$, $x \to -x$. As a consequence, changing the sign of x results in a reflection of the nodes τ_ν^C with respect to the origin, the weights λ_ν^C being transferred unchanged. For the purpose of generating the coefficients α_k, β_k, and the quadrature rule (2.8), therefore, it suffices to consider $x > 0$.

While the quadrature rule (2.9) has the distinct advantage of being easily generated, independently of the values of the parameters x and a, it has some drawbacks that make it less attractive in certain cases. For one thing, it is rather difficult to estimate the error, be it by real variable techniques or contour integration in the complex plane, owing to the appearance of the weight function w in the remainder term R_n^L. In addition, when a is large, the first few poles $x \pm i\pi/a$, $x \pm 2i\pi/a, \dots$ of the integrand lie relatively close to the interval $[-1, 1]$, causing convergence of the Gauss-Legendre quadrature rule to slow down. For these reasons, (2.9) cannot be recommended unless a is small or moderately large, or only modest accuracy is required. Some guidelines in this regard can be obtained from Example 3.2 below.

The quadrature rule (2.8), on the other hand, must be generated afresh for each new $|x|$ and a. In return, however, one gets fast convergence (if g is smooth), irrespective of the magnitude of a. Furthermore, as will be shown in Section 4, the estimation of the error is greatly simplified.

Both formulas, in conjunction with (2.6), have one weakness in common: They require the evaluation of the function $g(t; x)$ in (2.7). If a node τ_ν happens to be very close to x, some cancellation error will result. This calls for special care in programming the function g. Another approach not subject to this inconvenience will be described in Section 5.

3. Examples.

We illustrate the relative merits of the formulae (2.8) and (2.9) in the case of the simple function $f(t) = \exp(t)$.

EXAMPLE 3.1. $I_1(x) = \displaystyle⨍_{-1}^{1} e^t \coth(t-x)\,dt, \qquad -1 < x < 1.$

This integral can be evaluated in closed form. By (2.5), (2.6) and (2.7), with $f(x) = e^x$ and $a = 1$, one has indeed

$$I_1(x) = e^x \ln \frac{\sinh(1-x)}{\sinh(1+x)} + \int_{-1}^{1} \frac{e^t - e^x}{t-x}(t-x)\coth(t-x)dt. \tag{3.1}$$

The integral on the right is

$$e^x \int_{-1}^{1} (e^{t-x}-1)\coth(t-x)dt = e^x \int_{-1}^{1} \frac{e^{2(t-x)}+1}{e^{t-x}+1}dt,$$

which, on substituting $u = \exp(t-x)$, yields an elementary integral. Evaluating it, and substituting the result in (3.1), gives

$$I_1(x) = 2\sinh 1 + e^x \ln \left\{ \frac{\sinh 2 - \sinh 2x + 2\sinh(1-x)}{\sinh 2 + \sinh 2x + 2\sinh(1+x)} \right\}. \tag{3.2}$$

Table 3.1 shows the relative errors incurred when the Gauss-Christoffel formula (2.8) is applied to the integral in (2.6) to compute $I_1(x)$ for $x = 0$. (Numbers in parentheses denote decimal exponents.)

Table 3.1. *Relative errors in approximating* $I_1(0)$ *by* (2.6), (2.8).

n	rel.error	n	rel.error
1	5.808(− 2)	6	5.913(−14)
2	7.515(− 4)	7	7.062(−17)
3	4.573(− 6)	8	6.507(−20)
4	1.605(− 8)	9	4.766(−23)
5	3.673(−11)	10	3.140(−26)

Similar results have been obtained for $x = \pm.2, \pm.4, \pm.6, \pm.8, \pm.9, \pm.99, \pm.999, \pm.9999, \pm.99999$: For fixed n, the relative error is almost constant for negative x, though slowly increasing with x, while for positive x it continues to increase slightly before decreasing to a value, at $x = .99999$, somewhat smaller than at $x = -.99999$. It is seen, therefore, that convergence of (2.8) is indeed very fast, uniformly in x.

The Gauss-Legendre formula (2.9) performs almost equally well when $x = 0$ (the relative errors being consistently about one decimal order larger than those in Table 3.1), but considerably worse (though still satisfactorily) when $x \neq 0$. The results for $x = .2$, shown in Table 3.2, are typical.

Table 3.2. *Relative errors in approximating* $I_1(.2)$ *by* (2.6), (2.9).

n	rel.error	n	rel.error
1	1.723(−1)	6	3.264(−10)
2	4.387(−3)	7	8.752(−12)
3	2.075(−5)	8	2.295(−13)
4	5.590(−7)	9	5.895(−15)
5	1.131(−8)	10	1.487(−16)

EXAMPLE 3.2. $I_a(0) = \displaystyle\fint_{-1}^{1} e^t \coth(at)dt, \qquad a > 0.$

It does not seem possible to evaluate this integral in closed form. We computed it for $a = .25, .5, 1., 2., 4., 8., 16.$, using (2.8) in (2.6). When $a = 1$, we are in the case considered in Table 3.1 and thus have the convergence behavior shown there. It was found that the same convergence behavior persists, almost identically, for all other values of a tried. Thus again, (2.8) converges very fast, uniformly in a.

The Gauss-Legendre formula (2.9), on the other hand, converges comparably fast only when $a \leq 1$, and much slower when a becomes large, for reasons already explained. Some typical results are shown in Table 3.3.

Table 3.3. *Relative errors in approximating* $I_a(0)$ *by* (2.6), (2.9).

n	$a = 1$	$a = 2$	$a = 4$	$a = 8$	$a = 16$
1	1.49(− 1)	3.26(− 1)	5.80(−1)	7.75(−1)	8.86(−1)
2	3.28(− 3)	2.93(− 3)	4.57(−2)	9.73(−2)	1.17(−1)
3	2.97(− 5)	7.54(− 4)	1.21(−2)	4.63(−2)	8.03(−2)
4	1.26(− 7)	6.08(− 5)	2.77(−3)	1.80(−2)	3.24(−2)
5	3.51(−10)	5.22(− 6)	6.55(−4)	8.34(−3)	2.35(−2)
6	6.67(−13)	4.46(− 7)	1.55(−4)	3.72(−3)	1.24(−2)
7	9.19(−16)	3.80(− 8)	3.67(−5)	1.73(−3)	8.86(−3)
8	9.60(−19)	3.23(− 9)	8.68(−6)	7.96(−4)	5.32(−3)
9	7.85(−22)	2.75(−10)	2.06(−6)	3.70(−4)	3.71(−3)
10	5.15(−25)	2.34(−11)	4.87(−7)	1.72(−4)	2.37(−3)

4. The remainder term R_n^C.

Combining (2.6) and (2.8), we can write

$$(I_a f)(x) = \frac{1}{a}\left\{ f(x) \ln \frac{\sinh a(1-x)}{\sinh a(1+x)} + \sum_{\nu=1}^{n} \lambda_\nu^C g(\tau_\nu^C) + R_n^C(g) \right\}, \quad -1 < x < 1, \tag{4.1}$$

where g is given by (2.7). The remainder R_n^C admits well-known representations, either in real variable form, involving derivatives, or in terms of contour integration in the complex plane. We briefly discuss both these representations.

4.1. *The remainder for smooth real-valued functions.*

Assuming $g \in C^{2n}[-1, 1]$, we have

$$R_n^C(g) = \frac{\|\pi_n\|_w^2}{(2n)!} g^{(2n)}(\xi), \qquad -1 < \xi < 1, \tag{4.2}$$

where $\pi_n(\cdot\,; wdt)$ is the monic nth degree orthogonal polynomial relative to the measure $w(t)dt$ and $\|\cdot\|_w$ the associated L_2-norm. For the latter we have

$$\|\pi_n\|_w^2 = \beta_0\beta_1 \cdots \beta_n, \tag{4.3}$$

where $\{\beta_k\}$ are the recursion coefficients in (2.11) for the polynomials $\{\pi_k(\cdot\,; wdt)\}$, with $\beta_0 = \int_{-1}^{1} w(t)dt$.

Now recall that $g(t)$ is the first divided difference $[t, x]f$ of f; cf. (2.7). Assuming $f \in C^{2n+1}[-1, 1]$ and following [1, Sec. 2], we write $g(t) = \int_0^1 f'(x+s(t-x))ds$ and use $2n$ differentiations and the mean-value theorem to obtain

$$g^{(2n)}(\xi) = \int_0^1 s^{2n} f^{(2n+1)}(x+s(\xi-x))ds = \frac{1}{2n+1} f^{(2n+1)}(\eta),$$

with η between x and ξ. Therefore,

$$R_n^C(g) = \gamma_n^C f^{(2n+1)}(\eta), \qquad -1 < \eta < 1, \qquad \text{where} \tag{4.4}$$

$$\gamma_n^C = \frac{\beta_0\beta_1 \cdots \beta_n}{(2n+1)!}. \tag{4.5}$$

The error constant in (4.5) is easily computed, once the recursion coefficients $\beta_0, \beta_1, \ldots, \beta_n$ have been obtained. These (except for β_n) are required anyhow to compute the Gauss-Christoffel formula (2.8). For $a = 1$ and $x = 0$, a few of these constants are shown in Table 4.1.

The relative errors in Table 3.1 are

Table 4.1. *Error constants γ_n^C in the case $a = 1$, $x = 0$.*

n	γ_n^C	n	γ_n^C	n	γ_n^C
1	1.323(− 1)	6	1.368(−13)	15	4.040(− 43)
2	1.715(− 3)	7	1.637(−16)	20	9.735(− 62)
3	1.049(− 5)	8	1.510(−19)	25	2.055(− 81)
4	3.697(− 8)	9	1.107(−22)	30	6.143(−102)
5	8.482(−11)	10	6.608(−26)	35	3.584(−123)

$\gamma_n^C e^\eta/(2\sinh 1)$, $-1 < \eta < 1$, and are thus within a factor Θ, $.1565 < \Theta < 1.157$, of γ_n^C. Very similar values of γ_n^C are obtained for other values of x, even very close to $x = 1$. When $x = 0$ and a increases, the values of γ_n^C grow slowly, but are still within approximately one order of magnitude of those in Table 4.1 when $a = 16$. Again, γ_n^C is almost constant, as a function of x, for fixed a and n.

4.2. *The remainder for holomorphic functions.*

Assuming f holomorphic in $|z| \le r$, $r > 1$, we can express the remainder in (4.1) as a contour integral extended over the circle $C_r = \{z : |z| = r\}$ (cf., e.g., [2, Eq. (3.11)]),

$$R_n^C(g) = \frac{1}{2\pi i}\int_{C_r} \frac{K_n(z; w dt)}{z - x} f(z)dz, \qquad \text{where} \tag{4.6}$$

$$K_n(z; w dt) = \frac{\varrho_n(z)}{\pi_n(z)}, \qquad \varrho_n(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z - t}\, w(t)dt, \tag{4.7}$$

$\pi_n(\cdot) = \pi_n(\cdot\,; w dt)$ being the nth degree orthogonal polynomial relative to the weight function w. There follows

$$|R_n^C(g)| \le \frac{r}{r - |x|} \max_{|z| = r} |K_n(z; w dt)| \cdot \max_{|z| = r} |f(z)|. \tag{4.8}$$

It can easily be verified from (4.7) that

$$|K_n(z; w(t; a, -x))dt| = |K_n(-\bar{z}; w(t; a, x))dt|, \tag{4.9}$$

so that changing the sign of x corresponds to a reflection in the complex plane with respect to the imaginary axis, as far as the magnitude of K_n is concerned.

It is furthermore known that [4, Thm. 3.1]

$$\max_{|z| = r} |K_n(z; w dt)| = \begin{cases} K_n(r; w dt) & \text{or} \\ |K_n(-r; w dt)|, & \end{cases} \tag{4.10}$$

in the case that $w(t)/w(-t)$ is nondecreasing or nonincreasing on $[-1, 1]$, respectively. We now show, in part by numerical computation, that *if* $a < a^*$, *where* $a^* = 1.33803698\ldots$ *is the unique positive root of*

$$\frac{4a^2\cosh 2a}{(\sinh 2a)^2} = 1, \tag{4.11}$$

then the first relation in (4.10) *holds if* $x < 0$ *and the second if* $x > 0$.

In view of (4.9) it suffices to consider $x > 0$. To check the required monotonicity property, we examine the logarithmic derivative of $w(t)/w(-t)$,

$$\frac{d}{dt}\left\{\log\frac{w(t)}{w(-t)}\right\} = \frac{2x}{t^2-x^2} + 2a\left\{\frac{1}{\sinh 2a(t+x)} - \frac{1}{\sinh 2a(t-x)}\right\}. \tag{4.12}$$

This is clearly an even function of t, so that it suffices to consider $t \geq 0$. Assume first $0 \leq t < x$. Since $\sinh u > u$ for $u > 0$, the right side of (4.12) is less than

$$\frac{2x}{t^2-x^2} + 2a\left\{\frac{1}{2a(t+x)} + \frac{1}{2a(x-t)}\right\} = \frac{2x}{t^2-x^2} + \frac{2x}{x^2-t^2} = 0,$$

so that $w(t)/w(-t)$ is nonincreasing. To show the same for $0 < x < t \leq 1$, we note by an elementary computation that the logarithmic derivative (4.12) is negative if

$$\frac{(\sinh u/u)\quad \cosh v}{[\sinh(u+v)/(u+v)][\sinh(v-u)/(v-u)]} > 1, \qquad 0 < u < v, \tag{4.13}$$

where $u = 2ax$, $v = 2at$. Numerical computations reveal that the function on the left of (4.13), as u varies between 0 and v, either decreases monotonically, or changes from an increasing to a decreasing function, the limit as $u \uparrow v$ being obviously 1. The inequality (4.13) therefore holds, if it holds for $u = 0$, i.e., if

$$\frac{v^2 \cosh v}{\sinh^2 v} > 1, \qquad v = 2at. \tag{4.14}$$

Since $t \leq 1$, we want (4.14) to hold for $v = 2a$, which is equivalent to $a < a^*$, where a^* is the positive root of (4.11). Thus, under this assumption, $w(t)/w(-t)$ is nonincreasing on $[-1, 1]$, hence the second relation in (4.10) holds if $x > 0$. By (4.9), the first relation holds if $x < 0$. If $x = 0$, either formula in (4.10) holds, since $w(t)$ is an even function in this case.

In Table 4.2 we display $K_n(r; wdt)$ in the case $a = 1$, $x = 0$, for selected values of r. The numbers were obtained by backward recursion, as described in [4, Sec. 4].

Table 4.2. $K_n(r; wdt)$ *for* $w(t) = t \coth t$, $n = 1(1)10$.

n	$r = 1.5$	$r = 2.0$	$r = 3.0$	$r = 5.0$	$r = 10.0$	$r = 20.0$
1	3.323(−1)	1.181(− 1)	3.161(− 2)	6.514(− 3)	7.990(− 4)	9.941(− 5)
2	5.242(−2)	9.026(− 3)	9.768(− 4)	6.924(− 5)	2.084(− 6)	6.452(− 8)
3	7.967(−3)	6.697(− 4)	2.961(− 5)	7.266(− 7)	5.381(− 9)	4.149(−11)
4	1.188(−3)	4.894(− 5)	8.853(− 7)	7.527(− 9)	1.372(−11)	2.635(−14)
5	1.757(−4)	3.554(− 6)	2.633(− 8)	7.757(−11)	3.481(−14)	1.665(−17)
6	2.588(−5)	2.571(− 7)	7.806(−10)	7.970(−13)	8.805(−17)	1.049(−20)
7	3.801(−6)	1.857(− 8)	2.310(−11)	8.175(−15)	2.223(−19)	6.595(−24)
8	5.575(−7)	1.339(− 9)	6.827(−13)	8.375(−17)	5.607(−22)	4.143(−27)
9	8.167(−8)	9.647(−11)	2.016(−14)	8.573(−19)	1.413(−24)	2.600(−30)
10	1.195(−8)	6.946(−12)	5.951(−16)	8.770(−21)	3.559(−27)	1.631(−33)

We note that monotonicity of $w(t)/w(-t)$ is only a sufficient condition for one of the formulae (4.10) to hold, and is by no means necessary. Limited numerical evidence (for $a = 2, 4, 8, 16$; $x = .2, .4, .6, .8$; $r = 1.5, 2, 3, 5$; and $n = 1(1)10$) indeed suggests that (4.10) *always* holds for $x < 0$ and $x > 0$, respectively, regardless of the value of a.

5. Procedures based on interpolatory quadratures.

The inconvenience of having to compute the divided difference (2.7) can be circumvented if an n-point interpolatory quadrature rule is employed based on the nodes τ_ν^C of (2.8). While this formula has only polynomial degree of exactness n, not $2n$ as (2.8), cf. (4.4), it can be implemented in a stable manner (see, e.g., [2, §3.2.3] or [1, Sec. 3]).

With $p_{n-1}^C(\cdot) = p_{n-1}^C(f;\cdot)$ denoting the polynomial of degree $\leq n-1$ interpolating f at the nodes τ_ν^C, $\nu = 1, 2, \ldots, n$, we have

$$(5.1) \qquad (I_a f)(x) = (I_a p_{n-1}^C)(x) + R_n^I(f),$$

where the first term on the right is the desired approximation, and the second the remainder. For the latter, assuming $f \in C^{2n+1}[-1, 1]$, we have the following representation ([1, Eq. (2.9)]),

$$(5.2) \qquad R_n^I(f) = \frac{1}{a}\left\{\gamma_n^I(x) f^{(n)}(\eta_1) + \gamma_n^C f^{(2n+1)}(\eta_2)\right\}, \qquad -1 < \eta_1, \eta_2 < 1, \qquad \text{where}$$

$$(5.3) \qquad \gamma_n^I(x) = \frac{1}{n!} \int_{-1}^{1} \frac{\pi_n(t; wdt)}{t-x} w(t)dt,$$

and γ_n^C is as defined in (4.5). Numerical values of γ_n^C have been given in Table 4.1; selected numerical values of $\gamma_n^I(x)$ in the case $a = 1, x = .2$ are shown in Table 5.1. Similar values of $|\gamma_n^I|$

Table 5.1. *Selected values of $\gamma_n^I(x)$ for $a = 1$, $x = .2$.*

n	γ_n^I	n	γ_n^I	n	γ_n^I
1	2.112(0)	6	1.001(−4)	15	1.120(−16)
2	2.924(−1)	7	−2.440(−7)	20	1.598(−24)
3	−7.186(−2)	8	−4.551(−7)	25	4.001(−33)
4	−9.845(−3)	9	−9.263(−9)	30	−1.972(−42)
5	5.179(−4)	10	1.083(−9)	35	−8.549(−51)

prevail for other values of x, even very close to $x = 1$, except when $x = 0$ and n is even, in which case $\gamma_n^I(0) = 0$ and (5.1) indeed reduces to (2.8). Numbers of roughly the same order of magnitude were observed for other values of a as large as $a = 16$. Note also that $|\gamma_n^I(-x)| = |\gamma_n^I(x)|$.

To compute the approximation in (5.1), one expresses $p_{n-1}^C(f;\cdot)$ in terms of the orthogonal polynomials $\pi_k(\cdot) = \pi_k(\cdot\,; wdt)$,

$$(5.4)\qquad p_{n-1}^C(f;t) = \sum_{k=0}^{n-1} a_k^C \pi_k(t), \qquad \text{where}$$

$$(5.5)\qquad a_k^C = \frac{1}{\|\pi_k\|_w^2} \int_{-1}^{1} p_{n-1}^C(f;t)\pi_k(t)w(t)dt, \qquad k = 0, 1, \ldots, n-1.$$

Applying the Gauss-Christoffel formula (2.8) to the integral on the right yields (as is well known)

$$(5.6)\qquad a_k^C = \frac{1}{\|\pi_k\|_w^2} \sum_{\nu=1}^{n} \lambda_\nu^C \pi_k(\tau_\nu^C) f(\tau_\nu^C), \qquad k = 0, 1, \ldots, n-1.$$

Substituting (5.4) in (2.6) then gives

$$(I_a p_{n-1}^C)(x) = \frac{1}{a} \sum_{k=0}^{n-1} a_k^C \left\{ \pi_k(x) \ln \frac{\sinh a(1-x)}{\sinh a(1+x)} + \int_{-1}^{1} \frac{\pi_k(t) - \pi_k(x)}{t-x} w(t)dt \right\}.$$

Here, the integral on the right is the "polynomial of the second kind" associated with wdt,

$$\sigma_k(x) = \sigma_k(x; wdt) = \int_{-1}^{1} \frac{\pi_k(t) - \pi_k(x)}{t-x} w(t)dt.$$

It is well known that $\{\sigma_k(x)\}$ satisfies the same recurrence relation (2.11) as $\{\pi_k(x)\}$, i.e.,

$$(5.7)\qquad y_{k+1} = (x - \alpha_k)y_k - \beta_k y_{k-1}, \qquad k = 0, 1, 2, \ldots,$$

where $y_{-1} = 0$, $y_0 = 1$ for $\{\pi_k(x)\}$, and $y_{-1} = -1$, $y_0 = 0$ for $\{\sigma_k(x)\}$. (As before, we assume $\beta_0 = \int_{-1}^{1} w(t)dt$.) Therefore,

$$(5.8)\qquad (I_a p_{n-1}^C)(x) = a^{-1} \sum_{k=0}^{n-1} a_k^C \tau_k(x),$$

where $\{\tau_k(x)\}$ is the solution of (5.7) with

$$(5.9)\qquad \tau_{-1}(x) = -1, \qquad \tau_0(x) = \ln \frac{\sinh a(1-x)}{\sinh a(1+x)},$$

and the coefficients a_k^C are given by (5.6). The sum (5.8) is conveniently evaluated by Clenshaw's algorithm.

A similar procedure can be developed using interpolation at the nodes τ_ν^L and employing the Legendre polynomials $\pi_k^L(\cdot) = \pi_k(\cdot\,; dt)$ in place of the orthogonal polynomials $\pi_k(\cdot\,; w dt)$. Unfortunately, this procedure requires the "modified" moments

$$m_k = \int_{-1}^{1} \pi_k^L(t) w(t) dt, \qquad k = 0, 1, 2, \ldots, \tag{5.10}$$

which are not easily computed, short of again using the quadrature rule (2.8). Specifically, with $p_{n-1}^L(f;\cdot)$ denoting the polynomial of degree $\le n-1$ interpolating f at the nodes τ_ν^L, $\nu = 1, 2, \ldots, n$, one now has

$$(I_a p_{n-1}^L)(x) = \sum_{k=0}^{n-1} a_k^L \tau_k^L(x), \tag{5.11}$$

where a_k^L are as in (5.6), with superscripts "C" replaced by "L" and π_k replaced by π_k^L, and where $\{\tau_k^L(x)\}$ is the solution of the inhomogeneous difference equation

$$z_{k+1} = (x - \alpha_k^L) z_k - \beta_k^L z_{k-1} + m_k, \qquad k = 0, 1, 2, \ldots, \tag{5.12}$$

corresponding to initial values

$$\tau_{-1}^L(x) = 0, \quad \tau_0^L(x) = \ln \frac{\sinh a(1-x)}{\sinh a(1+x)}. \tag{5.13}$$

The coefficients $\alpha_k^L = 0$, β_k^L are now those in the recurrence relation (2.11) for the (monic) Legendre polynomials (with $\beta_0^L = 2$).

6. Infinite interval.

In applications one usually encounters the integral (1.1) extended over the whole real line, i.e.,

$$\begin{aligned} (I_{[-\infty,\infty]}\varphi)(\xi) &= \int_{-\infty}^{\infty} \varphi(\tau) \coth \frac{\tau - \xi}{2} d\tau \\ &= \lim_{T\to\infty} \int_{-T}^{T} \varphi(\tau) \coth \frac{\tau - \xi}{2} d\tau. \end{aligned} \tag{6.1}$$

For this integral to be meaningful, one must assume that

$$\varphi(\tau) \to \varphi_\infty \text{ as } \tau \to \pm\infty \tag{6.2}$$

for some constant φ_∞. Let $[\alpha, \beta]$ be an interval centered at ξ,

$$\xi-\alpha = \beta-\xi = (\beta-\alpha)/2. \tag{6.3}$$

For $T > \max(-\alpha, \beta)$ write

$$(J_T\phi)(\xi) = \left(\int_{-T}^{\alpha} + \int_{\beta}^{T}\right)\phi(\tau)\coth\frac{\tau-\xi}{2}\,d\tau, \tag{6.4}$$

so that

$$(I_{[-\infty,\infty]}\phi)(\xi) = \int_{\alpha}^{\beta}\phi(\tau)\coth\frac{\tau-\xi}{2}\,d\tau + \lim_{T\to\infty}(J_T\phi)(\xi).$$

We approximate the limit on the right (assumed to exist) by

$$\lim_{T\to\infty}(J_T\phi_\infty)(\xi) = \lim_{T\to\infty} 2\phi_\infty \ln\frac{\sinh\frac{1}{2}(T-\xi)}{\sinh\frac{1}{2}(T+\xi)} = -2\phi_\infty\xi,$$

and write

$$(I_{[-\infty,\infty]}\phi)(\xi) = \int_{\alpha}^{\beta}\phi(\tau)\coth\frac{\tau-\xi}{2}\,d\tau - 2\phi_\infty\xi + (R\phi)(\xi), \tag{6.5}$$

where

$$(R\phi)(\xi) = \lim_{T\to\infty}\{(J_T\phi)(\xi)-(J_T\phi_\infty)(\xi)\} \tag{6.6}$$

denotes the error term. The integral on the right of (6.5), after transformation to standard form,

$$\int_{\alpha}^{\beta}\phi(\tau)\coth\frac{\tau-\xi}{2}\,d\tau = \int_{-1}^{1} f(t)\coth(at)\,dt,$$

$$a = \tfrac{1}{4}(\beta-\alpha),\quad f(t) = \tfrac{1}{2}(\beta-\alpha)\phi(\xi+\tfrac{1}{2}(\beta-\alpha)t),$$

can be computed by the methods of Sections 2 and 5.

In order to estimate the error $(R\phi)(\xi)$ in (6.5), we assume that ϕ approaches ϕ_∞ exponentially fast and that $[\alpha,\beta]$ has been chosen such that

$$|\phi(\tau)-\phi_\infty| \le \delta\exp(-|\tau-\xi|),\qquad \delta > 0, \tag{6.7}$$

for all $\tau < \alpha$ and all $\tau > \beta$. Then, using (6.4), there follows

$$|(J_T\phi)(\xi)-(J_T\phi_\infty)(\xi)| = |(J_T(\phi-\phi_\infty))(\xi)|$$

$$\le 2\delta\left\{\ln\frac{1-e^{-(T-\xi)}}{1-e^{-(\beta-\xi)}} + \ln\frac{1-e^{-(T+\xi)}}{1-e^{-(\xi-\alpha)}} - \tfrac{1}{2}e^{-(\beta-\xi)} - \tfrac{1}{2}e^{-(\xi-\alpha)}\right\},$$

hence, as $T \to \infty$, by virtue of (6.3) and (6.6),

$$|(R\phi)(\xi)| \le 2\delta\{2\ln[(1-e^{-(\beta-\alpha)/2})]^{-1} - e^{-(\beta-\alpha)/2}\} < \frac{2\delta}{e^{(\beta-\alpha)/2}-1}. \tag{6.8}$$

The error, therefore, can be made arbitrarily small by choosing δ small enough and/or $[\alpha, \beta]$ large enough.

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