Invited paper

QUADRATURE PROCESSES FOR SEVERAL TYPES OF QUASI-SINGULAR INTEGRALS APPEARING IN THE ELECTROMAGNETIC FIELD PROBLEMS

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Abstract: The paper shows one very general, exact and simple procedure for numerical integration of one class of quasi-singular integrals appearing in the convenient formulas of electromagnetic field components and potential functions of linear source distributions.

Key words: Quasi-singular integral, Numerical integration, Electromagnetic field.

INTRODUCTION

In the electromagnetic field theory and its applications it is often necessary to numerical calculate the value of nonelementer quasi-singular integrals having the following form

$$\mathrm{I} = \int_0^h \mathrm{f}(z')\,\mathrm{K_n}(z',z)\,\mathrm{d}z', \tag{1}$$

where:

$f(z')$ is a continuous function, which defines the electromagnetic field linear sources distribution (distribution of line charge and current on the axis of the thin linear conductor);

$K_n(z',z) = \dfrac{\mathrm{e}^{-\mathrm{j}kR}}{R^n}$, $n = 1,2$ and 3, is the kernel;

$k = \omega\sqrt{\varepsilon\mu}$ is phase constant;

ε and μ is electric and magnetic permittivity;

ω is angular frequency;

$R = \sqrt{r^2 + (z' - z)^2}$ is the distance between the field point and the source element on the linear conductor axis; and

$\mathrm{j} = \sqrt{-1}$ is imaginary unit.

For example, the magnetic vector potential of the thin linear conductor from Fig. 1 can be expressed as

$$\boldsymbol{A} = A\hat{\boldsymbol{z}}, \quad A = \frac{\mu}{4\pi}\int_0^h I(z')K_1(z',z)\,\mathrm{d}z', \tag{2}$$

where:

$I(z')$ defines equivalent current distribution on the linear conductor axis; and

r, θ, z are cylindrical coordinates and $\hat{z}$ presents the unit vector.

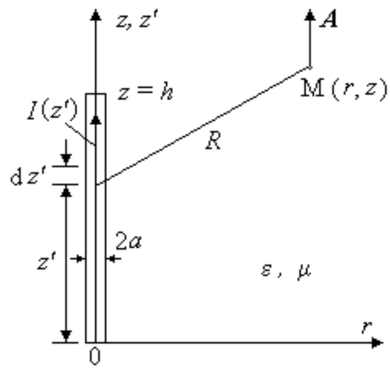

Fig. 1 – *Thin linear conductor.*

Using the standard procedure from the electromagnetic field theory, the electric potential, φ, and vectors of the electric, $\boldsymbol{E}$, and the magnetic field, $\boldsymbol{H}$, can be expressed in the form:

$$\varphi = \mathrm{j}\frac{v}{k}\operatorname{div}\boldsymbol{A} = \mathrm{j}\frac{v}{k}\frac{\partial A}{\partial z}, \; v = \frac{1}{\sqrt{\varepsilon\mu}}; \tag{3}$$

$$\boldsymbol{E} = -\operatorname{grad}\varphi - \mathrm{j}\omega\boldsymbol{A}; \tag{4}$$

and

$$\boldsymbol{H} = \frac{1}{\mu}\operatorname{rot}\boldsymbol{A}. \tag{5}$$

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Direct application of quadrature formulas

Because of the value r, representing radial distance from the linear conductor or the radius of the linear conductor, a, is not large, $10^{-3} < a/h < 10^{-1}$, it is evident that the kernel in formulas (2-4) has quasi-singular qualities when $z' = z$, including critical singularities for $z' = z \pm \mathrm{j}r$, and then the numerical integration of any kind produces significant error in numerical quadrature process, especially for $n = 3$, when the quasi-singular behaviors of sub-integral functions are the biggest. These problems are not neglectable in the numerical solving of antenna integral equations, when it is necessary to attend the biggest accuracy in numerical integration of several integrals having form (1) [1-9].

For example, a direct application of the *m*-point Gauss-Legendre quadrature formula gives the results as it is presented in Tables I, II and III.

Table I

The values of integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 10.$

m	Real part	Imaginary part
10	0.00008194794912479070	0.00003870825367223831
20	0.00008194794912479042	0.00003870825367223794
30	0.00008194794912479042	0.00003870825367223794

Table II

The values of integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 0.01.$

m	Real part	Imaginary part
10	0.178043490752242	- 0.568581404600254
20	2.139736200154773	- 1.115029648166676
30	4.700446741682397	- 1.583837600765169
40	7.441156273846891	- 1.961763534784023
50	10.01583755011203	- 2.251478748994386
100	17.18799932710139	- 2.850929071836605

Table III

The values of integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 0.001.$

m	Real part	Imaginary part
10	0.21126085996615	- 0.5897827082473286
20	2.44937999083527	- 1.2023012110631518
30	5.93703593483708	- 1.8140469298328049
40	10.70926194694546	- 2.4243638622539227
50	16.76656449035618	- 3.0327354621423115
100	65.77059173194277	- 6.0288471658697205

The exact value of integral from Table III is

1953.2057527904824 – j30.619089018279530.

It is evident that the values for small radius of the linear conductor are not correct, so a direct application of such quadrature formulas is not applicable for numerical evaluation of the integral (1).

Several methods for calculation of integral (1)

In order to eliminate these difficulties (slowly convergence of quadratures formulas, much computation work and numerical instability), in the cases when $f(z')$ can be expressed in a polynomial form, the following procedures are very useful:

1. Using singularity extraction method [3], the integral (1) can be presented in the form

$$I = \int_0^h f(z') \frac{\mathrm{e}^{-\mathrm{j}kR} - 1}{R^n} \mathrm{d}z' + \int_0^h \frac{f(z')}{R^n} \mathrm{d}z'. \tag{6}$$

The first integral is not elementary, but it can be very exactly numerically calculated using any quadratures, because sub-integral function is without singularities. The second term can be evaluated in the closed form, so the quasi-singularity in integrand has not influence to the realized accuracy.

2. In the case $n = 1$, semi-polynomial approximation of the kernel of the integral (6),

$$\frac{\mathrm{e}^{-\mathrm{j}kR} - 1}{R} \approx \sum_{m=0}^{M} A_m(z) z'^m , \tag{7}$$

gives the results in a closed form [10, 11]. The unknown functions $A_m(z)$ can be very easy determined using point matching method.

3. Very good results can be achieved by expanding the exponential part of the kernel of (1) in Taylor's series [12],

$$\mathrm{e}^{-\mathrm{j}kR} = \sum_{m=0}^{\infty} \frac{(-\mathrm{j}kR)^m}{m!}, \tag{8}$$

so that

$$\mathrm{I} = \sum_{m=0}^{\infty} \frac{(-\mathrm{j}k)^m}{m!} \int_0^h f(z') R^{m-n} \mathrm{d}z'. \tag{9}$$

The integrals in (9) can be given in the closed form using recurrent relations [16].

In this paper a new numerical approach for numerical calculation of the presented integral, when $f(z')$ is an arbitrary continuous function, is proposed. This procedure is based on a convenient transformation of the integral (1) into form for which the singularity influence is not significant and application of the standard Gauss-Legendre quadratures. This method is very exact, fast and general, even in the cases when the values for k and n are large and the radial distance r is small, and it can be successfully extended to the other problems of electromagnetic field theory.

CONSTRUCTION OF A QUADRATURE METHOD

Let f be an integrable function on $[0, h]$ Our method uses three successive transformations of the given integral (1).

1. Using transformation

$$\tau = \operatorname{arcsh}\frac{z'-z}{r}, \qquad (10)$$

the two existing critical singularities $z \pm \mathrm{j}r$ (Fig. 2) map to the poles of the order $n-1$ (Fig. 3),

$$\sigma_\nu = \mathrm{j}\pi(\nu + \tfrac{1}{2}), \ \ \nu = 0, \pm 1, \ldots.$$

Notice that for $n = 1$ the transformed integrand has no poles.

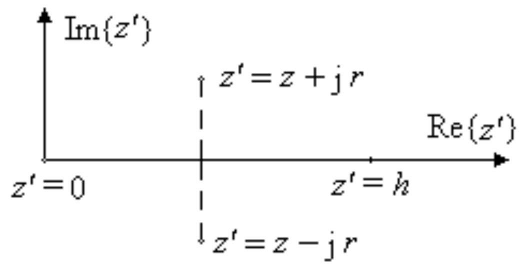

Fig. 2 - *Critical singularities in* z' *plane.*

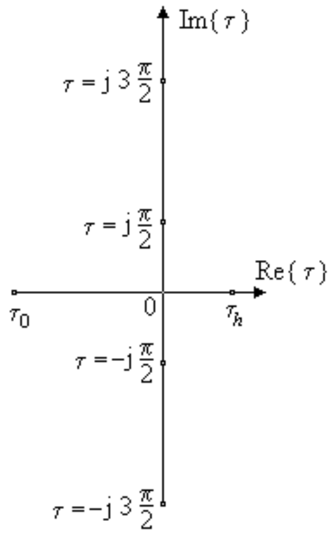

Fig. 3 - *Critical singularities in* τ *plane for* $n = 3$.

The corresponding lower and upper bound in the transformed integral,

$$\tau_0 = -\operatorname{arcsh}\frac{z}{r} \qquad (11)$$

and

$$\tau_h = \operatorname{arcsh}\frac{h-z}{r}, \qquad (12)$$

are presented in Fig. 4, when $a = 0.01$ and $h = 1$. As we can see, the segment of integration $[\tau_0, \tau_h]$ is an asymmetric interval around origin.

2. In order to reduce the influence of the poles σ_ν $(\nu = 0, \pm 1, \ldots)$, when $n > 1$, we use the transformation displayed in Fig. 5. Precisely, it means the following: If $|\tau_0| < \tau_h$ we take

$$\int_{\tau_0}^{\tau_h} F(t)\,\mathrm{d}t = \int_0^{|\tau_0|}\left[F(t) + F(-t)\right]\mathrm{d}t + \int_{|\tau_0|}^{\tau_h} F(t)\,\mathrm{d}t, \qquad (13)$$

and for $|\tau_0| > \tau_h$ we put

$$\int_{\tau_0}^{\tau_h} F(t)\,\mathrm{d}t = \int_{\tau_0}^{-\tau_h} F(t)\,\mathrm{d}t + \int_0^{\tau_h}\left[F(t) + F(-t)\right]\mathrm{d}t. \qquad (14)$$

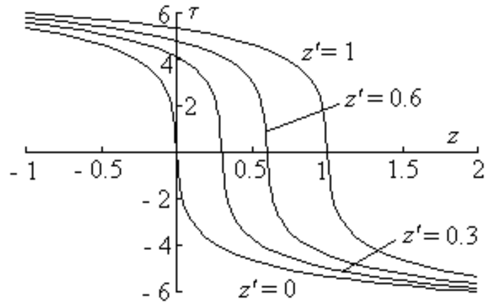

Fig. 4 - τ *versus* z *for* $a = 0.01$, $h = 1$ *and different values of* $z' = 0$, 0.3, 0.6, 1.

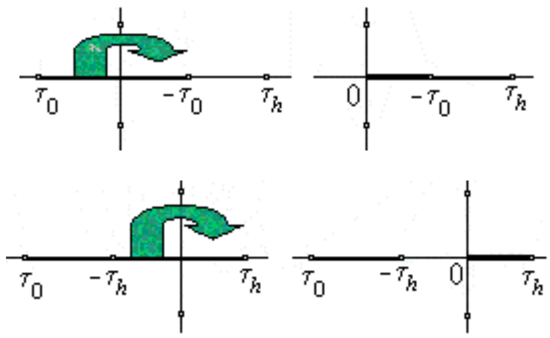

Fig. 5 - *Second transformation.*

3. Finally, the third transformation in reduction of both integrals (from 2) to the interval [0,1] and an application of the Gauss-Legendre quadratures formula.

Let

$$0 = h_0 < h_1 < \cdots < h_N = h, \qquad (15)$$

$$\tau_i := \operatorname{arcsh}\left(\frac{h_i - x}{a}\right) \quad (i = 0, 1, \ldots, N), \qquad (16)$$

$$\tau^i := \tau^i(t) = (\tau_i - \tau_{i-1})t + \tau_{i-1} \qquad (17)$$

and

$$R = \sqrt{a^2 + (x' - x)^2} = a\cosh\tau\,. \qquad (18)$$

Then

$$I = \int_0^h f(z')K_n(z', z)\,\mathrm{d}z' = \qquad (19)$$

$$= \sum_{i=1}^{N}(\tau_i - \tau_{i-1})\int_0^1 \frac{\mathrm{e}^{-\mathrm{j}ka\cosh\tau}}{(a\cosh\tau)^{n-1}} f(x + a\sinh\tau)\,\mathrm{d}t\,.$$

An application of the $m-$ point Gaussian quadrature formula gives

$$I_m \cong \sum_{\nu=1}^{m} A_\nu \sum_{i=1}^{N} \frac{(\tau_i-\tau_{i-1}) f(x+a\sinh\tau^i(t_\nu))}{[a\cosh\tau^i(t_\nu)]^{n-1}} \mathrm{e}^{-\mathrm{j}ka\cosh\tau^i(t_\nu)}. \quad (20)$$

Here, $(t_\nu, A_\nu),\ \nu = 1,\dots,m$ are Gaussian nodes and weights.

NUMERICAL RESULTS

The proposed method can be used for numerical solving of several integrals having different sub-integral functions, for example:

$$f(z') = z'^{p}; \quad (21)$$

$$f(z') = |z' - 0.6|^{p}; \quad (22)$$

$$f(z') = \sin(p\pi z'); \quad (23)$$

and

$$f(z') = |\sin(p\pi z')|, \quad (24)$$

where p is an arbitrary constant, $0 \le p \le 10$.

The relative error at the point z is

$$G_m = G_m(z) = \left|\frac{I_m - I}{I}\right|, \quad (25)$$

where I denotes the exact integral value and I_m is an approximation, determined by m-point Gauss-Legendre quadrature formula.

Maximal relative error on the segment $0 \le z \le 1$ is

$$g_{\max}(m) = \max_{0\le x\le 1} G_m(z). \quad (26)$$

The obtained results are presented in Tables IV – X.

Table IV

The values of the integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 10$.

m	Real part	Imaginary part
5	0.00008194795363948043	0.00003870826512259681
10	0.00008194794912479042	0.00003870825367223794
15	0.00008194794912479042	0.00003870825367223794

Table V

The values of the integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 0.01$.

m	Real part	Imaginary part
10	19.03038606937039	- 2.957895423422053
15	19.03032517164717	- 2.957222788242857
20	19.03032516863204	- 2.957222844450848
25	19.03032516863276	- 2.957222844444242
30	19.03032516863276	- 2.957222844444243

Table VI

The values of the integral (1) for
$f(z') = z'^{10}, h = 1, k = 10, n = 3, z = 0.5$ *and* $a = 0.001$.

m	Real part	Imaginary part
10	1953.187453906310	- 30.61892353598636
15	1953.205701496044	- 30.61909022441759
20	1953.205752721126	- 30.61908901583803
25	1953.205752790376	- 30.61908901828122
30	1953.205752790482	- 30.61908901827953
35	1953.205752790482	- 30.61908901827953
40	1953.205752790482	- 30.61908901827953

The exact value of the integral from Table VI is

1953.20575279048249487422275236 –
– j30.6190890182795304393932361547.

Table VII

Maximal relative error for
$f(z') = z'^{10}, n = 1, m = 30, 40.$

a	$k = 0$	$k = 1.5$	$k = 3$	$k = 10$
10	9.55(-16)	1.38(-15) 1.89(-15)	1.31(-14) 1.16(-15)	3.97(-15) 5.62(-14)
10^{-2}	0.(-15)	0.(-15) 0.(-15)	1.84(-15) 0.(-15)	2.55(-14) 7.72(-14)
10^{-3}	0.(-15)	0.(-14) 0.(-15)	1.45(-14) 0.(-14)	2.23(-14) 3.75(-14)

Table VIII

Maximal relative error for
$f(z') = z'^{10}, n = 2, m = 30, 40.$

a	$k = 0$	$k = 1.5$	$k = 3$	$k = 10$
10	7.20(-16)	1.26(-15) 2.04(-15)	1.30(-14) 1.09(-15)	3.45(-15) 5.56(-14)
10^{-2}	0.(-15)	0.(-15) 0.(-15)	3.98(-14) 0.(-15)	1.23(-14) 5.73(-14)
10^{-3}	0.(-15)	0.(-14) 0.(-15)	2.70(-14) 0.(-14)	6.00(-14) 1.12(-14)

Table IX

Maximal relative error for
$f(z') = z'^{10}, n = 3, m = 30, 40.$

a	$k = 0$	$k = 1.5$	$k = 3$	$k = 10$
10	1.20(-15)	9.78(-16) 1.74(-15)	9.78(-15) 9.37(-16)	3.64(-15) 3.73(-14)
10^{-2}	0.(-15)	0.(-15) 0.(-15)	3.58(-14) 0.(-15)	9.26(-14) 2.86(-14)
10^{-3}	0.(-15)	0.(-14) 0.(-14)	1.69(-14) 0.(-14)	8.98(-14) 0.(-14)

Table X

The values of the integral (1) for $f(z') = |z' - 0.6|^5$, $h = 1$, $k = 3,\ n = 2,\ z = 0.5$ *and* $a = 0.01$.

Critical point $z' = 0.6$ is ignored.		
m	Real part	Imaginary part
10	0.02847674799433376	-0.04943542548526180
20	0.02847684964282064	-0.04943535125031928
30	0.02847684868818208	-0.04943535101468958
40	0.02847684860546104	-0.04943535099065409
50	0.02847684859188430	-0.04943535098647378
60	0.02847684858869295	-0.04943535098546171
70	0.02847684858775104	-0.04943535098515739
Critical point $z' = 0.6$ is included in the method.		
m	Real part	Imaginary part
10	0.02847839229914865	-0.04944343706203532
20	0.02847684838746965	-0.04943535098520644
30	0.02847684858713270	-0.04943535098497954
40	0.02847684858721351	-0.04943535098497954
50	0.02847684858721347	-0.04943535098497954

Table XI

The values of the integral (1) for $f(z') = |z' - 0.6|^5$, $h = 1$, $k = 3,\ n = 2,\ z = 0$ *and* $a = 0.01$.

(Critical point $z' = 0.6$ *is not ignored.)*

m	Real part	Imaginary part
10	10.19566587675235	-0.5995567086084155
20	10.19566566029846	-0.5995568370502992
30	10.19566566029843	-0.5995568370502994
40	10.19566566029843	-0.5995568370502994

Table XII

The values of the integral (1) for $f(z') = |z' - 0.6|^5$, $h = 1$, $k = 10,\ n = 3,\ z = 1$ *and* $a = 0.001$.

(Critical point $z' = 0.6$ *is not ignored.)*

m	Real part	Imaginary part
10	10113.04106148979	-154.4488849438038
20	10113.17698319027	-154.4492924132789
30	10113.17698352062	-154.4492924135681
40	10113.17698352062	-154.4492924135681

Table XIII

The values of integral (1) for $f(z') = |z' - 0.6|^5$, $h = 1$, $k = 10,\ n = 3,\ z = 0.5$ *and* $a = 0.001$.

(Critical point $z' = 0.6$ *is not ignored.)*

m	Real part	Imaginary part
10	27.72381147834240	-0.3084923364152579
20	19.99165687009960	-0.3069499819030502
30	20.00258956009131	-0.3069661235280619
40	20.00260291624604	-0.3069661434336091
50	20.00260199061115	-0.3069661426902968
60	20.00260200121186	-0.3069661426969270
70	20.00260200113483	-0.3069661426968875
80	20.00260200113520	-0.3069661426968877

The exact value of integral from Table XIII is

$$20.0026020011351981411614740052\ -$$
$$-\ j0.3069661426968876829959025352064.$$

Table XIV

The values of the integral (1) for $f(z') = \sin(10\pi z')$, $h = 1$, $k = 3,\ n = 2,\ z = 0$ *and* $a = 0.01$.

m	Real part	Imaginary part
20	50.74163885536479	-3.875083850333138
25	50.79916034948494	-3.872708333064265
30	50.80025607977755	-3.872108313610875
35	50.80025922203297	-3.872098190561152
40	50.80025918982393	-3.872098156397293
45	50.80025918970359	-3.872098156450951
50	50.80025918970364	-3.872098156451187

Table XV

The values of the integral (1) for $f(z') = \sin(3\pi z')$, $h = 1$, $k = 3,\ n = 2,\ z = 0.5$ *and* $a = 0.01$.

m	Real part	Imaginary part
10	-285.9935340779567	14,26519539392197
15	-285.9934265434636	14.26514930753651
20	-285.9934265321161	14.26514932127759
25	-285.9934265321174	14.26514932127762
30	-285.9934265321174	14.26514932127762

Table XVI

The values of the integral (1) for $f(z') = |\sin(3\pi z')|$, $h = 1$, $k = 10,\ n = 2,\ z = 0.5$ *and* $a = 0.001$. (Points of non-differentiability $z' = 0, \frac{1}{3}, \frac{2}{3}, 1$ are not ignored.)

m	Real part	Imaginary part
10	3056.852971486942	-103.5976458478436
20	3102.125138130292	-103.5948512781994
30	3102.342816246379	-103.5948512739351
40	3102.343856949416	-103.5948512739351
50	3102.343861922294	-103.5948512739351
60	3102.343861946051	-103.5948512739351
70	3102.343861946165	-103.5948512739351
80	3102.343861946165	-103.5948512739351

Table XVII

The values of the integral (1) for $f(z') = |\sin(3\pi z')|$, $h = 1$, $k = 10,\ n = 2,\ z = 1$ *and* $a = 0.001$. (Points of non-differentiability $z' = 0, \frac{1}{3}, \frac{2}{3}, 1$ are included in the method.)

m	Real part	Imaginary part
10	40.45705342618568	-13.18410266352344
20	40.45822670156258	-13.18515155129837
30	40.45822670102173	-13.18515155156373
40	40.45822670102173	-13.18515155156373

Real and imaginary part of the integral (1) versus z, when $f(z') = \sin(3\pi z')$, $k = 10$, $h = 1$, $n = 2$ and $a = 0.001$ are presented in Figs. 6 and 7.

CONCLUSION

A new numerical approach for numerical evaluation of integrals having the form (1) is presented. The presented method is very fast, efficient and general. It can be applied for a wide class of integrals (1), when:

k is small (including zero) and large (e.g., $k = 10$);

$n = 1,2,3, \ldots$;

the parameter a could be small (e.g., $a = 0.001$) and also very large (distance r for far field calculation); and

the point z can be arbitrary (not only in the segment $[0, h]$).

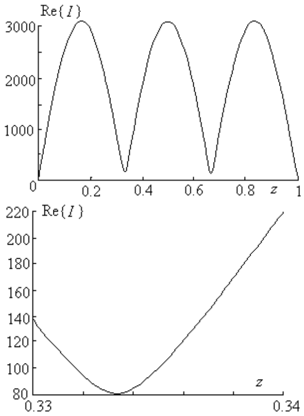

Fig. 6 – *Real part of the integral (1) when* $f(z') = |\sin(3\pi z')|,\ h = 1,\ k = 10,\ n = 2$ *and* $a = 0.001$.

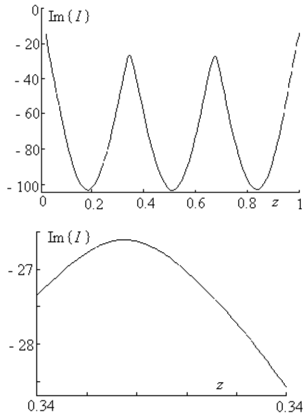

Fig. 7 – *Imaginary part of the integral (1) when* $f(z') = |\sin(3\pi z')|,\ h = 1,\ k = 10,\ n = 2$ *and* $a = 0.001$.

Thus the method is very useful for precise calculation of near and far electromagnetic filed components of linear antenna structures and it is very convenient for exact calculations in solving of integral equation systems governing currents distribution on linear antenna conductors.

It can be successfully extended to some other problems of the electromagnetic field theory, for solving problems of linear antennas in semi-conducting media [13]. Then the kernel in the integral (1) has the following form,

$$K_n(z', z) = \frac{e^{-\gamma R}}{R^n},\ n = 1,2 \text{ and } 3,$$

where $\gamma = \alpha + jk$ is a constant of propagation. Some numerical methods for strongly oscillatory and singular functions are given in [15].

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$$J_n = \int_0^{\pi/2} \left(A + B\sin^2\alpha\right) R^{-n}\, d\alpha,\ R = R_1\left(1 - F^2\sin^2\alpha\right)^{-1/2}",$$

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