Applied Mathematics and Computation 224 (2013) 74–87

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Properties of Boubaker polynomials and an application to Love's integral equation ☆

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ARTICLE INFO

Keywords:
Chebyshev polynomials
Boubaker polynomials
Zeros
Recurrence relation
Fredholm integral equation
Love's integral equation

ABSTRACT

The paper deals with three-term recurrence relations for Boubaker and related polynomials, as well as some properties including zero distribution of such kinds of polynomials. Also, an application of these polynomials for obtaining approximate analytical solution of Love's integral equation is presented. This Fredholm integral equation of the second kind appeared in an electrostatic problem analyzed for the first time by Love (1949) [13].

1. Introduction

There are several papers on the so-called Boubaker polynomials and on their applications in different problems arising in physics and other computational and applied sciences (cf. [2,3,9,26]). These polynomials $\{B_n(x)\}$ have a close relationship with the Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$, which are orthogonal on $(-1,1)$ with respect to the weights functions $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, respectively.

Solutions to several applied physics problems are based on the so-called Boubaker Polynomials Expansion Scheme (BPES), using only the subsequence $\{B_{4m}(x)\}$ of these polynomials (cf. [26] and references therein). Such polynomials satisfy the relation (cf. [9])

$$B_{4(m+1)}(x) = (x^4 - 4x^2 + 2)B_{4m}(x) - \beta_m B_{4(m-1)}(x), \quad m \geqslant 1, \tag{1.1}$$

with $B_0(x) = 1$ and $B_4(x) = x^4 - 2$, where $\beta_0 = 0, \beta_1 = -2$ and $\beta_m = 1$ for $m \geqslant 2$.

Recently, Kumar [11] has presented a method for obtaining an analytical solution of Love's integral equation (see [13,14]), with a positive parameter d,

$$f(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{d}{d^2 + (x-y)^2} f(y)\,\mathrm{d}y = 1, \quad -1 \leqslant x \leqslant 1 \tag{1.2}$$

for a particular electrostatical system, based on the Boubaker polynomials expansion scheme (BPES). However, a mistake has appeared in his approach.

The main goal of this paper is a correction of Kumar's approach, as well as the proof of most important properties of the Boubaker polynomials, including the zero distribution. The paper is organized as follows.

☆ The authors were supported in part by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174015 and No. #III 44006). A preliminary short form of some results of this paper was presented at the International Conference on Numerical Analysis and Applied Mathematics 2012 (Kos, Greece, September 19–25, 2012) and published as an extended abstract [17].

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.08.055

In Section 2 some properties of the polynomials $B_n(x)$ and certain related polynomials (the three-term recurrence relation, relations with Chebyshev polynomials, properties of zeros, etc.) are presented. The proofs of statements are given in Section 3. Finally, in Section 4 an application of these polynomials $B_n(x)$ in solving Love's integral equation is given, which appears in an electrostatic problem.

2. Properties of Boubaker polynomials

The well-known Chebyshev polynomials of the first and second kind for $x \in (-1,1)$ are defined by (cf. [15, p. 6])

$$T_n(x) = \cos(n \arccos x) \quad \text{and} \quad U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}},$$

respectively. Their explicit expressions are

$$T_0(x) = 1, \quad T_n(x) = \frac{n}{2}\sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}, \quad n \geq 1$$

and

$$U_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!}{k!(n-2k)!}(2x)^{n-2k}, \quad n \geq 0$$

and they satisfy the same three-term recurrence relation (cf. [15, p. 9]), i.e.,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geqslant 1,$$

with $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$.

In a similar way, the monic Boubaker polynomials are defined as (cf. [2,3,9,26])

$$B_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k}, \quad n \geq 1 \tag{2.1}$$

and $B_0(x) = 1$. Polynomials of even and odd orders are even and odd functions, respectively, i.e., $B_n(-x) = (-1)^n B_n(x), n \in \mathbb{N}_0$. For example, for $n \leqslant 11$ we have

$$\begin{aligned}
&B_1(x) = x, \quad B_2(x) = x^2 + 2, \quad B_3(x) = x^3 + x, \quad B_4(x) = x^4 - 2,\\
&B_5(x) = x^5 - x^3 - 3x, \quad B_6(x) = x^6 - 2x^4 - 3x^2 + 2, \quad B_7(x) = x^7 - 3x^5 - 2x^3 + 5x,\\
&B_8(x) = x^8 - 4x^6 + 8x^2 - 2, \quad B_9(x) = x^9 - 5x^7 + 3x^5 + 10x^3 - 7x,\\
&B_{10}(x) = x^{10} - 6x^8 + 7x^6 + 10x^4 - 15x^2 + 2,\\
&B_{11}(x) = x^{11} - 7x^9 + 12x^7 + 7x^5 - 25x^3 + 9x.
\end{aligned}$$

In the sequel we give some of most important properties of these polynomials. Some of them are known.

2.1. Three-term recurrence relations and connections with Chebyshev polynomials

The polynomials $B_n(x)$ can be alternatively represented by a three-term recurrence relation

$$B_{n+1}(x) = xB_n(x) - B_{n-1}(x), \quad n = 2,3,\ldots, \tag{2.2}$$

where $B_0(x) = 1, B_1(x) = x, B_2(x) = x^2 + 2$.

As we can see, the relation (2.2) is not true for $n = 1$. In order to provide a relation for each $n \in \mathbb{N}$, we can define a sequence $\{\beta_k\}_{k \in \mathbb{N}_0}$ by $\beta_0 = 0, \beta_1 = -2$ and $\beta_k = 1$ for $k \geqslant 2$. Then, we have the three-term recurrence relation in the following form

$$B_{k+1}(x) = xB_k(x) - \beta_k B_{k-1}(x), \quad k = 0,1,\ldots, \tag{2.3}$$

with $B_0(x) = 1$, $B_{-1}(x) = 0$.

The polynomials (2.1) can be expressed in terms of Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$.

Theorem 2.1. *For $n \geqslant 1$ the following formulas*

$$B_n(x) = 2T_n(x/2) + 4U_{n-2}(x/2) \tag{2.4}$$

and

$$B_n(x) = U_n(x/2) + 3U_{n-2}(x/2) \tag{2.5}$$

hold, where $U_{-1}(x) \equiv 0$.

Eliminating $U_{n-2}(x/2)$ from (2.4) and (2.5), we get

$$B_n(x) = 4U_n(x/2) - 6T_n(x/2).$$

Now, according to Eq. (1.1.9) in [15, p. 6], we obtain the expansions of $B_n(x)$ in terms of Chebyshev polynomials of the first kind,

$$B_{2n}(x) = 2T_{2n}(x/2) + 8T_{2n-2}(x/2) + 8T_{2n-4}(x/2) + \cdots + 8T_2(x/2) + 4T_0(x/2)$$

and

$$B_{2n+1}(x) = 2T_{2n+1}(x/2) + 8T_{2n-1}(x/2) + 8T_{2n-3}(x/2) + \cdots + 8T_3(x/2) + 8T_1(x/2).$$

In the following we recall a property of orthogonal polynomials on a symmetric interval $(-a, a), a > 0$, satisfing a three-term recurrence relation of the form (2.3). For such kind of polynomials, which are orthogonal with respect to an even weight function on $(-a, a)$, two new polynomial systems orthogonal on the interval $(0, a^2)$ can be defined (cf. [15, pp. 102–103]).

In a quite similar way, we can also introduce here two new (nonorthogonal) systems of monic polynomials $\{p_m(t)\}$ and $\{q_m(t)\}$ via Boubaker polynomials $B_m(x)$, so that

$$B_{2m}(x) = p_m(x^2) \quad \text{and} \quad B_{2m+1}(x) = xq_m(x^2) \tag{2.6}$$

and prove the following result:

Theorem 2.2. *Let* $\beta_m, m \geq 1$, *be recursive coefficients in the recurrence relation* (2.3). *Then,*

$$p_{m+1}(t) = (t - a_m)p_m(t) - b_m p_{m-1}(t) \quad \text{and} \quad q_{m+1}(t) = (t - c_m)q_m(t) - d_m q_{m-1}(t), \tag{2.7}$$

with $p_0(t) = q_0(t) = 1, p_{-1}(t) = q_{-1}(t) = 0$, *where the recursive coefficients are given by*

$$a_m = \beta_{2m} + \beta_{2m+1} = \begin{cases} -2, & m = 0, \\ 2, & m \geq 1, \end{cases} \quad b_m = \beta_{2m}\beta_{2m-1} = \begin{cases} -2, & m = 1, \\ 1, & m \geq 2 \end{cases}$$

and

$$c_m = \beta_{2m+1} + \beta_{2m+2} = \begin{cases} -1, & m = 0, \\ 2, & m \geq 1, \end{cases} \quad d_m = \beta_{2m}\beta_{2m+1} = 1, \; m \geq 1.$$

2.2. Determinatal form of polynomials and distribution of zeros

Using the recurrence relation (2.3) for $k = 0, 1, \ldots, n-1$ and defining the n-dimensional vector $\mathbf{b}_n(x)$ by

$$\mathbf{b}_n(x) = [B_0(x) \; B_1(x) \; \ldots \; B_{n-1}(x)]^T,$$

we obtain the equation

$$(xI_n - M_n)\mathbf{b}_n(x) = B_n(x)\mathbf{e}_n, \tag{2.8}$$

where I_n is the identity matrix of order $n, \mathbf{e}_n = [0 \; 0 \; \ldots \; 0 \; 1]^T$ is the last coordinate vector, and M_n is a tridiagonal matrix of order n, given by

$$M_n = \begin{bmatrix} 0 & 1 & & & O \\ \beta_1 & 0 & 1 & & \\ & \beta_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & \beta_{n-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & O \\ -2 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & 1 & 0 \end{bmatrix}.$$

According to (2.8) the monic polynomials $B_n(x)$ can be expressed in the following determinant form

$$B_n(x) = \det(xI_n - M_n), \quad n \geqslant 1.$$

It is a form which is well known in the theory of orthogonal polynomials (cf. [15, p. 100]). We also conclude that the zeros of the polynomial $B_n(x)$ are eigenvalues of the matrix M_n. Now we apply a result of Veselić [24] to the irreducible tridiagonal matrix M_n, for which we can form, according to [24, Eq. (3)], the following sequence

$$1, \; \underbrace{-2, -2, \ldots, \; -2}_{n-1}.$$

Since the numbers of positive and negative signs in this sequence are $k_+ = 1$ and $k_- = n-1$, respectively, and $k = \min(k_+, k_-) = 1$, we conclude that M_n for $n \geqslant 3$ has at least $|k_+ - k_-| = n-2$ different real spectral points. Furthermore, using Gerschgorin's theorem, it is easy to see that all eigenvalues are in the unit circle $|z| < 2$ (see also [26]). Thus, $B_n(x), n \geq 2$, can have only one pair of conjugate-complex zeros.

Precisely, the following result holds:

Theorem 2.3. *Every polynomial $B_n(x), n \geqslant 2$, has two complex conjugate zeros $\pm \mathrm{i}\sqrt{\gamma_n}$, $\gamma_n > 0$, and other zeros are real and symmetrically distributed in $(-2,2)$, where $\lim_{n\to+\infty}\gamma_n = 4/3$. The corresponding representations of these polynomials for $m \geq 2$ are*

$$B_{2m}(x) = (x^2+\gamma_{2m})\prod_{\nu=1}^{m-1}(x^2-\tau_{2m,\nu}), \quad B_{2m+1}(x) = x(x^2+\gamma_{2m+1})\prod_{\nu=1}^{m-1}(x^2-\tau_{2m+1,\nu}), \tag{2.9}$$

where $4 > \tau_{n,1} > \cdots > \tau_{n,m-1} > 0$ *and* $n = 2m$ *or* $n = 2m+1$.

Remark 2.1. Theorem 2.3 on zero distribution of $B_n(x)$ has been mentioned in [17, Theorem 4]. Recently, a similar result has appeared on arXiv:1211.0383 (see [10]).

The corresponding tridiagonal matrices (of Jacobi type) of order m for the polynomial sequences $\{p_m(t)\}$ and $\{q_m(t)\}$ are

$$J_m^p = \begin{bmatrix} -2 & 1 & & & O \\ -2 & 2 & 1 & & \\ & 1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & 1 & 2 \end{bmatrix} \quad \text{and} \quad J_m^q = \begin{bmatrix} -1 & 1 & & & O \\ -1 & 2 & 1 & & \\ & 1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & 1 & 2 \end{bmatrix},$$

respectively. Otherwise, these polynomial systems $\{p_m(t)\}$ and $\{q_m(t)\}$ consist of the following polynomials

$$\{1,\ t+2,\ t^2-2,\ t^3-2t^2-3t+2,\ t^4-4t^3+8t-2,\ t^5-6t^4+7t^3+10t^2-15t+2,\ \ldots\}$$

and

$$\{1,\ t+1,\ t^2-t-3,\ t^3-3t^2-2t+5,\ t^4-5t^3+3t^2+10t-7,\ t^5-7t^4+12t^3+7t^2-25t+9,\ \ldots\},$$

respectively.

In order to investigate zeros of the polynomials $B_n(z)$ on the imaginary axis we put $z = \mathrm{i}y$ and consider $B_n(\mathrm{i}y)/\mathrm{i}^n, n \geqslant 2$, i.e., the sequence of polynomials

$$y^2-2,\quad y(y^2-1),\quad y^4-2,\quad y(y^4+y^2-3),\quad y^6+2y^4-3y^2-2, y(y^6+3y^4-2y^2-5),\quad y^8+4y^6-8y^2-2,\quad y(y^8+5y^6+3y^4-10y^2-7),\ \text{etc.}$$

For $t > 0$ we define two sequences of polynomials $e_m(t)$ and $o_m(t)$, $m = 1, 2, \ldots$, by

$$e_m(t) = (-1)^m B_{2m}(\mathrm{i}\sqrt{t}) \quad \text{and} \quad o_m(t) = (-1)^m \frac{B_{2m+1}(\mathrm{i}\sqrt{t})}{\mathrm{i}\sqrt{t}}, \tag{2.10}$$

respectively. According to (2.1) and Theorem 2.2, it is clear that

$$e_m(t) = (-1)^m p_m(-t) = \sum_{k=0}^{m}\binom{2m-k}{k}\frac{2m-4k}{2m-k}t^{m-k} \tag{2.11}$$

and

$$o_m(t) = (-1)^m q_m(-t) = \sum_{k=0}^{m}\binom{2m-k}{k}\frac{2m-4k+1}{2m-2k+1}t^{m-k}. \tag{2.12}$$

Theorem 2.4. *For any $m \in \mathbb{N}$ the polynomials $e_m(t)$ and $o_m(t)$ have only one positive zero.*

In Fig. 2.1 we present the graphs of polynomials $e_m(t)$ (left) and $o_m(t)$ (right) for $t \in [0,2]$ and $m = 2,3,4$, and 5. Notice that $e_m(0) = -2$ and $o_m(0) = -(2m-1)$, and the unique positive zero ξ_m of $e_m(t)$ (and also η_m of $o_m(t)$) belongs to $(1, 3/2)$. Also, derivatives $e_m^{(\nu)}(t)$ and $o_m^{(\nu)}(t), \nu = 1, 2, \ldots$ (of course for a sufficiently large m) have the unique positive zeros $\xi_m^{(\nu)}$ and $\eta_m^{(\nu)}$, respectively, for which it is easy to see that

$$\xi_m^{(\nu+1)} < \xi_m^{(\nu)} < \xi_m \quad \text{and} \quad \eta_m^{(\nu+1)} < \eta_m^{(\nu)} < \eta_m. \tag{2.13}$$

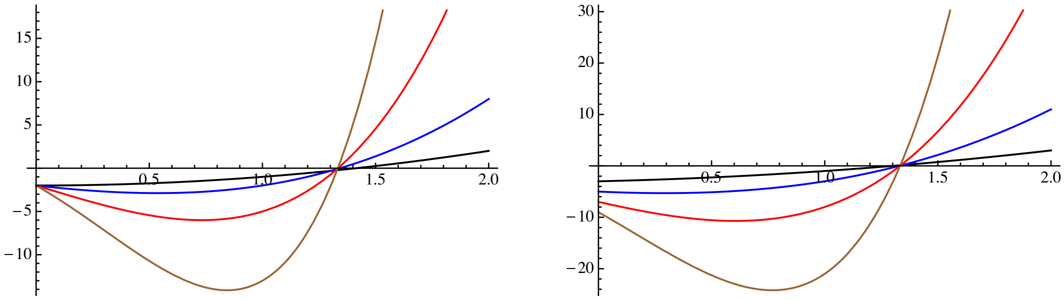

Fig. 2.1. The graphs of polynomials $e_m(t)$ (left) and $o_m(t)$ (right) for $m = 2$ (black line), $m = 3$ (blue line), $m = 4$ (red line) and $m = 5$ (brown line). (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

Remark 2.2. Other zeros of $e_m(t)$ and $o_m(t)$ are also real, but negative.

In the sequel we investigate the exact positions of these zeros and give their asymptotics when $m \to +\infty$. First we need some auxiliary results.

Lemma 2.5. *The values of polynomials $e_m(t)$ and $o_m(t)$, as well as their derivatives of the first and second order at $t = 4/3$ are*

$$e_m(4/3) = -\frac{2}{3^m}, \quad e'_m(4/3) = \frac{3}{32}\left[3^{m+1} + (8m-3)3^{-m}\right],$$

$$e''_m(4/3) = \frac{9}{1024}\left[(8m-7)3^{m+1} - (32m^2+16m-21)3^{-m}\right]$$

and

$$o_m(4/3) = \frac{1}{3^m}, \quad o'_m(4/3) = \frac{3}{64}\left[3^{m+2} - (8m+9)3^{-m}\right],$$

$$o''_m(4/3) = \frac{9}{2048}\left[(8m-11)3^{m+2} + (32m^2+112m+99)3^{-m}\right],$$

respectively.

Remark 2.3. Some interesting finite sums can be obtained from (2.11), (2.12), and Lemma 2.5. For example, we have

$$\sum_{k=0}^{m}\binom{m+k}{2k}\frac{m-2k}{m+k}\left(\frac{4}{3}\right)^k = \sum_{k=0}^{m}\binom{m+k}{2k}\frac{4k-2m+1}{2k+1}\left(\frac{4}{3}\right)^k = \frac{1}{3^m},$$

$$\sum_{k=1}^{m}\binom{m+k-1}{2k-1}(2k-m)\left(\frac{4}{3}\right)^k = \frac{1}{8}\left[3^{m+1} + (8m-3)3^{-m}\right], \quad \text{etc.}$$

The following result is related with positive zeros ξ_m and η_m of the polynomials $e_m(t)$ and $o_m(t)$, respectively.

Theorem 2.6. *For each $m \geqslant 1$, the following inequalities*

$$0 < \xi_m - \frac{4}{3} < \frac{64}{9^{m+1}} \cdot \frac{1}{1 + \frac{8m-3}{3^{2m+1}}} < \frac{64}{9^{m+1}} \tag{2.14}$$

and

$$\frac{4}{3} \cdot \frac{2m-1}{2m-1+3^{-m}} < \eta_m < \frac{4}{3} - \frac{64}{3} \cdot \frac{1}{9^{m+1}} \cdot \frac{1}{1 - \frac{8m+9}{9^{m+1}}} \tag{2.15}$$

hold.

3. Proofs

Proof of Theorem 2.1. First equality (2.4), which has been known earlier, can be proved, for example, by the mathematical induction, or by solving (2.2), i.e., $y_{n+1} - xy_n + y_{n-1} = 0, n \geqslant 1$, as a difference equation with respect to n, supposing that x is a fixed number in $(-2, 2)$, for example, $x = 2\cos\theta$. Then, the roots of the characteristic equation are $e^{\pm i\theta}$, so that the general solution of this equation is $y_n = C_1 \cos n\theta + C_2 \sin n\theta$. Taking the starting values

$$y_1 = B_1(x) = x = 2\cos\theta \quad \text{and} \quad y_2 = B_2(x) = x^2 + 2 = 2\cos 2\theta + 4,$$

we get $C_1 = -2$ and $C_2 = 4\cot\theta$, so that

$$B_n(x) = -2\cos n\theta + 4\frac{\cos\theta \sin n\theta}{\sin\theta} = 2\cos n\theta + 4\frac{\sin(n-1)\theta}{\sin\theta},$$

i.e., $B_n(x) = 2T_n(x/2) + 4U_{n-2}(x/2)$.

Since $2T_n(x) = U_n(x) - U_{n-2}(x)$, (2.5) follows directly from (2.4). □

Proof of Theorem 2.2. According to (2.3) we have

$$B_{2m+3}(x) = xB_{2m+2}(x) - \beta_{2m+2}B_{2m+1}(x) \quad \text{and} \quad B_{2m+2}(x) = xB_{2m+1}(x) - \beta_{2m+1}B_{2m}(x),$$

or, using (2.6),

$$xq_{m+1}(x^2) = xp_{m+1}(x^2) - \beta_{2m+2}xq_m(x^2) \quad \text{and} \quad p_{m+1}(x^2) = x^2q_m(x^2) - \beta_{2m+1}p_m(x^2).$$

Putting $x^2 = t$, it is easy to see that

$$p_{m+1}(t) + \beta_{2m+1}p_m(t) = tq_m(t) \tag{3.1}$$

and

$$q_{m+1}(t) + \beta_{2m+2}q_m(t) = p_{m+1}(t). \tag{3.2}$$

If we replace k by $k-1$ in (3.2), multiply it by t, and finally add it to the relation (3.1), we obtain

$$p_{m+1}(t) + (\beta_{2m+1} - t)p_m(t) + \beta_{2m}tq_{m-1}(t) = 0. \tag{3.3}$$

In a similar way, taking k instead of k in (3.1) and combining it with (3.3) we get

$$p_{m+1}(t) + (\beta_{2m} + \beta_{2m+1} - t)p_m(t) + \beta_{2m}\beta_{2m-1}p_{m-1}(t) = 0,$$

i.e., the first relation in (2.7). In a similar way we obtain the corresponding recurrence relation for polynomials $\{q_m\}$. □

Proof of Theorem 2.4. In the proof we use the number of sign variations (differences) between consecutive nonzero coefficients of a polynomial ordered by descending variable exponent. We note that the coefficients in (2.11) are positive for $k < m/2$ and negative for $k > m/2$, so that we have only one sign variation.

According to Descartes' Rule the number of positive zeros is either equal to the number of sign differences between consecutive nonzero coefficients, or lower than it by a multiple of 2.

Since $e_m(0) = -2 < 0$ and $e_m(T) > 0$ for each sufficiently large positive T, we conclude that $e_m(t)$ has only one positive zero.

A similar proof can be done for polynomials $o_m(t)$. □

Proof of Lemma 2.5. According to (2.7) and (2.11), i.e., (2.12), we have the following difference equations

$$e_{m+1}(t) - (t+2)e_m(t) + e_{m-1}(t) = 0 \quad \text{and} \quad o_{m+1}(t) - (t+2)o_m(t) + o_{m-1}(t) = 0,$$

as well as the ones for derivatives

$$e_{m+1}^{(\nu)}(t) - (t+2)e_m^{(\nu)}(t) + e_{m-1}^{(\nu)}(t) = \nu e_m^{(\nu-1)}(t)$$

and

$$o_{m+1}^{(\nu)}(t) - (t+2)o_m^{(\nu)}(t) + o_{m-1}^{(\nu)}(t) = \nu o_m^{(\nu-1)}(t),$$

where $\nu = 1, 2, \ldots$. In particular, we are interested only in solutions of these difference equations for $t = 4/3$. Since the characteristic equation $\lambda^2 - (10/4)\lambda + 1 = 0$ (with roots $\lambda_1 = 3$ and $\lambda_2 = 1/3$) is the same for each of these equations, we obtain the general solutions

$$e_m(4/3) = C_1 3^m + C_2 3^{-m} \quad \text{and} \quad o_m(4/3) = D_1 3^m + D_2 3^{-m},$$

where C_1, C_2, D_1, D_2 are arbitrary constants. Taking the starting values $e_1(4/3) = -2/3, e_2(4/3) = -2/9, o_1(4/3) = 1/3, o_2(4/3) = 1/9$, we get $C_1 = D_1 = 0, C_2 = -2$, and $D_2 = 1$, so that

$$e_m(4/3) = -\frac{2}{3^m} \quad \text{and} \quad o_m(4/3) = \frac{1}{3^m}, \quad m \geqslant 1. \tag{3.4}$$

For $\nu = 1$, the corresponding difference equations are

$$e'_{m+1}(4/3) - \frac{10}{4} e'_m(4/3) + e'_{m-1}(4/3) = -\frac{2}{3^m}$$

and

$$o'_{m+1}(4/3) - \frac{10}{4} o'_m(4/3) + o'_{m-1}(4/3) = \frac{1}{3^m}$$

and then, with starting values $e'_1(4/3) = 1, e'_2(4/3) = 8/3, e'_3(4/3) = 23/3$ and $o'_1(4/3) = 1, o'_2(4/3) = 11/3, o'_3(4/3) = 34/3$, we obtain

$$e'_m(4/3) = \frac{3}{32}\left[3^{m+1} + (8m-3)3^{-m}\right] \quad \text{and} \quad o'_m(4/3) = \frac{3}{64}\left[3^{m+2} - (8m+9)3^{-m}\right].$$

In a similar way, with starting values $e''_1(4/3) = 0, e''_2(4/3) = 2, e''_3(4/3) = 12, e''_4(4/3) = 160/3$ and $o''_1(4/3) = 0, o''_2(4/3) = 2, o''_3(4/3) = 14, o''_4(4/3) = 202/3$, we obtain the solutions of the following difference equations

$$e''_{m+1}(4/3) - \frac{10}{4} e''_m(4/3) + e''_{m-1}(4/3) = \frac{9}{16} 3^m + \frac{24m-9}{16} 3^{-m}$$

and

$$o''_{m+1}(4/3) - \frac{10}{4} o''_m(4/3) + o''_{m-1}(4/3) = \frac{27}{32} 3^m - \frac{24m+27}{32} 3^{-m},$$

in the form

$$e''_m(4/3) = \frac{9}{1024}\left[(8m-7)3^{m+1} - (32m^2 + 16m - 21)3^{-m}\right]$$

and

$$o''_m(4/3) = \frac{9}{2048}\left[(8m-11)3^{m+2} + (32m^2 + 112m + 99)3^{-m}\right],$$

respectively. □

Proof of Theorem 2.6. Since $e_m(4/3) = -2/3^m < 0$ we have that $\xi_m > 4/3$. Also, because of $\xi_m^{(2)} < \xi_m^{(1)} < \xi_m$ (see (2.13)) and $e''_m(4/3) \geqslant 0, m \geqslant 1$, it is clear that $e''_m(t) \geqslant e''_m(4/3) > e''_m(\xi_m^{(2)}) = 0$ for $t \geqslant 4/3$. Applying Taylor's formula

$$e_m(t) = e_m(4/3) + e'_m(4/3)\left(t - \frac{4}{3}\right) + \frac{1}{2} e''_m(\tau)\left(t - \frac{4}{3}\right)^2, \quad \frac{4}{3} < \tau < t,$$

for $t = \xi_m$, we conclude that

$$0 = e_m(\xi_m) > e_m(4/3) + e'_m(4/3)\left(\xi_m - \frac{4}{3}\right),$$

i.e.,

$$0 < \xi_m - \frac{4}{3} < -\frac{e_m(4/3)}{e'_m(4/3)} = \frac{2/3^m}{\frac{3}{32}\left[3^{m+1} + (8m-3)3^{-m}\right]},$$

which gives (2.14).

In order to prove (2.15) we note that $\eta_m < 4/3$, because of $o_m(4/3) = 1/3^m > 0$. Under similar arguments, $\eta_m^{(2)} < \eta_m^{(1)} < \eta_m < 4/3$ and $o''_m(4/3) \geqslant o''_m(\eta_m) > o''_m(\eta_m^{(2)}) = 0$, an application of Taylor's formula gives

$$0 = o_m(\eta_m) = o_m(4/3) + o'_m(4/3)\left(\eta_m - \frac{4}{3}\right) + \frac{1}{2} o''_m(\tau)\left(\eta_m - \frac{4}{3}\right)^2, \quad \eta_m < \tau < \frac{4}{3},$$

i.e.,

$$\frac{4}{3} - \eta_m > \frac{o_m(4/3)}{o'_m(4/3)} = \frac{1/3^m}{\frac{3}{64}\left[3^{m+2} - (8m+9)3^{-m}\right]} = \frac{64}{3} \cdot \frac{1}{9^{m+1}} \cdot \frac{1}{1 - \frac{8m+9}{9^{m+1}}}.$$

On the other side, using a secant method at the points $t = 0$ and $t = 4/3$, we obtain a lower bound for η_m. Namely, the line between the points $(0, -(2m-1))$ and $(4/3, 1/3^m)$ crosses the apscisa at

$$\widehat{x}_m = \frac{4}{3} \cdot \frac{2m-1}{2m-1+3^{-m}},$$

so that $\widehat{x}_m < \eta_m$. Combining it with the previous upper bound we obtain (2.15). $\square$

Finally, we can prove the result on the zero distribution of polynomials $B_n(x)$.

Proof of Theorem 2.3. Let ξ_m and η_m be unique positive zeros of $e_m(t)$ and $o_m(t)$, respectively. Then, according to (2.10), we conclude that $\mathrm{i}\sqrt{\xi_m}$ (but also $-\mathrm{i}\sqrt{\xi_m}$) is an imaginary zero of $B_{2m}(x)$, so that γ_{2m} in (2.9) must be $\gamma_{2m} = \xi_m$. In a quite similar way, we prove that $\gamma_{2m+1} = \eta_m$.

The other $m-1$ negative zeros of $e_m(t)$ (see Remark 2.1) generate $2m-2$ real zeros of $B_{2m}(x)$ symmetrically distributed on $(-2, 2)$. Similarly, negative zeros of $o_m(t)$ generate $2m-2$ real zeros of $B_{2m+1}(x)$ symmetrically distributed also on $(-2, 2)$. Thus, the representations (2.9) hold.

Using inequalities (2.14) and (2.15) from Theorem 2.6, we have

$$\lim_{m\to+\infty} \eta_m = \lim_{m\to+\infty} \xi_m = \frac{4}{3}, \tag{3.5}$$

so that $\lim_{n\to+\infty} \gamma_n = 4/3$. $\square$

4. Applications

As we mentioned in Section 1, the polynomials $\{B_{4m}(x)\}$ play an important role in applications. These polynomials satisfy the relation (1.1). However, if we need all even polynomials $B_{2m}(x)$, then the corresponding recurrence relation is

$$B_{2m+2}(x) = (x^2 - a_m)B_{2m}(x) - b_m B_{2m-2}(x), \quad m \geqslant 0, \tag{4.1}$$

where a_m and b_m are recursive coefficients defined in Theorem 2.2.

Recently, Kumar [11] has presented a method for obtaining an analytical solution of Love's integral equation (see [13,14])

$$f(x) - \frac{1}{\pi} \int_{-1}^{1} \frac{d}{d^2 + (x-y)^2} f(y)\,\mathrm{d}y = 1, \quad -1 \leqslant x \leqslant 1, \tag{4.2}$$

for a particular electrostatical system, based on the Boubaker polynomials expansion scheme (BPES). However, a mistake has appeared in his approach. Our goal in this section is to correct this Kumar's approach and give a much better approximation of the solution of Love's integral equation.

In 1949 Love (1912–2001) described the electrostatic potential in space, generated by a condenser consisting of two parallel equal circular plates of the radius R, separated by a distance h (see Fig. 4.1). Taking a normalization so that $h = Rd$, it can be considered with dimensionless variables as two unit disks, where d is a distance between them. Supposing the equal and opposite potentials at these disks, e.g., the upper at $V = +1$ and the lower one at $V = -1$, and the potential at infinity being taken as zero, Love [13, Theorem 1] used a coaxial symmetry of this electrostatical system and proved that the potential in an arbitrary point $M(r, \theta, z) \in \mathbb{R}^3$, outside the circular plates, is given by

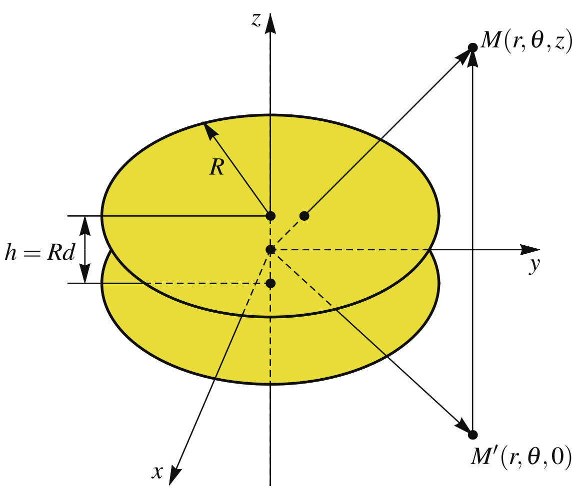

Fig. 4.1. Electrostatical system od two parallel equal circular plates.

$$V(r,z) = \frac{1}{\pi}\int_{-1}^{1}\left\{\frac{1}{\sqrt{r^2+\left(z-\frac{1}{2}d+\mathrm{i}x\right)^2}} - \frac{1}{\sqrt{r^2+\left(z+\frac{1}{2}d+\mathrm{i}x\right)^2}}\right\} f(x)\,\mathrm{d}x, \tag{4.3}$$

where both square roots in this integral have positive real part and f is the unique solution of the Fredholm integral equation of the second kind (4.2). He also proved that there exists a unique, continuous, real and even solution f of the Eq. (4.2) on the closed interval $[-1,1]$.

In the case when the potentials of the plates are equal in magnitude and sign, then the corresponding integral equation (4.2) becomes

$$f(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2} f(y)\,\mathrm{d}y = 1, \quad -1 \leqslant x \leqslant 1. \tag{4.4}$$

Also, in that case the $-$ sign between two terms on the right-hand side in (4.3) becomes a $+$ sign.

Recently, Norgren and Jonsson [19] have calculated the capacitance of the circular parallel plate capacitor by expanding the solution to the Love integral equation (4.2) into a Fourier cosine series. For some other approaches see [8,5,25,4,22].

Love's integral equations have the so-called difference kernel

$$k(x,y) = k(x-y) = \frac{1}{\pi}\cdot\frac{d}{d^2+(x-y)^2}, \quad d>0, \tag{4.5}$$

which has two complex conjugate poles $x \pm \mathrm{i}d$. We can see these poles approach the real axis when $d \to 0^+$, and therefore the kernel is quasi-singular.

Letting

$$(Kf)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2} f(y)\,\mathrm{d}y,$$

the operator form of Love's integral Eqs. (4.2) and (4.4) are

$$f \mp Kf = (I \mp K)f = g, \tag{4.6}$$

where I denotes the identity operator, and K is compact with (cf. [18]).

$$\|K\|_\infty = \frac{2}{\pi}\tan^{-1}\frac{1}{d} < 1.$$

There are many numerical methods for solving integral equations (cf. [1,6,7,12,21,23]). Sometimes, they are developed for specific type of kernels. Numerical methods for linear integral equations of the form (4.6) lead to algebraic systems of linear equations and sometimes the conditional number of the corresponding matrices are large. The solution of an integral equation can be done in a polynomial form, as a peacewise polynomial, spline, etc.

An approximation to the solution of (4.2), in the case $d = 1$, was given by Love [14],

$$f(x) \approx f_L(x) = 1.919200 - 0.311717x^2 + 0.015676x^4 + 0.019682x^6 - 0.000373x^8. \tag{4.7}$$

In this section we give a correction and extensions of Kumar's method [11], using Boubaker polynomials. In this approach we need to compute the integrals

$$J_{2m}(x,d) = (KB_{2m})(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}B_{2m}(y)\,\mathrm{d}y$$

for different values of d and x, which can be done numerically for specific values of these parameters. But, we give here an analytic (symbolic) form, using the recurrence relation (4.1). In fact, these integrals J_{2m} are moments of the (weight) function $y \mapsto k(x,y)$, given by (4.5), with respect to the Boubaker polynomial sequence. For J_{2m} we can prove the following recurrence relation:

Lemma 4.1. *We have*

$$J_{2m+2} + (d^2 + a_m - x^2)J_{2m} + b_m J_{2m-2} = \frac{2d}{\pi}I_{2m} + \frac{xd}{\pi}\left\{B_{2m}(1)\log\frac{d^2+(1-x)^2}{d^2+(1-x)^2} + K_{2m}\right\}, \tag{4.8}$$

where

$$J_0(x,d) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{d}\right) + \tan^{-1}\left(\frac{x+1}{d}\right)\right], J_2(x,d)$$
$$= \frac{1}{\pi}\left[2d + \left(2-d^2+x^2\right)\tan^{-1}\left(\frac{x+1}{d}\right) - 2xd\tanh^{-1}\left(\frac{2x}{d^2+x^2+1}\right) + \left(d^2-2\right)\tan^{-1}\left(\frac{x-1}{d}\right) + x^2\tan^{-1}\left(\frac{1-x}{d}\right)\right]$$

and

$$I_{2m} = \int_0^1 B_{2m}(y)\,dy, \quad K_{2m} = K_{2m}(x,d) = \int_0^1 \log\frac{d^2+(x+y)^2}{d^2+(x-y)^2} B'_{2m}(y)\,dy. \tag{4.9}$$

The value of the first integral in (4.9) can be done in the form

$$I_{2m} = \frac{6\sin\left(\frac{\pi}{6}(4m+1)\right)}{2m-1} + \frac{2\cos\left(\frac{\pi}{3}(2m+1)\right)}{2m+1}$$

and the second integral can be expressed as a linear combination of the integrals

$$S_k = S_k(x,d) = \int_0^1 \log\frac{d^2+(x+y)^2}{d^2+(x-y)^2} y^{2k-1}\,dy, \quad k\in\mathbb{N}.$$

Their values are

$$S_k(x,d) = P_{2k-1}(x;d) + \left(\frac{1}{2k} + Q_{2k}(x;d)\right)\log\frac{d^2+(1+x)^2}{d^2+(-1+x)^2} + R_{2k-1}(x;d)\left(\tan^{-1}\left(\frac{1+x}{d}\right) - \tan^{-1}\left(\frac{-1+x}{d}\right)\right),$$

where $P_{2k-1}(x;d)$ and $R_{2k-1}(x;d)$ are odd polynomials of degree $2k-1$ in x, and $R_{2k}(x;d)$ is an even polynomial of degree $2k$ in x. Moreover, their forms are

$$P_{2k-1}(x;d) = \sum_{\nu=1}^{k}\left(\sum_{j=0}^{k-\nu}(-1)^j a_{\nu,j}^{(k)} d^{2j}\right)x^{2\nu-1},$$

$$Q_{2k}(x,d) = \sum_{j=0}^{k}(-1)^{j+1} b_j^{(k)} d^{2j} x^{2k-2j}, \quad R_{2k-1}(x;d) = \sum_{j=0}^{k-1}(-1)^{j+1} c_j^{(k)} d^{2j+1} x^{2k-2j-1},$$

with coefficients $a_{\nu,j}^{(k)}, b_j^{(k)}, c_j^{(k)}$, respectively, for which there is a symmetry $b_{k-j}^{(k)} = b_j^{(k)}$ and $c_{k-j-1}^{(k)} = c_j^{(k)}$, so that

$$Q_{2k}(x;d) = (-1)^k Q_{2k}(d;x) \quad \text{and} \quad R_{2k-1}(x;d) = (-1)^{k-1} R_{2k-1}(d;x).$$

For example, for $k=1, k=2$, and $k=3$, we have

$$P_1(x;d) = 2x, \quad Q_2(x;d) = \frac{1}{2}d^2 - \frac{1}{2}x^2, \quad R_1(x;d) = -2dx, \quad P_3(x;d) = \left(\frac{1}{3} - 3d^2\right)x + x^3,$$

$$Q_4(x;d) = -\frac{1}{4}x^4 + \frac{3}{2}d^2x^2 - \frac{1}{4}d^4, \quad R_3(x;d) = -2dx^3 + 2d^3x$$

and

$$P_5(x;d) = \left(\frac{2}{15} - \frac{2}{3}d^2 + \frac{10}{3}d^4\right)x + \left(\frac{2}{9} - \frac{20}{3}d^2\right)x^3 + \frac{2}{3}x^5,$$

$$Q_4(x;d) = \frac{1}{6}d^6 - \frac{5}{2}d^4x^2 + \frac{5}{2}d^2x^4 - \frac{1}{6}x^6, \quad R_5(x;d) = -2d^5x + \frac{20}{3}d^3x^3 - 2dx^5,$$

respectively. The integrals S_k can be obtained in a symbolic form in the MATHEMATICA Package by the command

```
intS[k_]:=Assuming[-1<=x<=1&&d>0,Integrate[Log[(d^2+(x+y)^2) / (d^2+(x-y)^2)]y^(2k-1),
{y,0,1}]]
Simplify
```

As an approximate solution of Love's equation (4.2) or (4.4) in the set of polynomials of degree at most $4n$ (in notation $\mathscr{P}_{4n}$), Kumar [11] used a linear combination of polynomials $B_4(x), B_8(x), \ldots, B_{4n}(x)$, i.e., the expansion

$$f_{4n}^{(4)}(x) = \sum_{m=1}^{n} c_m B_{4m}(x), \tag{4.10}$$

but his approach contained a serious mistake. The corrected version of this method leads to the equation

$$\sum_{m=1}^{n} c_m B_{4m}(x) \mp \frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}\sum_{m=1}^{n} c_m B_{4m}(y)\,dy = 1,$$

i.e.,

$$\sum_{m=1}^{n}\left(B_{4m}(x) \mp \frac{1}{\pi}\int_{-1}^{1}\frac{dB_{4m}(y)}{d^2+(x-y)^2}\,dy\right)c_m = 1.$$

The indices in $f_{4n}^{(4)}(x)$ indicate to minimal and maximal degrees of basis polynomials in (4.10).

Since the solution of Love's equation is an even function on $[-1,1]$, we can take n mutually different nonnegative points in $[0,1]$ as collocation points $\tau_k, k=1,\ldots,n$.

Thus, for $x=\tau_k, k=1,\ldots,n$, we get a system of linear equations for determining the coefficients $c_m, m=1,\ldots,n$,

$$\begin{aligned} &a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n = 1,\\ &a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n = 1,\\ &\vdots\\ &a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nn}c_n = 1, \end{aligned} \tag{4.11}$$

with the matrix $A_n^{(1)} = [a_{km}]_{k=1,m=1}^{n,n}$, where

$$a_{km} = B_{4m}(\tau_k) \mp \frac{1}{\pi}\int_{-1}^{1}\frac{dB_{4m}(y)}{d^2+(\tau_k-y)^2}\,dy, \quad k,m=1,\ldots,n \tag{4.12}$$

and the signs $\mp$ correspond to the ones in (4.6). The upper index in $A_n^{(1)}$ indicates the starting value for m in the expansion (4.10).

4.1. Love's integral equation (4.2)

As we have mentioned above, Kumar [11] obtained a wrong system of Eqs. (4.11) and then used an equidistant system of collocation points on $[0,1]$. A better condition number of the corresponding matrix $A_n^{(1)}$ can be obtained, for example, using the positive zeros or positive extremal points of the Chebyshev polynomials of the first kind as collocation points. All computations were performed in MATHEMATICA, Ver. 9.0, on MacBook Pro Retina, OS X 10.8.2.

Taking collocation points as the positive zeros of $T_{2n}(x)$, in the same case considered by Love [14], i.e., when $d=1$, from the system of Eqs. (4.11) for $n=1$ and $n=2$, we obtain the corresponding solutions

$$f_4^{(4)}(x) = -1.01362B_4(x) = 2.02725 - 1.01362x^4$$

and

$$f_8^{(4)}(x) = -1.01062B_4(x) + 0.140162B_8(x) = 1.74091 + 1.1213x^2 - 1.01062x^4 - 0.560649x^6 + 0.140162x^8,$$

respectively.

However, we can get some better solutions taking the constant term in the corresponding expansion $(B_0(x)=1)$ of the approximate polynomial solution. Namely, if we take

$$f_{4n}^{(0)}(x) = \sum_{m=0}^{n} c_m B_{4m}(x)$$

instead of (4.10), then using the positive zeros of $T_{2n+2}(x)$ as collocation points, we obtain the following approximative solutions

$$f_4^{(0)}(x) = 1.32192B_0(x) - 0.279362B_4(x) = 1.88064 - 0.279362x^4$$

and

$$\begin{aligned} f_8^{(0)}(x) &= 1.63647B_0(x) - 0.106254B_4(x) - 0.0339144B_8(x)\\ &= 1.91681 - 0.271315x^2 - 0.106254x^4 + 0.135658x^6 - 0.0339144x^8. \end{aligned}$$

Here,

$$\tau_k = \cos\frac{(2k+1)\pi}{4(n+1)}, \quad k=0,1,\ldots,n$$

and the matrix $A_n^{(0)} = [a_{km}]_{k=0,m=0}^{n,n}$ of the corresponding system of equations is of order $n+1$.

Moreover, in the previous set of polynomials (of degree at most $4n$) we can get much better results if we take the complete basis of all even polynomials. Thus, in order to find an approximate solution in the set $\mathscr{P}_{2n}$, we put

$$\tilde{f}_{2n}^{(0)}(x) = \sum_{m=0}^{n} c_m B_{2m}(x). \tag{4.13}$$

In this case, the matrix of the corresponding system of equations is $\tilde{A}_n^{(0)} = [\tilde{a}_{km}]_{k=0,m=0}^{n,n}$, where

Table 4.1
Condition numbers of the matrix $\tilde{A}_n^{(0)}, n = 2(2)12$, for different systems of collocation points.

n	2	4	6	8	10	12
Chebyshev zeros	1.46(2)	1.08(4)	8.98(5)	1.26(8)	2.06(10)	3.75(12)
Chebyshev extremal points	1.06(2)	8.55(3)	5.76(5)	8.24(7)	1.37(10)	2.41(12)
equidistant points	1.31(2)	3.19(4)	1.04(7)	5.64(9)	3.89(12)	2.53(15)

Table 4.2
Maximal relative errors of the approximate solutions.

Approximate solution	Maximal relative errors			
	$n=1$	$n=2$	$n=3$	$n=4$
$f_L(x)$		1.69(−3)		
$f_{4n}^{(4)}(x)$	3.82(−1)	1.27(−1)	3.57(−2)	1.22(−2)
$f_{4n}^{(0)}(x)$	2.34(−2)	1.16(−3)	1.52(−4)	1.07(−5)
$\tilde{f}_{4n}^{(0)}(x)$	2.43(−4)	1.37(−6)	9.65(−9)	2.41(−10)

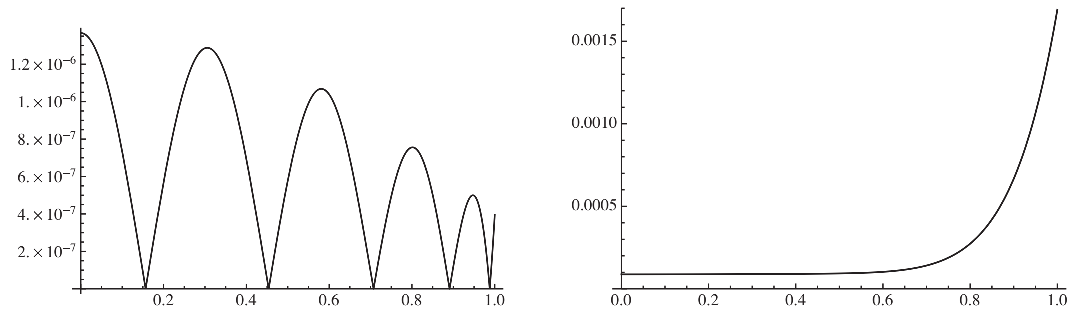

Fig. 4.2. (left) Relative errors of the approximate solutions of degree eight, $\tilde{f}_8^{(0)}(x)$ (left) and $f_L(x)$ (right).

$$\tilde{a}_{km} = B_{2m}(\tau_k) - J_{2m}(\tau_k, d) = B_{2m}(\tau_k) - \frac{1}{\pi}\int_{-1}^{1} \frac{dB_{2m}(y)}{d^2 + (\tau_k - y)^2}\,\mathrm{d}y, \quad k, m = 0, 1, \ldots, n,$$

with collocation points, for example,

$$\tau_k = \cos\frac{(2k+1)\pi}{4(n+1)}, \quad k = 0, 1, \ldots, n.$$

We use Lemma 4.1 for calculating integrals J_{2m}.

The condition number of the matrix $\tilde{A}_n^{(0)}$ (in the 1-norm) for $n = 2(2)12$ is given in Table 4.1. Numbers in parentheses indicate decimal exponents. We can see that for these values of n we have $\mathrm{cond}(\tilde{A}_n^{(0)}) \sim 10^n$, which means that the corresponding system of linear equations for determining the coefficients c_m in (4.13) becomes ill-conditioned when n increases. It causes a loss of about n decimal digits in the coefficients c_m. A recent progress in symbolic computation and variable-precision arithmetic enables overcoming of this numerical instability, by seting `WorkingPrecision` to be sufficiently large.

Taking nonnegative extremal points of the Chebyshev polynomial $T_{2n}(x)$ as collocation points, i.e., $\tau_k = \cos\frac{k\pi}{2n}, k = 0, 1, \ldots, n$, as well as equidistant points $\tau_k = \frac{k}{n}, k = 0, 1, \ldots, n$, the corresponding condition numbers are also presented in the same Table 4.1. As we can see the condition number in the case of equidistant collocation points is much bigger that the ones for Chebyshev points (zeros or extremal points)!

In the sequel we use Chebyshev zeros as collocation points. Using the previous procedure, for $n = 2$ and $n = 4$, we find

$$\tilde{f}_4^{(0)}(x) = 2.63989B_0(x) - 0.32014B_2(x) + 0.0400543B_4(x) = 1.919498 - 0.320140x^2 + 0.0400543x^4$$

and

$$\begin{aligned}\tilde{f}_8^{(0)}(x) &= 2.46662B_0(x) - 0.264159B_2(x) + 0.0160255B_4(x) + 0.000730762B_6(x) - 0.00565549B_8(x)\\ &= 1.91903 - 0.311595x^2 + 0.014564x^4 + 0.0233527x^6 - 0.00565549x^8.\end{aligned}$$

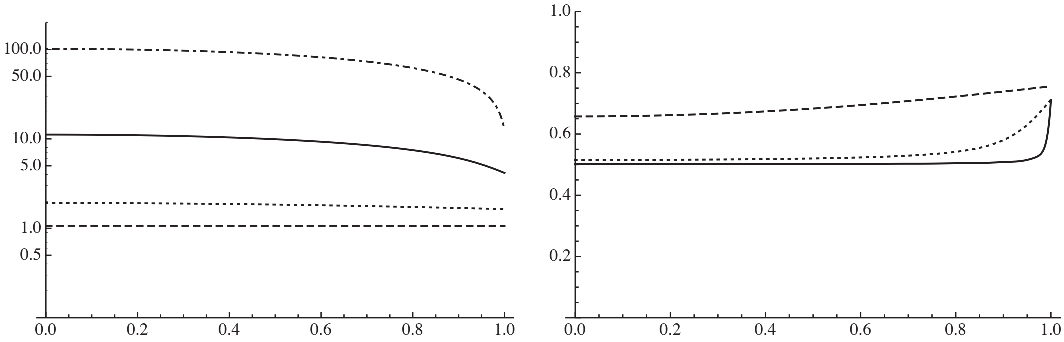

Fig. 4.3. (left) The solutions of Love's Equation (4.2) as log-plots for $d = 10$ (dashed line), $d = 1$ (dotted line), $d = 1/10$ (solid line), and $d = 1/100$ (dot-dashed line); (right) The solutions of Love's equation (4.4) for $d = 1$ (dashed line), $d = 1/10$ (dotted line) and $d = 1/100$ (solid line).

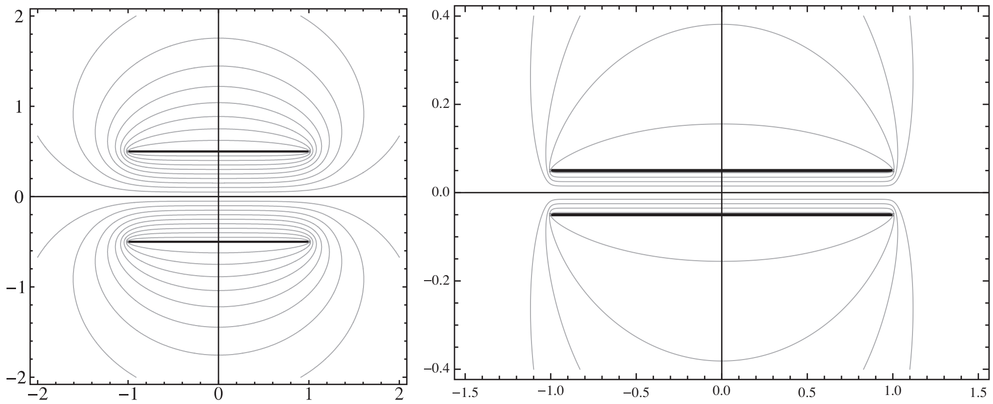

Fig. 4.4. Equipotential lines for $\pm V = 0.1(0.1)0.9$ when $d = 1$ (left) and for $\pm V = 0.3(0.2)0.9$ when $d = 1/10$ (right).

Maximal relative errors of the previous approximate solutions, including Love's solution (4.7), are presented in Table 4.2, where we used as the exact solution the one obtained by an efficient method for solving Fredholm integral equations of the second kind [16]). Alternatively, we can use $\tilde{f}_{32}^{(0)}(x)$ as $f(x)$.

Also, graphs of the relative errors $|(\tilde{f}_8^{(0)}(x) - f(x))/f(x)|$ and $|(f_L(x) - f(x))/f(x)|$ are displayed in Fig. 4.2. Notice that the both approximate solutions $\tilde{f}_8^{(0)}(x)$ and $f_L(x)$ are polynomials of the same degree eight.

The solutions $\tilde{f}_8^{(0)}(x)$ for different values of the distance d ($d = 0.01, d = 0.1, d = 1$, and $d = 10$) are presented in Fig. 4.3 (left). In the case when $d \to \infty$ the solution of the Love's equation (4.2) tends to the constant $f(x) = 1$. For example for $d = 10$, the corresponding solutions $\tilde{f}_{2n}^{(0)}(x)$ for $n = 1$ and $n = 2$ are

$$\tilde{f}_2^{(0)}(x) = 1.067734116 - 0.00065980x^2,$$

$$\tilde{f}_4^{(0)}(x) = 1.067734911236 - 0.00066617763x^2 + 6.3737080810 \cdot 10^{-6}x^4,$$

with maximal relative errors on $[-1, 1]$, $7.40(-7)$ and $1.79(-9)$, respectively. Now, using (4.3), we can calculate and plot the equipotential lines (see Fig. 4.4 for two cases $d = 1$ and $d = 1/10$).

4.2. Love's integral equation (4.4)

Finally, we give some results for Love's integral equation (4.4), for which the matrix of the corresponding system of equations is given by $[\tilde{a}'_{km}]_{k=0,m=0}^{n,n}$, where

$$\tilde{a}'_{km} = B_{2m}(\tau_k) + J_{2m}(\tau_k, d) = B_{2m}(\tau_k) + \frac{1}{\pi}\int_{-1}^{1} \frac{dB_{2m}(y)}{d^2 + (\tau_k - y)^2}\,dy, \quad k, m = 0, 1, \ldots, n,$$

with collocation points,

$$\tau_k = \cos\frac{(2k+1)\pi}{4(n+1)}, \quad k = 0, 1, \ldots, n.$$

The solutions for different values of the distance d ($d = 1, d = 1/10$, and $d = 1/100$) are presented in Fig. 4.3 (right). A problem in approximation can appear when $d \to 0^+$. Namely, in that case we have

$$(Kf)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{d}{d^2 + (x-y)^2} f(y)\, dy \to f(x), \quad d \to 0^+$$

which means that for $-1 < x < 1$, the solution $f(x)$ of the Eq. (4.4) is nearly equal to 1/2, but at the endpoints $f(\pm 1) \approx 3/4$. Thus, in this case with small parameter d some difficulties in approximation, especially by polynomials, have appeared. An efficient procedure for a very small value of the parameter d in the Eq. (4.4) has recently been introduced by Pastore in [20].

Acknowledgement

We are very grateful to the reviewer for remarks and suggestions, which have improved the quality of the paper.

References

[1] K.E. Atkinson, The numerical solution of integral equations of the second kind, Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1997.
[2] K. Boubaker, Boubaker polynomials expansion scheme (BPES) solution to Boltzmann diffusion equation in the case of strongly anisotropic neutral particles forward–backward scattering, Ann. Nucl. Energy 38 (2011) 1715–1717.
[3] K. Boubaker, L. Zhang, Fermat-linked relations for the Boubaker polynomial sequences via Riordan matrices analysis, J. Assoc. Arab Univ. Basic Appl. Sci. 12 (2012) 74–78.
[4] G.T. Carlson, B.L. Illman, The circular disk parallel plate capacitor, Am. J. Phys. 62 (12) (1994) 1099–1105.
[5] W.C. Chew, J.A. Kong, Asymptotic formula for the capacitance of two oppositely charged discs, Math. Proc. Cambridge Philos. Soc. 89 (2) (1981) 373–384.
[6] D. Elliott, A Chebyshev series method for the numerical solution of Fredholm integral equations, Comput. J. 6 (1) (1963) 102–112.
[7] L. Fox, E.T. Goodwin, The numerical solution of non-singular linear integral equations, Philos. Trans. R. Soc. Lond. Ser. A 245 (1953) 501–534.
[8] V. Hutson, The circular plate condenser at small separations, Proc. Cambridge Philos. Soc. 59 (1963) 211–224.
[9] B. Karem Ben Mahmoud, Temperature 3D profiling in cryogenic cylindrical devices using Boubaker polynomials expansion scheme (BPES), Cryogenics 49 (2009) 217–220.
[10] S.-H. Kim, L. Zhang, K. Boubaker, Q. Lei, On zeros of Boubaker polynomials. <arXiv:1211.0383v2[math.FA]>.
[11] A.S. Kumar, An analytical solution to applied mathematics-related Love's equation using the Boubaker polynomials expansion scheme, J. Franklin Inst. 347 (2010) 1755–1761.
[12] P.K. Kythe, P. Puri, Computational Methods for Linear Integral Equations, Birkhäuser, Boston, Basel, Berlin, 2002.
[13] E.R. Love, The electrostatic field of two equal circular co-axial conducting disks, Quart. J. Mech. Appl. Math. 2 (1949) 428–451.
[14] E.R. Love, The potential due to a circular parallel plate condenser, Mathematika 37 (1990) 217–231.
[15] G. Mastroianni, G.V. Milovanović, Interpolation Processes – Basic Theory and Applications, Springer Monographs in Mathematics, Springer-Verlag, Berlin, Heidelberg, 2008.
[16] G. Mastroianni, G.V. Milovanović, Well-conditioned matrices for numerical treatment of Fredholm integral equations of the second kind, Numer. Linear Algebra Appl. 16 (2009) 995–1011.
[17] G.V. Milovanović, D. Joksimović, Some properties of Boubaker polynomials and applications, AIP Conf. Proc. 1479 (2012) 1050–1053.
[18] G. Monegato, A. Palamara Orsi, Product formulas for Fredholm integral equations with rational kernel functions, in: Numerical Integration, III Oberwolfach, 1987, Internat. Schriftenreihe Numer. Math., vol. 85, Birkhäuser, Basel, 1988, pp. 140–156.
[19] M. Norgren, B.L.G. Jonsson, The capacitance of the circular parallel plate capacitor obtained by solving the Love integral equation using an analytic expansion of the kernel, Progress In Electromagnetics Research, PIER 97 (2009) 357–372.
[20] P. Pastore, The numerical treatment of Love's integral equation having very small parameter, J. Comput. Appl. Math. 236 (6) (2011) 1267–1281.
[21] S. Prössdorf, B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Birkhäuser Verlag, Basel, 1991.
[22] E. Reich, E.R. Love's integral equation for the circular plate condenser, ANZIAM 46 (2004) 85–93.
[23] A. Sengupta, M. Rama Mohana Rao, On approximate solution of linear singular integral equations, Appl. Math. Comput. 36 (3) (1990) 147–161.
[24] K. Veselić, On real eigenvalues of real tridiagonal matrices, Linear Algebra Appl. 26 (1979) 167–171.
[25] H.J. Wintle, S. Kurylowicz, Edge corrections for strip and disc capacitors, IEEE Trans. Instr. Meas. 34 (1) (1985) 41–47.
[26] T.G. Zhao, L. Naing, W.X. Yue, Some new features of the Boubaker polynomials expansion scheme BPES, Math. Notes V87 (2) (2010) 165–168.