Convergence of Gaussian Quadrature Rules for Approximation of Certain Series*

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Dedicated to the Memory of Professor Nikolay Pavlovich Korneichuk

Abstract

In this paper we consider an application of Gaussian quadrature rules to numerical series. Beside the general properties of orthogonal polynomials related to linear functionals L (complex discrete measures μ), we study the convergence of the corresponding Gaussian-type quadrature rules and present some numerical examples.

1 Introduction

Define linear functional L, acting on the space of all polynomials $\mathcal{P}$, using some complex discrete measure μ, supported at the complex points $\lambda_\nu,\ \nu \in \mathbb{N}_0$, with complex masses $w_\nu,\ \nu \in \mathbb{N}_0$, i.e.,

$$L(p) = \int p d\mu = \sum_{\nu=0}^{+\infty} w_\nu p(\lambda_\nu), \quad w_\nu,\ \lambda_\nu \in \mathbb{C}\backslash\{0\},\ \lim_{i\to+\infty} \lambda_\nu = 0. \tag{1.1}$$

We assume that the measure μ is such that all polynomials are absolutely μ-integrable, i.e., for all n-series

$$\mu_n = \int x^n d\mu = \sum_{\nu=0}^{+\infty} w_\nu \lambda_\nu^n, \tag{1.2}$$

is absolutely convergent.

Since sequence $\{\lambda_\nu\}_{\nu\in\mathbb{N}_0}$, converging to zero, for an absolute convergence of the series representing moments, it is enough to have an absolute summability of the sequence of masses $\{w_\nu\}_{\nu\in\mathbb{N}_0}$. Hence, we assume that

$$\sum_{\nu=0}^{+\infty} |w_\nu| = W < +\infty. \tag{1.3}$$

As a second condition we suppose a regularity of the linear functional L (or complex discrete measure μ), i.e., we assume that the sequence of orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}_0}$ exists (uniquely) and that the following three term recurrence relation

$$xp_n(x) = \beta_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x),\ p_0(x) = 1,\ p_{-1}(x) = 0, \tag{1.4}$$

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holds, which is provided by $\beta_n \neq 0$ for each $n \in \mathbb{N}_0$. The monic version of the polynomial p_n is denoted by P_n.

The second linearly independent solution of (1.4) is denoted by q_n, $n \in \mathbb{N}_0$, and it satisfies the initial conditions $q_{-1} = -1$ and $q_0 = 0$ (cf. [5], [2]). The monic version of this solution is denoted by Q_n. We also refer to this polynomials as numerator polynomials of the sequence p_n. Note that $\deg(q_n) = n - 1$.

Let the measure μ be normalized such that $\beta_0 = 1$, i.e., $\sum_{\nu\in\mathbb{N}_0} w_\nu = 1$.

We assume also that three term recurrence coefficients have the following limits

$$\lim_{n\to+\infty} \alpha_n = 0, \qquad \lim_{n\to+\infty} \beta_n = 0. \tag{1.5}$$

Following [2], we give the following definition:

Definition 1.1 *For every two uniformly bounded sequences of complex numbers $\{\alpha_k\}_{k\in\mathbb{N}_0}$ and $\{\beta_k\}_{k\in\mathbb{N}_0}$ we associate an infinite tridiagonal complex Jacobi matrix*

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & & \\ \beta_1 & \alpha_1 & \beta_2 & \\ & \beta_2 & \alpha_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}. \tag{1.6}$$

This Jacobi matrix can be interpreted as a linear operator acting on the Hilbert space ℓ^2 of all complex square-summable sequences with usual scalar product $\langle u, v\rangle = \sum_{\nu\in\mathbb{N}_0} u_\nu \overline{v_\nu}$. The value of the operator can be defined as a result of matrix multiplication of the infinite matrix given in (1.6) with an infinite vector representing an element from ℓ^2. We refer to Jacobi matrix when we mean to refer to associated linear operator and vice versa.

Definition 1.2 *For a linear functional L, the symbol J represents a complex Jacobi matrix and the corresponding linear operator acting on ℓ^2 constructed from sequences given in* (1.4).

It is known (cf. [2], [1]) that the condition (1.5) is sufficient for the related Jacobi operator to be compact. Even more, under assumption that J is compact, the condition (1.5) holds.

In the sequel we assume that the linear functional L (or complex discrete measure μ) is such that the conditions given in (1.3), (1.4) and (1.5) are satisfied.

Under the assumption that the measure μ is positive, i.e., the sequences $\{\lambda_\nu\}_{\nu\in\mathbb{N}_0}$ and $\{w_\nu\}_{\nu\in\mathbb{N}_0}$ are real and positive, respectively, it is known that the corresponding Jacobi operator is compact, i.e., three term recurrence coefficients satisfy the condition (1.5). Even more, under condition that the compact Jacobi operator is self-adjoint, the measure of orthogonality can be obtained from the spectral representation theorem (see [8]). This means that set of linear functionals L, satisfying the previous conditions (1.3), (1.4) and (1.5), is not empty.

According to [12], the linear functional $L_a^{p,q}$, given by

$$L_a^{p,q}(f) = \frac{p-1}{p} \sum_{\nu=0}^{+\infty} \frac{1}{p^\nu} f\left(\frac{a}{q^\nu}\right), \quad |p| > 1, \quad |q| > 1, \tag{1.7}$$

is regular, has absolutely summable masses and three term recurrence coefficients are given by

$$\begin{aligned} \alpha_k &= aq^k \frac{p+q-2pq^k(1+q)+pq^{2k}(p+q)}{(pq^{2k-1}-1)(pq^{2k+1}-1)}, && k \geq 0, \\ \beta_0^2 &= 1, && \\ \beta_k^2 &= a^2 p\, q^{2k} \frac{(q^k-1)^2(pq^{k-1}-1)^2}{(pq^{2k-2}-1)(pq^{2k-1}-1)^2(pq^{2k}-1)}, && k \geq 1. \end{aligned} \tag{1.8}$$

Hence, the related Jacobi operator is compact.

That the set of complex discrete measures satisfying all three conditions is not empty was noted by Carlitz (see [4]). At the end of his paper [4], there is a short note explaining that results are true also for all real α, except for negative integers or zero[1]. The following linear functional has been studied

$$L^\alpha(p) = \int p\, d\mu = \frac{\alpha e^{-\alpha}}{2} \sum_{k\in\mathbb{N}_0} e^{-k} \frac{(k+\alpha)^{k-1}}{k!} \Big(p((k+\alpha)^{-1/2}) + p(-(k+\alpha)^{-1/2}) \Big). \tag{1.9}$$

It can be proved that under condition α is not negative integer or zero, the orthogonal polynomials exists (uniquely), i.e., the functional L^α is regular. It is easy to check that masses are absolutely summable, and also that related Jacobi operator is compact, since the coefficients in the three-term recurrence relation (1.4) are

$$\alpha_k = 0 \ \ (k \in \mathbb{N}_0), \quad \beta_0^2 = 1, \quad \beta_k^2 = \frac{k}{(k+\alpha)(k+\alpha-1)} \ \ (k \in \mathbb{N}).$$

There are also several other examples in this direction.

This paper is organized as follows. Section 2 is devoted to general properties of orthogonal polynomials related to the linear functionals L (complex discrete measures μ), which satisfy all three conditions mentioned before. The convergence results of Gaussian-type quadrature rules are presented in Section 3. Finally, some numerical examples are given in Section 4.

2 General properties of polynomials orthogonal w.r.t. linear functional L (complex discrete measure μ)

In this section we need a general result from [2], which states that all zeros of all orthogonal polynomials are contained if the closure of the numerical range of Jacobi operator J. The numerical range of J is defined as the set $\{\langle Jx, x\rangle \mid x \in \ell^2,\ ||x|| = 1\}$. Following [2] we refer to its closure as $\Gamma(J)$. We can express the mentioned statement as all zeros of all orthogonal polynomials are contained in $\Gamma(J)$.

It is easy to check (see [11]) that the set $\Gamma(J)$ is bounded, provided Jacobi operator J is bounded and has the property $\operatorname{diam}(\Gamma(J)) \le 2||J||$. This means all zeros of all orthogonal polynomials are bounded in their modulus by $||J||$.

Since we are interested only in compact Jacobi operators, we know their spectrum is at most countable set with zero as the only accumulation point (see [8]), also all points except zero, contained in the spectrum, are eigenvalues of Jacobi operator J. Under three conditions imposed on L, (1.3), (1.4) and (1.5), we can state the following result:

Theorem 2.1 *The spectrum $\sigma(J)$ of the Jacobi operator J and $\operatorname{supp}(\mu)$ (the support of the related measure μ), satisfy the following equality $\sigma(J) = \operatorname{supp}(\mu) \cup \{0\}$.*

Proof. It is known that the Weyl function, given by

$$\phi(z) = \langle (zI - J)^{-1} e_0, e_0\rangle = \frac{1}{z} \sum_{\nu=0}^{+\infty} \frac{\langle J^\nu e_0, e_0\rangle}{z^\nu}, \qquad |z| > ||J||,$$

is analytic on the set $\mathbb{C}\backslash\sigma(J)$ and on no larger set. Since the quantities $\langle J^\nu e_0, e_0\rangle$, $\nu \in \mathbb{N}_0$, represent the moments (see [1], [2]), the Weyl function ϕ coincides with the function

$$\hat{\phi}(x) = \int \frac{d\mu}{z - x} = \frac{1}{z} \sum_{\nu=0}^{+\infty} \frac{\mu_\nu}{z^\nu},$$

[1] The paper has an error introduced in masses w_k. Instead of $\frac{1}{2}\alpha e^{-\alpha} \frac{k+\alpha^{k-1}}{k!}$ it should be $\frac{1}{2}\alpha e^{-\alpha-k} \frac{k+\alpha^{k-1}}{k!}$.

on the open set defined by $|z| > ||J||$. However, the function $\hat{\phi}$ can be calculated from

$$\hat{\phi} = \int \frac{d\mu}{z-x} = \sum_{\nu=0}^{+\infty} \frac{w_\nu}{z-\lambda_\nu}, \qquad z \in \mathbb{C}\backslash\overline{\{\lambda_\nu \mid \nu \in \mathbb{N}_0\}}.$$

It is easy to check that the series, representing function $\hat{\phi}$, is absolutely convergent on any compact $C \subset \mathbb{C}\backslash\overline{\{\lambda_\nu \mid \nu \in \mathbb{N}_0\}}$, since for any $z \in C$

$$\left|\frac{w_\nu}{z-\lambda_\nu}\right| \leq \frac{|w_\nu|}{\operatorname{dist}(C, \operatorname{supp}(\mu))},$$

and, as we know from (1.3), the sequence $\{w_\nu\}_{\nu\in\mathbb{N}_0}$ is absolutely convergent. The previous means that, using Weierstrass theorem, the functional series is uniformly convergent on any compact $C \subset \mathbb{C}\backslash\overline{\{\lambda_\nu \mid \nu \in \mathbb{N}_0\}}$ and hence, it represents an analytic function. This means that the function $\hat{\phi}$ is an analytic continuation of the Weyl function ϕ.

It remains to prove that this analytic continuation is unique. For any given point $a \notin \overline{\{\lambda_\nu \mid \nu \in \mathbb{N}_0\}}$ we can construct an open set containing a and which intersection with $\overline{\{\lambda_\nu \mid \nu \in \mathbb{N}_0\}}$ is empty, such that the intersection with $|z| > ||J||$ is not empty. The previous is true, because there are only finitely many points λ_ν with the property $|\lambda_\nu| > |a| \neq 0$. The mentioned set can be constructed to be an open neighborhood of half line originating in a and tending to infinity, which intersection with the set $|z| < |a|$ is empty. This open set has nonempty intersection with the region $|z| > ||J||$. Thus, using Weierstrass theorem of analytic continuation, we know that there is unique analytic continuation in a and that continuation is $\hat{\phi}$.

It is known that the Weyl function has singularities at the points which are spectrum of J. Even more, if a singularity ζ of ϕ is isolated and is a pole of multiplicity m, related Jacobi operator has at ζ an eigenvalue of multiplicity m. If the point ζ is an essential singularity of ϕ, then the point ζ belongs to the essential spectrum of the corresponding Jacobi operator[2] (see [2]). Since our Weyl function has simple poles at λ_ν, $\nu \in \mathbb{N}_0$, they are eigenvalues of multiplicity 1 of J. The only point which belongs to the essential spectrum of J is zero. $\square$

Since the spectrum is point with eigenvalues λ_ν, $\nu \in \mathbb{N}_0$, it is clear that respected eigenvectors are made of the values of the orthonormal polynomials p_n, $n \in \mathbb{N}_0$, at points λ_ν as their components. Denote by $p(x)$ the infinite vector $(p_n(x))_{n\in\mathbb{N}_0}$. Taking $n = 0, 1, \ldots$ in (1.4) we get an equivalent operator equation

$$Jp(x) = xp(x).$$

Choosing for x the values λ_ν, $\nu \in \mathbb{N}_0$, it is clear that $p(\lambda_\nu)$, $\nu \in \mathbb{N}_0$, are eigenvectors. Hence, $p(\lambda_\nu)$ belongs to ℓ^2. As a consequence, they are square-summable $\langle p(\lambda_\nu), p(\lambda_\nu)\rangle < +\infty$ and $\lim_{n\to+\infty} p_n(\lambda_\nu) = 0$, which implies

$$P_n(\lambda_\nu) = o(||p_n||) = o\left(\prod_{k=0}^{n} \beta_k\right).$$

It means that at the points λ_ν we have the convergence

$$\lim_{n\to+\infty} \frac{P_{n+1}(\lambda_\nu)}{P_n(\lambda_\nu)} = \lim_{n\to+\infty} \beta_n = 0.$$

For other points in the complex plane, we have:

Lemma 2.1 *For every $z \in \mathbb{C}\backslash\sigma(J)$, we have*

$$\lim_{n\to+\infty} \frac{P_{n+1}(z)}{P_n(z)} = z.$$

[2]The essential spectrum of Jacobi operator is the set $\sigma_{\text{ess}}(J) = \{z \mid \text{range of } zI - J \text{ is not closed}\}$ (cf. [8], [2]).

Proof. From Poincare theorem (see [10]), there are two possibilities

$$\lim_{n\to+\infty} \frac{P_{n+1}(z)}{P_n(z)} = \begin{cases} 0, \\ z, \end{cases}$$

under condition $z \neq 0$, since the condition (1.5) holds.

Suppose that this limit is 0 for some $z \in \mathbb{C}\backslash\sigma(J)$. Then, from

$$\frac{P_{n+1}(z)}{P_n(z)} = z - \alpha_n - \beta_n^2 \frac{P_{n-1}(z)}{P_n(z)},$$

we can conclude that

$$\frac{P_n(z)}{P_{n-1}(z)} = O\left(\frac{\beta_n^2}{z}\right),$$

from where

$$P_n(z) = O\left(z^{-n}\prod_{k=0}^{n}\beta_k^2\right) \qquad \text{or} \qquad p_n(z) = O\left(z^{-n}\prod_{k=0}^{n}\beta_k\right).$$

Now, we find that $p(z)$ is square-summable, since $|p_n(z)|^{1/n} = O(|z|^{-1}|\beta_n|) \to 0 < 1$. This means z is an eigenvalue of J, which is a contradiction. □

Denote by (x, y) the bilinear functional $\langle x, \mathcal{C}y\rangle$, where $\mathcal{C}$ is the conjugation operator, i.e.,

$$\mathcal{C}(x_n)_{n\in\mathbb{N}_0} = (\overline{x}_n)_{n\in\mathbb{N}_0} \qquad \text{and} \qquad (x, y) = \langle x, \mathcal{C}y\rangle.$$

If vectors x and $\mathcal{C}y$ are orthogonal in ℓ^2, we say that x and y are *weakly orthogonal* or orthogonal with respect to $(\,\cdot\,,\,\cdot\,)$.

Lemma 2.2 *There holds*

$$(p(\lambda_i), p(\lambda_j)) = \sum_{n=0}^{+\infty} p_n(\lambda_i)p_n(\lambda_j) = \frac{\delta_{i,j}}{w_i}, \tag{2.10}$$

i.e., the vectors $p(\lambda_i)$, $i \in \mathbb{N}_0$, are weakly orthogonal.

Proof. Using Christoffel-Dabroux identity we have

$$\sum_{k=0}^{n} p_k(\lambda_j)p_k(\lambda_i) = \beta_{n+1}\frac{p_{n+1}(\lambda_j)p_n(\lambda_i) - p_{n+1}(\lambda_i)p_n(\lambda_j)}{\lambda_j - \lambda_i},$$

provided $i \neq j$. For the absolute value of the left side in this equality we get

$$\left|\sum_{k=0}^{n} p_k(\lambda_i)p_k(\lambda_j)\right| \leq |\beta_{n+1}|\frac{|p_{n+1}(\lambda_j)|\,|p_n(\lambda_i)| + |p_{n+1}(\lambda_i)|\,|p_n(\lambda_j)|}{|\lambda_j - \lambda_i|}.$$

Taking the limit it can be seen that

$$(p(\lambda_j), p(\lambda_i)) = \langle p(\lambda_j), Cp(\lambda_i)\rangle = \sum_{n=0}^{+\infty} p_n(\lambda_j)p_n(\lambda_i) = 0.$$

Now, we consider the function

$$K(x, y) = (p(x), p(y)) = \sum_{n=0}^{+\infty} p_n(x)p_n(y), \quad x, y \in \{\lambda_i \mid i \in \mathbb{N}_0\},$$

for which is clear that

$$\int K(x,y)d\mu(x) = \int p_0^2(x)d\mu(x) = 1.$$

On the other hand

$$\int K(x,\lambda_j)d\mu(x) = \sum_{i=0}^{+\infty} w_i K(\lambda_i,\lambda_j) = w_j(p(\lambda_j),p(\lambda_j)) = 1,$$

and the weak orthogonality condition holds. $\square$

Consider now the case in which only a compact Jacobi operator J is given, with infinitely many complex eigenvalues $\lambda_i,\ i \in \mathbb{N}_0$. We are interested in a construction of the measure of orthogonality. At first we note that the proof on weak orthogonality of vectors $p(\lambda_i),\ i \in \mathbb{N}_0$, still holds. For this general case we can prove the following auxiliary result:

Lemma 2.3 *Let J be a Jacobi operator with infinitely many eigenvectors $p(\lambda_i),\ i \in \mathbb{N}_0$, and let X be the set $X = \{x \mid (x,p_i) = 0,\ i \in \mathbb{N}_0\}$. Then, $J(X) \subset X,\ \overline{X} = X$ and X is a (closed) linear subspace of ℓ^2.*

Proof. If $x, y \in X$, then obviously

$$(\alpha x + \beta y, p(\lambda_i)) = \alpha(x,p(\lambda_i)) + \beta(y,p(\lambda_i)) = 0.$$

It means X is a linear subspace of ℓ^2. By continuity of $(\,\cdot\,,\,\cdot\,)$, since both $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\mathcal{C}$ are continuous, it is easy to conclude that if $x_n \in X$ tends to some x, then also $(x_n, p(\lambda_i)) \to 0$ and $(x,p(\lambda_i)) = 0$. It means $x \in X$ and X is closed.

If $x \in X$ then $(Jx, p(\lambda_i)) = (x, J(p(\lambda_i)) = \lambda_i(x,p(\lambda_i)) = 0$, which means $Jx \in X$ and $J(X) \subset X$. $\square$

However, we cannot conclude that subspace X of ℓ^2 is free of eigenvectors, since we have no argument against the case $(p(\lambda_i),p(\lambda_i)) = 0$, for some (possibly for all) $i \in \mathbb{N}_0$. Hence, obvious solution with masses $w_i = 1/(p(\lambda_i),p(\lambda_i)),\ i \in \mathbb{N}_0$, cannot be used safely.

Also note that even under assumption $(p(\lambda_i),p(\lambda_i)) \neq 0,\ i \in \mathbb{N}_0$, set X need not be the trivial subspace $\{0\}$, since constructive argument for eigenvectors, which can be successfully applied to self-adjoint J cannot be applied in a general case. We recall, for example, the Bessel polynomials having compact associated Jacobi operator without eigenvalues.

Denote by $J^{(k)}$ Jacobi matrix obtained from J by deleting first k columns and rows, i.e., an operator which matrix is created from delayed sequences $\alpha_n^{(k)} = \alpha_{n+k},\ \beta_n^{(k)} = \beta_{n+k}$. Denote by $\Gamma_{\text{ess}}(J)$ the following intersection

$$\Gamma_{\text{ess}}(J) = \bigcap_{k=0}^{+\infty} \Gamma(J^{(k)}).$$

For our compact operator, we have

Lemma 2.4 $\Gamma_{\text{ess}}(J) = \{0\}$.

The previous can be understood easily since sequences of recursive coefficients are decreasing and since (using bound for example given in [2] and [8])

$$\Gamma(J^{(k)}) \leq ||J^{(k)}|| \leq \sup_{n\geq k}\Big(|\beta_{n+1}| + |\alpha_n| + |\beta_n|\Big),$$

which gives $\lim_{k\to+\infty} ||J^{(k)}|| = 0$, with a direct consequence $\text{diam}(\Gamma(J^{(k)})) \to 0$ as $k \to +\infty$.

Denote by $\Omega(J) = \mathbb{C} \backslash \sigma(J)$, i.e., $\Omega(J)$ is the resolvent set of J. There is a result proved in [2], which states that for any compact set $C \subset \Omega(J) \backslash \Gamma_{ess}(J)$, there exists $N(C) \in \mathbb{N}$ such that there are no zeros of $p_n,\ n > N(C)$, in C.

From [13], it is know that some zeros of orthogonal polynomials need not be connected to the spectrum of J for a general complex Jacobi operator J. This can be true even for self-adjoint operators. It is enough to consider the measure $d\mu = \chi_{[-2,-1]}\chi_{[1,2]}dx$ to conclude, by the symmetry argument, that polynomials of odd degree must have zero at the origin. This sequence of zeros at the origin is not connected with the spectrum of the corresponding Jacobi operator, since its spectrum must be $[-2,-1] \cup [1,2]$.

For bounded Jacobi operator, we can give the following definition:

Definition 2.3 *Suppose there exists a sequence of zeros $\{x_n\}_{n \in N'}$, where N' is an infinite subset of $\mathbb{N}_0$ and x_n is a zero of p_n, such that*

$$\lim_{n \to +\infty,\ n \in N'} x_n = \zeta \notin \sigma(J).$$

The point ζ is called the spurious zero for a sequence of orthogonal polynomials $\{p_n\}_{n \in \mathbb{N}_0}$.

Lemma 2.5 *If a Jacobi operator J is compact, there are no spurious zeros of the sequence $\{p_n\}_{n \in \mathbb{N}_0}$. In another words, if a complex measure of orthogonality μ, given by (1.1), is such that the corresponding Jacobi operator is compact, the sequence of orthonormal polynomials $\{p_n\}_{n \in \mathbb{N}_0}$, does not have spurious zeros.*

Proof. Suppose there exists a spurious zero ζ and consider compact set F defined by

$$F = \{z \mid |z - \zeta| \leq \operatorname{dist}(\zeta, \sigma(J))/2\} \subset \Omega(J) \backslash \Gamma_{\text{ess}}(J).$$

¿From the previous it is clear that there is $N(F) \in \mathbb{N}$, such that there are no zeros of $p_n,\ n > N(F)$, in F, which is a contradiction. □

One consequence is that zeros of orthogonal polynomials related with a compact Jacobi operator are asymptotically simple.

Theorem 2.2 *Zeros of orthonormal polynomials $p_n,\ n \in \mathbb{N}_0$, are asymptotically simple in the neighborhood of each $\lambda_i,\ i \in \mathbb{N}_0$.*

Proof. Since a sequence of Pade approximants Q_n/P_n is converging uniformly on any compact $\Omega(J) \backslash \Gamma_{\text{ess}}(J)$ (see [2]), and the Weyl function has simple poles at points $\lambda_i,\ i \in \mathbb{N}_0$ (see Theorem 2.1), it is clear that Q_n/P_n, in a limit, must have a simple pole at every λ_i.

Q_n/P_n can have simple pole at λ_i, as $n \to +\infty$, under a condition that there is one zero more in P_n than in Q_n near λ_i. In another words, if P_n has $k+1$ zeros near λ_i, then Q_n must have k zeros near λ_i, and obviously all zeros must converge to λ_i. Denote zeros of Q_n near λ_i by $x_j^{n,i},\ j = 1, \ldots, k$. It is clear that $\lim_{n \to \infty} x_j^{n,i} = \lambda_i$.

Since the matrix $J^{(1)}$ is compact and the sequence of related orthonormal polynomials is given by $\{q_n\}_{n \in \mathbb{N}_0}$, it is clear that since it does not have spurious zeros, the sequence $x_j^{n,i}$, which is convergent, must converge to the spectral point of $J^{(1)}$. The previous means that λ_i is an eigenvalue of $J^{(1)}$ and the respected eigenvector is $q(\lambda_i)$. Thus, the both vectors $p(\lambda_i)$ and $q(\lambda_i)$ are elements of ℓ^2 and J is indeterminate (see [14], [2], [3], [11]), but this is impossible since J is bounded and hence determinate. □

The previous argument means that q_n cannot have zeros converging to any of the points $\lambda_i,\ i \in \mathbb{N}_0$, and that p_n must have only one zero in the neighborhood of every $\lambda_i,\ i \in \mathbb{N}_0$, provided n is sufficiently large integer.

3 Convergence of Gaussian quadrature rules

Since it cannot be claimed that zeros of orthonormal polynomials $p_n,\ n \in \mathbb{N}_0$, are simple, the related Gaussian quadrature rules have to be of the form

$$G_n(f) = \sum_{k=1}^{N} \sum_{i=0}^{M_k-1} w_{i,k}^n f^{(i)}(x_k^n), \tag{3.11}$$

where x_k^n are zeros of p_n, with multiplicities M_k, with $\sum_{i=1}^{N} M_k = n$. The weights $w_{i,k}^n$ can be found from the fractional decomposition

$$\frac{Q_n(z)}{P_n(z)} = \sum_{k=1}^{N} \sum_{i=0}^{M_k} \frac{i! w_{i,k}^n}{(z - x_k^n)^{j+1}}.$$

The following result holds (cf. [7], [11]):

Theorem 3.3 *Gaussian quadrature rule of the form* (3.11), *has algebraic degree of exactness* $2n-1$, *i.e., it is exact on the space of all algebraic polynomials of degree at most* $2n-1$, *denoted by* $\mathcal{P}_{2n-1}$.

Since orthogonality of polynomials p_n can be established on the contour $C = \{z \mid |z| = R > ||J||\}$ (see [11], [7], [13]), it can be easily verified that

$$\oint_C \phi(z)\, P_n(z) z^k \, dz = \begin{cases} 0, & 0 \le k < n, \\ ||P_n||^2, & k = n, \end{cases}$$

where ϕ is the Weyl function mentioned already in Section 2. The previous is due to the well-known result for the Pade approximation, which states (see [9], [13])

$$\phi(z) - \frac{Q_n(z)}{P_n(z)} = O\left(z^{-2n-1}\right), \quad z \to \infty,$$

which can be also restated in the following form

$$\phi(z)P_n(z) - Q_n(z) = O(z^{-n-1}), \quad z \to \infty.$$

Using this property it can be easily checked that

$$\oint_C \left(\phi - \frac{Q_n}{P_n}\right) p \, dz = 0, \quad p \in \mathcal{P}_{2n-1},$$

from which it is clear that

$$\oint_C \phi \, p \, dz = \oint_C \frac{Q_n}{P_n} p \, dz = G_n(p), \quad p \in \mathcal{P}_{2n-1}.$$

Consider now an analytic function f, which is μ-integrable, analytic in connected open set $\mathcal{D}$, with rectifiable bound C, which contains $\sigma(J)$. We assume f is continuous on C. In [11] the following error bound can be found (see also [7] for a similar formula)

$$\left| \int f d\mu - G_n(f) \right| \le \frac{\ell(C) ||f||_C \int |P_n|^2 d|\mu|}{2\pi d \min\limits_{z\in C} |P_n(z)|^2}, \tag{3.12}$$

where $C = \partial\mathcal{D}$, $\ell(C)$ is the length of the curve C, $||f||_C$ is the usual sup norm on the compact set C, $|\mu|$ is the measure supported on $\mathrm{supp}(\mu)$ with masses $|w_i|,\ i \in \mathbb{N}_0$, and $d = \mathrm{dist}(C, \sigma(J))$.

In the sequel it is assumed points $\lambda_i,\ i \in \mathbb{N}_0$, are ordered such that $|\lambda_i| \geq |\lambda_{i+1}|,\ i \in \mathbb{N}_0$. We construct a curve C_ρ in the following way. Choose $\rho > 0$ and $\rho \neq |\lambda_i|,\ i \in \mathbb{N}_0$. Denote by K_ρ circle $|z| = \rho$. If $\rho > \max\{|\lambda_i| \mid i \in \mathbb{N}_0\}$, we have $C_\rho = K_\rho$. Suppose the previous is not true. Then, there are $k,\ k \in \mathbb{N}$, points from $\sigma(J)$ such that $\rho < |\lambda_i|$. Denote by M the minimal distance between points $\lambda_i,\ i < k$, i.e., $M = \min_{i,j<k,\ i\neq j} |\lambda_i - \lambda_j|$ and construct the circles $C_\rho^i = \{z \mid |z - \lambda_i| = M\rho/3\},\ i < k$. Finally connect the circle C_ρ^i with K_ρ by some (rectifiable) arc ℓ_ρ^i and denote by $\ell_{\rho,+}^i$ arc oriented form K_ρ to C_ρ^i and by $\ell_{\rho,-}^i$ arc oriented from C_ρ^i to K_ρ. Curve C_ρ is positive oriented curve given by

$$C_\rho = K_\rho \bigcup \left(\bigcup_{i<k} (C_\rho^i \cup \ell_{\rho,+}^i \cup \ell_{\rho,-}^i) \right). \tag{3.13}$$

Theorem 3.4 *Suppose function f is analytic in $\mathrm{int}(C_r)$ for some $r > 0$ and continuous on C_r. For every $\rho > 0,\ \rho \neq |\lambda_i|,\ i \in \mathbb{N}_0$, there exists $n_0(\rho) \in \mathbb{N}$, such that for every $n > n_0(\rho)$*

$$\left| \int f d\mu - G_n(f) \right|^{1/2n} \approx a_n \frac{2\rho}{r}, \tag{3.14}$$

where $a_n > 0$ and $a_n \to 1$ as $n \to +\infty$.

Proof. For any given ρ we consider the compact set

$$F = \Big\{ z \ \Big|\ |z| \leq 2||J|| \Big\} \setminus \left(\mathrm{int}(K_\rho) \bigcup \left(\bigcup_{|\lambda_i|>\rho} \mathrm{int}(C_\rho^i) \right) \right).$$

As before there exists $N_1 = N(F) \in \mathbb{N}$, such that there are no zeros of orthonormal polynomials p_n in F for $n > N_1$.

Number of zeros of the sequence $\{p_n\}_{n\in\mathbb{N}_0}$, in the set $\cup_{|\lambda_i|>\rho} \mathrm{int}(C_\rho^i)$ is uniformly bounded in n by some $N_2 \in \mathbb{N}$.

There exists also $N_3 \in \mathbb{N}$, such that for $|\lambda_i| > \rho$ and $n > N_3$ we have $|P_n(\lambda_i)| \approx o\left(\prod_{j=0}^n |\beta_j|\right)$.

Finally there exists $N_4 \in \mathbb{N}$ such that $(2\rho)^{-2n} \left| ||P_n||^2 \right| < 1$ for $n > N_4$, since $\left| ||P_n||^2 \right|$ decreases faster then exponentially.

Now we suppose $n > n_0(\rho) = \max\{N_1, N_2, N_3, N_4\}$. Then we have

$$\begin{aligned}
\int |P_n|^2 d|\mu| &= \sum_{i=k}^{+\infty} |w_i||P_n(\lambda_i)|^2 + \sum_{i=0}^{k-1} |w_i||P_n(\lambda_i)|^2 \\
&\leq \sum_{i=k}^{+\infty} |w_i| \prod_{j=1}^{n} |\lambda_i - x_j^n|^2 + o\left(\left|||P_n||^2\right|\right) \sum_{i=0}^{k-1} |w_i| \\
&\leq \sum_{i=k}^{+\infty} |w_i| \prod_{|x_n^j|<\rho} |\lambda_i - x_n^j|^2 \prod_{|x_n^j|>\rho} |\lambda_i - x_n^j|^2 + o\left(\left|||P_n||^2\right|\right) W \\
&\leq \sum_{i=k}^{+\infty} |w_i| (2\rho)^{2(n-N_2)} (2||J||)^{2N_2} + o\left(\left|||P_n||^2\right|\right) W \\
&\leq \left((2\rho)^{2(n-N_2)} (2||J||)^{2N_2} + o\left(\left|||P_n||^2\right|\right) \right) W \\
&= (2\rho)^{2(n-N_2)} (2||J||)^{2N_2} \left(1 + o\left((2\rho)^{-2(n-N_2)} \left|||P_n||^2\right| \right) \right) W,
\end{aligned}$$

which gives the upper bound for the integral $\int |P_n|^2 d|\mu|$.

We adopt the following short notation

$$B = \frac{\ell(C_r)||f||_{C_r} W}{2\pi d}$$

Applying (3.12) to $C = C_r$ with $\mathcal{D} = \mathrm{int}(C_r)$, using lemma 2.1, we get

$$\left| \int f d\mu - G_n(f) \right|^{1/2n} \leq \left(B(1 + o((2\rho)^{-2(n-N_2)} \left| ||P_n||^2 \right|)) \right)^{1/2n} \times$$
$$\times \left(\frac{||J||}{\rho} \right)^{N_2/n} \frac{2\rho}{\min_{z \in C_r} |P_n(z)|^{1/n}} \approx a_n \frac{2\rho}{r},$$

where

$$a_n = B^{1/2n}(1 + o((2\rho)^{-2(n-N_2)} \left| ||P_n||^2 \right|) \left(\frac{||J||}{\rho} \right)^{N_2/n} > 0.$$

It is obvious $a_n \to 1$ as $n \to +\infty$. $\square$

4 Numerical examples

Example 4.1 We consider a positive definite case which appears for linear functional $L_a^{p,q}$ given by (1.7) for $p,\ q > 1,\ a = 1$. We take $p = q = 2$ and

$$f(x) = \frac{1}{x - 3^{-20}}$$

The corresponding quadrature rules can be constructed using QR-algorithm (cf. [6]). Gaussian approximations $G_n(f)$ and the corresponding relative errors (r. err.) are given in Table 4.1 (m.p. stands for machine precision $\approx 10^{-16}$ in double precision).

The same behavior we get for the choice $a = 1,\ p = q = 2i,\ i = \sqrt{-1}$, when the quadrature rule is applied for integration of the same function f. Table 4.2 displays the corresponding results for this case.

The behavior of the relative errors can be fully understood using Theorem 3.4. While zeros of orthogonal polynomials have modulus grater then 3^{-20}, the convergence of our quadrature rules is bad since ρ/r is greater then 1. In this case it can be rather said that quadrature rules are diverging. When some zeros drop below 3^{-20}, ρ can also be improved and hence ρ/r is smaller then 1, having as a consequence very fast convergence. For example, the smallest modulus of zeros for $n = 40$ is of order 10^{-13} and for $n = 30$ is of order 10^{-10}. Note that $3^{-20} \approx 3 \times 10^{-10}$.

n	G_n	r. err.
10	6.605719151641912	0.54
20	11.60759673863332	0.20
30	18.19943971672188	0.26
40	14.46129284956265	m.p.

Table 4.1: Gaussian approximations G_n and relative errors for $L_1^{2,2}(1/(. - 3^{-20}))$

n	G_n	r.err.
10	$9.577889690630307 + 4.0332355954016054i$	0.66
20	$19.57775686236893 + 9.0341888460741501i$	0.30
30	$29.71613796045620 + 14.066446517362046i$	0.07
40	$27.52570186363645 + 13.689702133806741i$	m.p.

Table 4.2: Gaussian approximations G_n and relative errors for $L_1^{2i,2i}(1/(. - 3^{-20}))$

Example 4.2 Consider the functional of the form

$$L(p) = -\frac{1}{\zeta(2)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^k}{k^2} p\left(\frac{1}{k}\right). \tag{4.15}$$

The moments are given by

$$\mu_{2k} = 2^{-2k}(2^{2k+1} - 1)\frac{\zeta(2k+2)}{\zeta(2)}, \quad \mu_{2k+1} = 0, \quad k \in \mathbb{N}_0.$$

Numerical examples shows that (note that $\alpha_k = 0$, $k \in \mathbb{N}_0$, by the symmetry) β coefficients in three term recurrence relation (1.4) are such that condition (1.5) holds. However, the convergence of $\{\beta_k\}_{k\in\mathbb{N}_0}$ to zero is much slower then it is for three term recurrence coefficients for a linear functional given by (1.7).

In the case of entire function, the convergence is rather fast and not disrupted. An application to the function $\cos(x)$ gives result with machine precision with only 10 nodes in G_n. However, when some singularities in the integrand are introduced, more numerical work is needed. We use this case to illustrate an example when the exact result is really *unreachable*. In order to show this phenomenon, we integrate the function

$$f(x) = \frac{1}{x - 3^{-20}/2},$$

with respect to L. It is clear f is μ-integrable. However, the value of the sum is *unreachable*, since zeros of orthogonal polynomials (nodes of the quadrature rules) have asymptotic behavior as $1/k$, and it is clear that we need roughly $3^{20}/2$ nodes in quadrature rule in order to achieve $\rho/r < 1$. For the present state of hardware this is *unreachable*.

More drastic example can be given in the case we are trying to integrate a function as in Example 4.1, having singularity at 3^{-20}, with respect to the functional L. It is obvious that this function is not integrable, since term with $k = 3^{20}$ is not determined in this case, however quadrature rules with say $n \leq 50$ converge.

We can state that quadrature rules can be safely applied, without encountering *unreachable* phenomenon, for functions with singularities which are greater then $|\lambda_n|$ with n less then for example few hounders.

Example 4.3 In the case of the functional L^α given by (1.9), we have that λ sequence is decreasing even more slower then for the functional L from previous example, which means that absolute value of the singularity, in order to be *reachable*, has to be even smaller. In Table 4.3 an illustration of this fact is given. Quadrature rules are constructed for $\alpha = i = \sqrt{-1}$ and applied to the function

$$f(x) = \frac{1}{x - 20^{-1/3}}.$$

Note that in this case we have very slow convergence of the zeros to the values λ_i, which is creating significant errors in the quadrature rules.

n	G_n	r. err.
10	$-0.8558888269296514 + 0.2466457390261486i$	0.21
20	$-0.9848001824924506 + 0.01584612614940328i$	0.07
30	$-0.9534034679656761 + 0.07406429506233932i$	0.01
40	$-0.9483952773180323 + 0.06708359861933309i$	$3(-6)$
50	$-0.9483979586973889 + 0.06708279145101483i$	$5(-12)$
60	$-0.9483979587018742 + 0.06708279145343425i$	m.p

Table 4.3: Gaussian approximation G_n and relative errors for $L^i(1/(. - 20^{-1/3}))$

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