ORTHOGONAL POLYNOMIALS ON THE RADIAL RAYS IN THE COMPLEX PLANE AND APPLICATIONS*

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We consider polynomials orthogonal on the radial rays in the complex plane as well as some applications of such polynomials. Beside the problems on existence and uniqueness of such polynomials, we give a numerical construction of these polynomials. We describe a discretized procedure, in particular for Jacobi weights on the rays, and analyze a special case of so-called two-sided Chebyshev polynomials. A distribution of zeros in a general case is included.

Some interesting special classes of orthogonal polynomials when the rays are distributed equidistant in the complex plane, with equal lengths and the same weights on the rays are also considered. A recurrence relation for these polynomials, a connection with standard polynomials orthogonal on the real line, and a differential equation are derived. It is shown that their zeros are simple and distributed symmetrically on the radial rays, with the possible exception of a multiple zero at the origin. An analogue of the Jacobi polynomials and the corresponding problem with the generalized Laguerre polynomials are also treated. Finally, some applications of these polynomials in physics and electrostatic are discussed.

1. Introduction

In this paper we study orthogonal polynomials on the radial rays in the complex plane. Let $M \in \mathbb{N}$, $a_s > 0$, $s = 1, 2, \dots, M$, and $0 \le \theta_1 < \theta_2 < \cdots < \theta_M < 2\pi$. Putting

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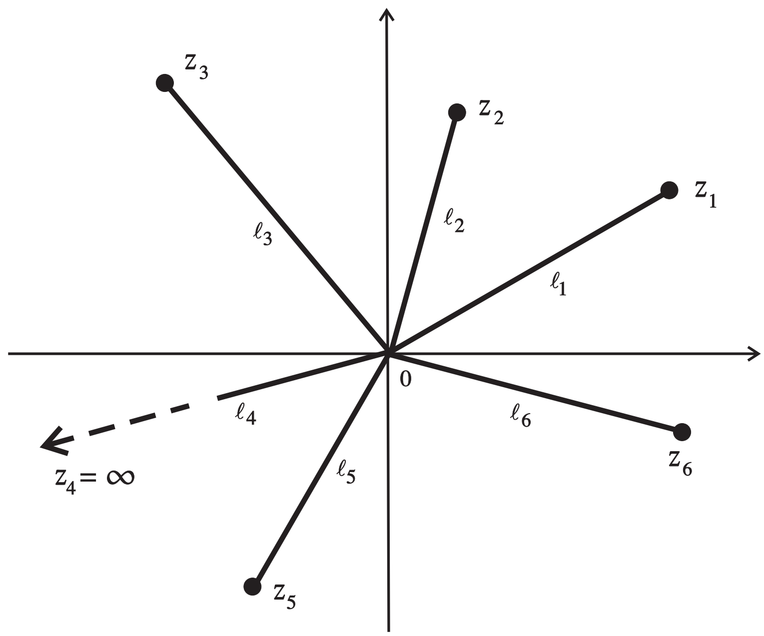

FIG. 1.1: The rays in the complex plane ($M = 6$)

$\varepsilon_s = e^{i\theta_s}$, $s = 1, 2, \ldots, M$, we consider M points in the complex plane, $z_s = a_s\varepsilon_s \in \mathbb{C}$, $s = 1, 2, \ldots, M$, with different arguments θ_s. Some of a_s (or all) can be ∞. The case $M = 6$ is shown in Figure 1.1.

We define an inner product (f, g) by

(1.1)
$$(f, g) = \sum_{s=1}^{M} e^{-i\theta_s} \int_{\ell_s} f(z)\overline{g(z)}\, |w_s(z)|\, dz,$$

where ℓ_s are the radial rays in the complex plane which connect the origin $z = 0$ and the points z_s, $s = 1, 2, \ldots, M$, and $z \mapsto w_s(z)$ are suitable complex (weight) functions. Precisely, we suppose that the functions $x \mapsto \omega_s(x) = |w_s(x\varepsilon_s)| = |w_s(z)|$ ($z \in \ell_s;\ s = 1, 2, \ldots, M$) are weight functions on $(0, a_s)$, i.e., they are nonnegative on $(0, a_s)$ and $\int_0^{a_s} \omega_s(x)\, dx > 0$. In the case when $a_s = +\infty$ all moments must exist and be finite.

The inner product (1.1) can be also expressed in the form

(1.2)
$$(f, g) = \sum_{s=1}^{M} \int_0^{a_s} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\, \omega_s(x)\, dx.$$

In a simple case when $M = 2$, $\theta_1 = 0$, $\theta_2 = \pi$, (1.2) becomes

$$(f, g) = \int_0^{a_1} f(x)\overline{g(x)}\,\omega_1(x)\,dx + \int_0^{a_2} f(-x)\overline{g(-x)}\,\omega_2(x)\,dx,$$

i.e.,

$$(f, g) = \int_a^b f(x)\overline{g(x)}\,\omega(x)\,dx, \tag{1.3}$$

where we put $a = -a_2$, $b = a_1$, and

$$\omega(x) = \begin{cases} \omega_1(x), & 0 < x < b, \\ \omega_2(-x), & a < x < 0. \end{cases}$$

Thus, it reduces to the standard case of polynomials orthogonal on (a, b) with respect to the weight function $x \mapsto \omega(x)$.

Using the characteristic function of a set A, defined by

$$\chi(A; z) = \begin{cases} 1, & z \in A, \\ 0, & z \notin A, \end{cases}$$

and putting $L = \ell_1 \cup \ell_2 \cup \ldots \cup \ell_M$, the inner product (1.1) can be expressed in an usual form

$$(f, g) = \int_L f(z)\overline{g(z)}\,d\mu(z), \tag{1.4}$$

where the measure $d\mu(z)$ is given by

$$d\mu(z) = \sum_{s=1}^{M} \varepsilon_s^{-1} |w_s(z)|\,\chi(\ell_s; z)\,dz. \tag{1.5}$$

The paper is organized as follows. In Section 2 we develop preliminary material on existence and uniqueness of the orthogonal polynomials on the radial rays in the complex plane, and in Section 3 we consider numerical construction of these polynomials. We describe a discretized procedure, in particular for Jacobi weights on the rays, and analyze a special case of so-called two-sided Chebyshev polynomials. A distribution of zeros is discussed in Section 4. In Section 5 we consider the complete symmetric cases when the rays are distributed equidistant in the complex plane, with equal lengths and the same weights on the rays. We give the recurrence relation for polynomials in such cases, as well as a connection with standard polynomials orthogonal on the real line. Also, we show that their zeros are simple and distributed symmetrically on the radial rays, with the possible exception of a multiple zero at the origin. An analogue of the Jacobi polynomials and the corresponding problem with the generalized Laguerre polynomials are treated in Section 6. Finally, some applications of these polynomials in physics and electrostatic are discussed in Section 7.

2. Moments, Existence, and Uniqueness

First we see that

$$\|f\|^2 = (f,f) = \sum_{s=1}^{M} \int_0^{a_s} |f(x\varepsilon_s)|^2 \omega_s(x)\, dx > 0,$$

except when $f(z) = 0$. The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \sum_{s=1}^{M} \varepsilon_s^{p-q} \int_0^{a_s} x^{p+q} \omega_s(x)\, dx,$$

i.e.,

$$\mu_{p,q} = \sum_{s=1}^{M} \varepsilon_s^{p-q} \mu_{p+q}^{(s)}, \tag{2.1}$$

where $\mu_p^{(s)}$ are single moments which correspond to the weight function $x \mapsto \omega_s(x)$ on the rays ℓ_s,

$$\mu_p^{(s)} = \int_0^{a_s} x^p \omega_s(x)\, dx, \qquad s = 1, 2, \ldots, M.$$

Notice that

$$\mu_{p,p} = \sum_{s=1}^{M} \mu_{2p}^{(s)} > 0 \qquad \text{and} \qquad \overline{\mu}_{q,p} = \mu_{p,q}.$$

Using the moment determinants

$$\Delta_0 = 1, \quad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \quad N \geq 1,$$

where the moments are given by (2.1), we can state the following existence result for the (monic) orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (1.2).

Theorem 2.1. *If* $\Delta_N > 0$ *for all* $N \geq 1$ *the monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$, *orthogonal with respect to the inner product* (1.2), *exist uniquely and the norm is given by*

$$\|\pi_N\| = \sqrt{\frac{\Delta_{N+1}}{\Delta_N}}\,. \tag{2.2}$$

Proof. Write

$$\pi_N(z) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} z^\nu, \qquad \alpha_N^{(N)} = 1,$$

and consider the orthogonality conditions

$$(\pi_N, z^k) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} (z^\nu, z^k) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} \mu_{\nu,k} = K_N \delta_{kN}, \quad k \leq N,$$

where $K_N = \|\pi_N\|^2 \neq 0$ and δ_{kN} is the Kronecker delta. These conditions are equivalent to the system of linear equations

$$\begin{bmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N1} \\ \vdots & & & \\ \mu_{0N} & \mu_{1N} & \cdots & \mu_{NN} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0^{(N)} \\ \alpha_1^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ K_N \end{bmatrix}. \tag{2.3}$$

Since $\Delta_{N+1} \neq 0$ the system (2.3) has a unique solution for the coefficients $\alpha_\nu^{(N)}$. For the monic polynomials we have $\alpha_N^{(N)} = 1$ and

$$\alpha_N^{(N)} = \frac{K_N \Delta_N}{\Delta_{N+1}} = \|\pi_N\|^2 \frac{\Delta_N}{\Delta_{N+1}} = 1,$$

i.e., $\|\pi_N\|^2 = \Delta_{N+1}/\Delta_N$. $\square$

It is clear that such monic orthogonal polynomials can be expressed in the form

$$\pi_0(z) = 1,$$

$$\pi_N(z) = \frac{1}{\Delta_N} \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} & 1 \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} & z \\ \vdots & & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} & z^{N-1} \\ \mu_{0,N} & \mu_{1,N} & \cdots & \mu_{N-1,N} & z^N \end{vmatrix}, \quad N \geq 1.$$

In some cases it is possible to evaluate the moment determinants in an explicit form and so get the corresponding orthogonal polynomials. The following symmetric case was considered in [13].

Let $a_s = 1$, $\theta_s = 2\pi(s-1)/M$, $s = 1, 2, \ldots, M$, with Legendre weight on the rays ($\omega_s(x) = 1$ for each s). Then, the inner product (1.2) reduces to

$$(f, g) = \sum_{s=1}^{M} \int_0^1 f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\, dx,$$

and the moments (2.1) to

$$\mu_{p,q} = (z^p, z^q) = \begin{cases} \dfrac{M}{p+q+1}, & p \equiv q (\text{mod } M), \\ 0, & \text{otherwise}, \end{cases}$$

i.e.,

$$\mu_{Mi+\nu, Mj+\nu} = \frac{M}{M(i+j)+2\nu+1}, \quad 0 \le \nu \le M-1, \quad i, j \ge 0.$$

The corresponding moment-determinants can be evaluated as

$$\begin{aligned} \Delta_{Mn} &= E_n^{(0)} E_n^{(1)} \cdots E_n^{(M-1)}, \\ \Delta_{Mn+\nu} &= \prod_{i=0}^{\nu-1} E_{n+1}^{(i)} \prod_{j=\nu}^{M-1} E_n^{(j)}, \qquad 0 < \nu < M, \end{aligned}$$

where $E_0^{(\nu)} = 1$ and

$$E_n^{(\nu)} = \begin{vmatrix} \mu_{\nu,\nu} & \mu_{M+\nu,\nu} & \cdots & \mu_{M(n-1)+\nu,\nu} \\ \mu_{\nu,M+\nu} & \mu_{M+\nu,M+\nu} & \cdots & \mu_{M(n-1)+\nu,M+\nu} \\ \vdots & & & \\ \mu_{\nu,M(n-1)+\nu} & \mu_{M+\nu,M(n-1)+\nu} & \cdots & \mu_{M(n-1)+\nu,M(n-1)+\nu} \end{vmatrix}.$$

The value of $E_n^{(\nu)}$ is

$$E_n^{(\nu)} = M^{n^2} \frac{[0!1!\cdots(n-1)!]^2}{\prod_{i,j=0}^{n-1} [M(i+j)+2\nu+1]}.$$

Using (2.2) we find

$$\|\pi_{Mn+\nu}\|^2 = \frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} = \begin{cases} \dfrac{M}{2\nu+1}, & n = 0, \\ \dfrac{M}{2nM+2\nu+1} \left(\prod_{k=n}^{2n-1} \dfrac{M(k-n+1)}{Mk+2\nu+1} \right)^2, & n \ge 1, \end{cases}$$

where $0 \le \nu \le M-1$.

These formulas enable us to obtain the explicit expressions for coefficients in the corresponding recurrence relation for polynomials orthogonal on the symmetric rays. Such considerations will be given in Section 5. For more details see the references [11] and [13].

3. Numerical Construction of Polynomials

In this section we present a numerical method for constructing orthogonal polynomials as well as a method for zero finding of such polynomials. We suppose that for a given inner product the corresponding orthogonal polynomials $\{\pi_k(z)\}_{k=0}^{+\infty}$ exist. In particular, some special cases of Jacobi weights will be considered.

Thus, let $\pi_k(z)$ $(k \in \mathbb{N}_0)$ be monic polynomials orthogonal with respect to the inner product (1.2). Since $\pi_k(z) - z\pi_{k-1}(z)$ is a polynomial of degree at most $k-1$ and $\pi_j(z)$, $j = 0, 1, \ldots, k-1$, form a basis of $\mathcal{P}_{k-1}$, it is clear that for any integer $k \in \mathbb{N}$, there exist constants β_{kj}, $j = 1, \ldots, k$, such that

$$\pi_k(z) = z\pi_{k-1}(z) - \sum_{j=1}^{k} \beta_{kj}\pi_{j-1}(z), \qquad k \in \mathbb{N}. \tag{3.1}$$

Because of orthogonality, the coefficients β_{kj} are given by

$$\beta_{kj} = \frac{(z\pi_{k-1}, \pi_{j-1})}{(\pi_{j-1}, \pi_{j-1})} \qquad (1 \le j \le k;\ k \in \mathbb{N}). \tag{3.2}$$

For a fixed $N \in \mathbb{N}$ we put

$$\mathbf{q}_N(z) = [\pi_0(z)\ \ \pi_1(z)\ \ \ldots\ \ \pi_{N-1}(z)]^T, \qquad \mathbf{e}_N = [0\ \ 0\ \ \ldots\ \ 0\ \ 1]^T.$$

Then, the previous relations can be represented in a matrix form

$$z\mathbf{q}_N(z) = B_N\mathbf{q}_N(z) + \pi_N(z)\mathbf{e}_N, \tag{3.3}$$

where B_N is the following lower Hessenberg matrix

$$B_N = \begin{bmatrix} \beta_{11} & 1 & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ \beta_{N-1,1} & \beta_{N-1,2} & & \ddots & 1 \\ \beta_{N,1} & \beta_{N,2} & & \cdots & \beta_{N,N} \end{bmatrix}. \tag{3.4}$$

Let $\xi_j \equiv \xi_j^{(N)}$ $(j = 1, \ldots, N)$ be zeros of $\pi_N(z)$. Then (3.3) reduces to the eigenvalue problem

$$\xi_j \mathbf{q}_N(\xi_j) = B_N \mathbf{q}_N(\xi_j).$$

Thus, ξ_j are eigenvalues of the matrix B_N and $\mathbf{q}_N(\xi_j)$ are the corresponding eigenvectors.

According to (3.3) one can obtain the following determinant representation of the monic polynomials

$$\pi_N(z) = \det(zI_N - B_N),$$

where I_N is the identity matrix of the order N.

For computing zeros of $\pi_N(z)$ as the eigenvalues of the matrix B_N we use the EISPACK routine COMQR [17, pp. 277–284]. Notice that this routine needs an upper Hessenberg matrix, i.e., B_N^T. Also, the MATLAB can be used.

We remark that in the case of the standard orthogonal polynomials on the real line, the matrix B_N reduces to a tridiagonal matrix and (3.1) is then the well-known three-term recurrence relation. The reason for this is the basic property $(zf, g) = (f, zg)$ satisfied by the inner product. Therefore, $(z\pi_{k-1}, \pi_{j-1}) = (\pi_{k-1}, z\pi_{j-1}) = 0$, i.e., $\beta_{kj} = 0$ for $j < k - 1$.

Regarding to the inner product (1.2), we have (see [13]):

Lemma 3.1. *If there exists any $\rho \in \mathbb{N}$ such that $\varepsilon_s^{2\rho} = 1$ for each $s = 1, \ldots, M$, then the inner product* (1.2) *has the property*

$$(z^\rho f, g) = (f, z^\rho g). \tag{3.5}$$

Theorem 3.2. *Let the inner product* (1.2) *satisfies* (3.5)*. Then the monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *satisfy a* $(2\rho+1)$*-recurrence relation*

$$z^\rho \pi_N(z) = \pi_{N+\rho}(z) + \sum_{j=N-\rho}^{N+\rho-1} \alpha_j^{(N)} \pi_j(z), \quad N \geq \rho. \tag{3.6}$$

Proof. Writing

$$z^\rho \pi_N(z) = \pi_{N+\rho}(z) + \sum_{j=0}^{N+\rho-1} \alpha_j^{(N)} \pi_j(z), \quad N \geq 0,$$

we see that $(z^\rho \pi_N, \pi_j) = \alpha_j^{(N)}(\pi_j, \pi_j)$ for $0 \leq j \leq N+\rho-1$. However, because of (3.5) and orthogonality, we have $(z^\rho \pi_N, \pi_j) = (\pi_N, z^\rho \pi_j) = 0$ for $\rho + j < N$. Thus, $\alpha_j^{(N)} = 0$ for $j < N - \rho$. □

Of course, the minimal ρ with the previous property is interesting in applications. Such ρ provides the simplest form of the recurrence relation (3.6).

Regarding to the previous remark on the standard orthogonal polynomials on the real line and the basic property $(zf, g) = (f, zg)$, we see that in this case we have two rays ($M = 2$) with $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. It is easy to see that $\rho = 1$ satisfies the condition of Lemma 3.1, so that (3.6) becomes a three-term recurrence relation.

3.1. Discretized Stieltjes-Gautschi procedure

In order to determine $\pi_0(z), \pi_1(z), \ldots, \pi_N(z)$, in a general case of the inner product (1.2), we must compute the matrix B_N. Because of that, we consider here an effective method for numerical calculation of the elements β_{kj} in (3.2).

At first we suppose that all a_s are finite. Then, we transform the integrals in (1.2) to the interval $(0, 1)$ and obtain

$$(f, g) = \sum_{s=1}^{M} \int_0^1 a_s f(a_s \varepsilon_s x) \overline{g(a_s \varepsilon_s x)}\, \Omega_s(x)\, dx. \tag{3.7}$$

where $\Omega_s(x) = \omega_s(a_s x)$, $s = 1, 2, \ldots, M$.

In order to evaluate the integrals in (3.7) we need n-point Gaussian quadratures

$$\int_0^1 \phi(x) \Omega_s(x)\, dx = \sum_{\nu=1}^{n} \lambda_\nu^{(n,s)} \phi(\tau_\nu^{(n,s)}) + R_{n,s}(\phi), \tag{3.8}$$

which are exact for all algebraic polynomials of degree at most $2n-1$. Thus, in this way we get a discretized form of the inner product

$$(f,g)\approx\sum_{s=1}^{M}a_s\sum_{\nu=1}^{n}\lambda_\nu^{(n,s)}f\big(a_s\varepsilon_s\tau_\nu^{(n,s)}\big)\overline{g\big(a_s\varepsilon_s\tau_\nu^{(n,s)}\big)}\,, \tag{3.9}$$

which is very easy for computing.

In the cases when some of a_s (or all) are infinity we should use certain quadratures over $(0,+\infty)$ depending on the corresponding weight functions.

Since the maximal degree of polynomials which appear in the inner products in (3.2) is $1+(k-1)+(j-1)\le 2N-1$, it is enough to take $n=N$ nodes in the previous quadratures. In that case, the all elements β_{kj} in the lower Hessenberg matrix (3.4) will be computed exactly, except for rounding errors.

In numerical construction of the Hessenberg matrix B_N we use some kind of the Stieltjes procedure (cf. [4]) and call it as the *discretized Stieltjes-Gautschi procedure*. Namely, we apply (3.2), with inner products in discretized form (3.9), in tandem with the basic linear relations (3.1).

Since $\pi_0(z)=1$, we can compute β_{11} from (3.2). Having obtained β_{11}, we then use (3.1) with $k=1$ to compute $\pi_1(z)$ for all $\{a_s\varepsilon_s\tau_\nu^{(N,s)}\}$ to obtain its values needed to reapply (3.2) with $k=2$. This yields β_{21} and β_{22}, which in turn can be used in (3.1) to obtain the corresponding values of $\pi_2(z)$ needed to return to (3.2) for computing β_{31}, β_{32}, and β_{33}. Thus, in this way, alternating between (3.2) and (3.1), we can 'bootstrap' ourselves up to any desired order N.

In a numerical implementation of the previous procedure it is very convenient to use the MATLAB.

3.2. Jacobi weights on the rays

In this section we consider polynomials orthogonal with respect to the inner product (1.2), i.e., (3.7), when the corresponding weight functions $w_s(z)$ on the radial rays ℓ_s, $s=1,2,\dots,M$, are defined by

$$w_s(z)=(z_s-z)^{\alpha_s}z^{\beta_s},\qquad \alpha_s,\beta_s>-1\qquad (s=1,2,\dots,M).$$

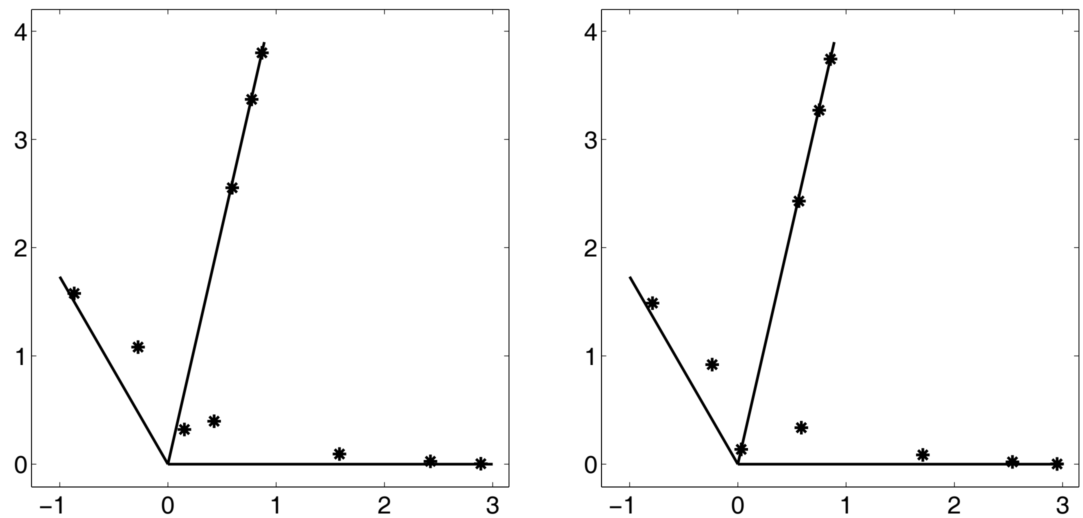

FIG. 3.1: Zeros of $\pi_{10}(z)$ for Legendre weight (left) and different Jacobi weights (right) on the rays

In other words, on the each of the rays ($z = xa_s\varepsilon_s,\ z_s = a_s\varepsilon_s,\ \varepsilon_s = e^{i\theta_s}$), we have a Jacobi weight with parameters α_s, β_s, i.e.,

$$|w_s(z)| = |a_s\varepsilon_s - z|^{\alpha_s}|z|^{\beta_s} = a_s^{\alpha_s}(1-x)^{\alpha_s}x^{\beta_s} \qquad (z \in \ell_s)$$

for each $s = 1, 2, \ldots, M$.

Then (3.7) becomes

$$(f, g) = \sum_{s=1}^{M} a_s^{1+\alpha_s} \int_0^1 f(xa_s\varepsilon_s)\overline{g(xa_s\varepsilon_s)}\,(1-x)^{\alpha_s}x^{\beta_s}\,dx. \tag{3.10}$$

Thus, in this case (3.8) are the Gauss-Jacobi quadratures transformed to the interval $(0, 1)$. For a selected N, using the discretized Stieltjes-Gautschi procedure we can compute the matrix B_N. It gives us access to the first $N + 1$ orthogonal polynomials $\pi_k(z)$, $k = 0, 1, \ldots, N$. As we have seen on the beginning of this section, the eigenvalues of the matrix B_N are zeros of the polynomial $\pi_N(z)$. Of course, the zeros of $\pi_k(z)$ $(1 \le k \le N)$ are the eigenvalues of the leading principal submatrix B_k.

The case of a polynomial of degree 10, which is orthogonal on the rays defined by $z_1 = 3$, $z_2 = 4e^{3\pi i/7}$, $z_3 = 2e^{2\pi i/3}$, is presented in Figure 3.1. The zeros of the polynomial are denoted by $*$. We considered two cases of the weights: (a) the Legendre weight on each of the rays (the left figure); and (b) the Jacobi weights with parameters: $\alpha_1 = \beta_1 = -1/2$, $\alpha_2 = \beta_2 = 1/2$, and $\alpha_3 = -\beta_3 = 1/2$ (the right figure).

3.3. Two-sided Chebyshev polynomials

In this subsection we consider some special cases when $M = 2$, $a_1 = a_2 = 1$, $\theta_1 = 0$, $\theta_2 = \pi$, and weights on the rays are pairs of the Chebyshev weights of the first, second, third, and fourth kind, shifted to $(0,1)$, i.e.,

$$\begin{aligned} w^{(1)}(x) &= \frac{1}{\sqrt{(1-x)x}}, & w^{(2)}(x) &= \sqrt{(1-x)x}, \\ w^{(3)}(x) &= \sqrt{\frac{x}{1-x}}, & w^{(4)}(x) &= \sqrt{\frac{1-x}{x}}. \end{aligned}$$

Thus, for the weights on the rays we take the pair $(w^{(p)}, w^{(q)})$, where $p, q \in \{1,2,3,4\}$. According to (1.3) we have standard polynomials orthogonal on $(-1,1)$ with respect to the weight function

$$\omega(x) = \omega^{(p,q)}(x) = \begin{cases} w^{(p)}(x), & 0 < x < 1, \\ w^{(q)}(-x), & -1 < x < 0. \end{cases}$$

The corresponding orthogonal polynomials will be called the *two-sided Chebyshev polynomials of* (p,q)*-type* and denoted by $C_k^{(p,q)}(x)$, where $p, q \in \{1,2,3,4\}$. These (monic) polynomials satisfy the basic three-term recurrence relation

$$\begin{aligned} C_{k+1}^{(p,q)}(x) &= (x - \alpha_k^{(p,q)}) C_k^{(p,q)}(x) - \beta_k^{(p,q)} C_{k-1}^{(p,q)}(x), \quad k = 0, 1, \ldots, \\ C_{-1}^{(p,q)}(x) &= 0, \quad C_0^{(p,q)}(x) = 1. \end{aligned}$$

Notice that $\omega^{(q,p)}(-x) = \omega^{(p,q)}(x)$, and therefore

$$\alpha_k^{(q,p)} = -\alpha_k^{(p,q)} \quad \text{and} \quad \beta_k^{(q,p)} = \beta_k^{(p,q)} \qquad (k = 0, 1, \ldots).$$

Thus, it is enough to investigate only cases when $p \le q$. In Figure 3.2 we display the weights $x \mapsto \omega^{(p,q)}(x)$, when (p,q) run over the following sequence of pairs: $(1,1)$, $(1,2)$; $(1,3)$, $(1,4)$; $(2,2)$, $(2,3)$; $(2,4)$, $(3,3)$; $(3,4)$, $(4,4)$.

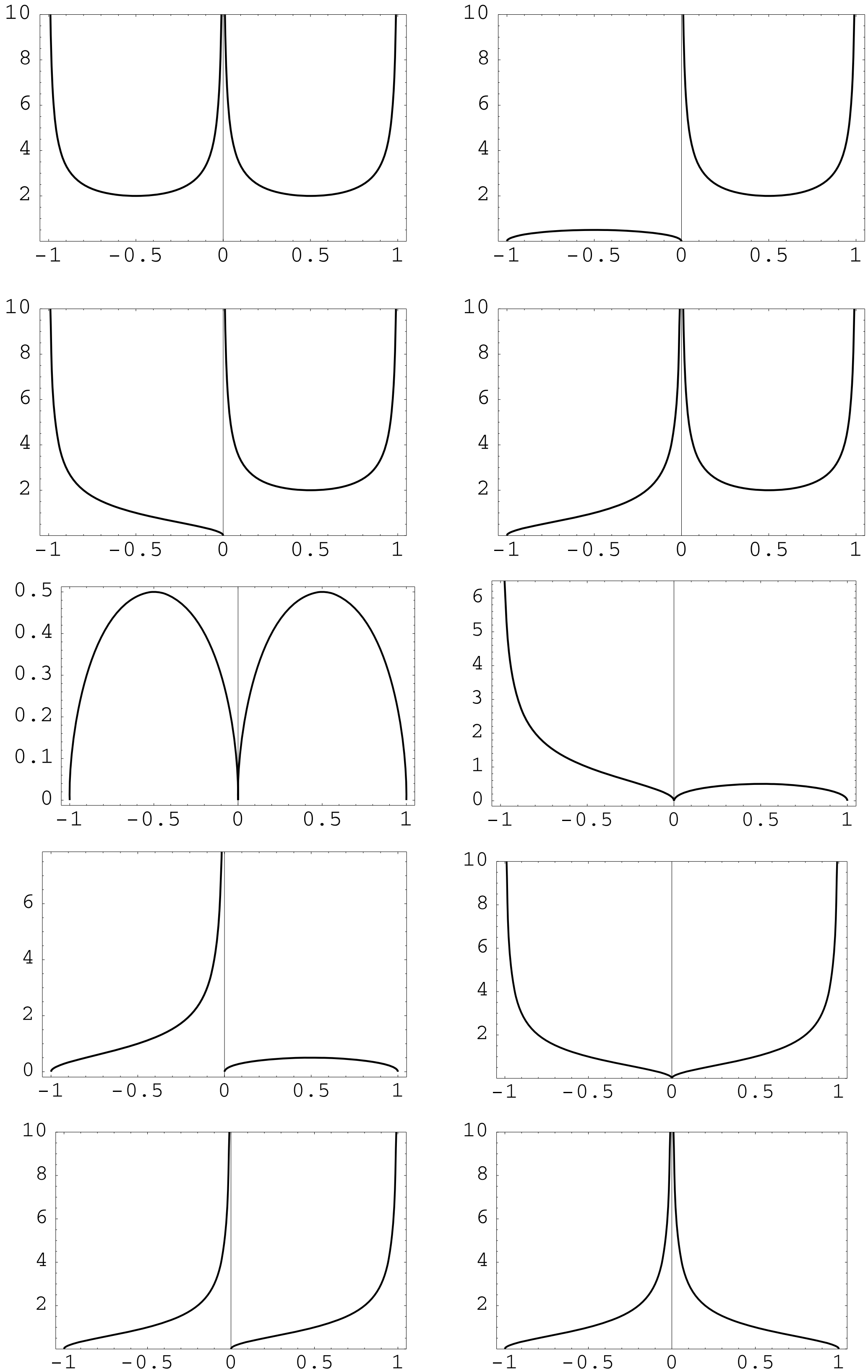

FIG. 3.2: The weight functions $x \mapsto \omega^{(p,q)}(x)$ for $1 \le p \le q \le 4$

Evidently, $\alpha_k^{(p,q)} = 0$ for $p = q$, because the weight function is even. By $A^{(p,q)}$ and $B^{(p,q)}$ we denote the sequences of recursion coefficients $\{\alpha_k^{(p,q)}\}_{k=0}^{+\infty}$ and $\{\beta_k^{(p,q)}\}_{k=0}^{+\infty}$, respectively.

For an even weight function, the recursion coefficients $B^{(p,p)}$, $p = 1,2,3,4$, are

$$
\begin{aligned}
B^{(1,1)} &= \left\{2\pi, \frac{3}{8}, \frac{17}{48}, \frac{161}{816}, \frac{12723}{43792}, \frac{2413235}{10924816}, \frac{91312277}{332155120}, \dots\right\},\\
B^{(2,2)} &= \left\{\frac{\pi}{4}, \frac{5}{16}, \frac{17}{80}, \frac{381}{1360}, \frac{23525}{103632}, \frac{7746713}{28681680}, \frac{8004121533}{34304267920}, \dots\right\},\\
B^{(3,3)} &= \left\{\pi, \frac{5}{8}, \frac{13}{80}, \frac{321}{1040}, \frac{14185}{66768}, \frac{4092673}{14570832}, \frac{3233397369}{14290354832}, \dots\right\},\\
B^{(4,4)} &= \left\{\pi, \frac{1}{8}, \frac{5}{16}, \frac{17}{80}, \frac{381}{1360}, \frac{23525}{103632}, \frac{7746713}{28681680}, \dots\right\},
\end{aligned}
$$

respectively. Also, we give coefficients for two non-symmetric cases ($(p,q) = (1,2)$ and $(p,q) = (1,3)$):

$$
\begin{aligned}
A^{(1,2)} &= \left\{\frac{7}{18}, \frac{607}{5058}, -\frac{518309}{5069802}, \frac{8485627223}{85691074386}, -\frac{49034107172287781}{524482597302736618}, \dots\right\},\\
B^{(1,2)} &= \left\{\frac{9\,\pi}{8}, \frac{281}{1296}, \frac{405945}{1263376}, \frac{1334618773}{6510275280}, \frac{498086602437333}{1804645097447120}, \dots\right\};\\
A^{(1,3)} &= \left\{\frac{1}{12}, -\frac{43}{390}, \frac{95157}{882310}, -\frac{1592897243}{29467538694}, \frac{380934910863589}{14965099689812742}, \dots\right\},\\
B^{(1,3)} &= \left\{\frac{3\,\pi}{2}, \frac{65}{144}, \frac{20361}{67600}, \frac{141107265}{737013904}, \frac{23393297835017}{75403589058576}, \dots\right\}.
\end{aligned}
$$

In Figure 3.3 we presented the graphics of two-sided polynomials $C_k^{(1,1)}(x)$ and $C_k^{(1,2)}(x)$ for $k = 2,3,4,5$.

4. Distribution of Zeros

One of the most relevant questions relating to orthogonal polynomials is their zero distribution. A general result on this subject and a few numerical examples are presented here.

Let $\pi_N(z)$ ($N \in \mathbb{N}_0$) be monic polynomials orthogonal with respect to the inner product (1.1), i.e., (1.4), where the measure $d\mu(z)$ is given by (1.5). Let $\mathrm{Co}(B)$ denotes

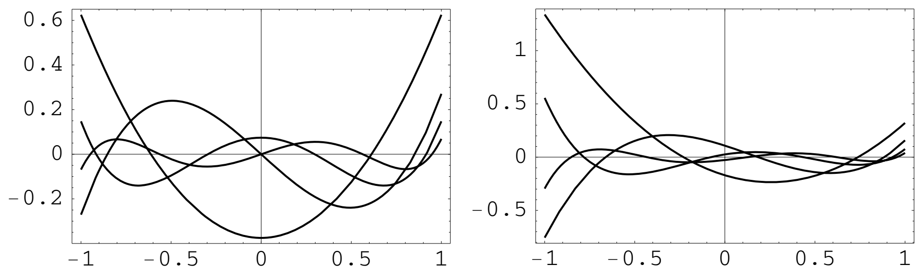

FIG. 3.3: Two-sided polynomials $C_k^{(1,1)}(x)$ (left) and $C_k^{(1,2)}(x)$ (right) for $k = 2(1)5$

the convex hull of a set $B \in \mathbb{C}$ (i.e., the smallest convex set containing B) and let $E = \mathrm{supp}(d\mu)$ be the support of the measure $d\mu(z)$. Since $E \subset L = \ell_1 \cup \ell_2 \cup \ldots \cup \ell_M$, using a result of Fejér (see [15] and [22]), we can state:

Theorem 4.1. *All the zeros of the polynomial $\pi_N(z)$, $N \in \mathbb{N}$, orthogonal with respect to* (1.4) *lie in the convex hull of the rays $L = \ell_1 \cup \ell_2 \cup \ldots \cup \ell_M$.*

Furthermore, an improvement of Fejér's result holds (see [15]):

Theorem 4.2. *If* $\mathrm{Co}(\mathrm{supp}(d\mu))$ *is not a line segment, then all the zeros of the polynomial $\pi_N(z)$ lie in the interior of* $\mathrm{Co}(\mathrm{supp}(d\mu)) \subset \mathrm{Co}(L)$.

In order to illustrate the previous results, we consider again the example from Subsection 3.2 with three non-symmetric rays defined by $z_1 = 3$, $z_2 = 4e^{3\pi i/7}$, $z_3 = 2e^{2\pi i/3}$. As we can see in Figure 3.1, the zeros of $\pi_{10}(z)$ are in the convex hull of the rays. Notice also that some of zeros lie on the rays, and others not.

More precise results on zero distribution of polynomials $\pi_N(z)$ can be obtained in some special (symmetric) cases. Such kind of investigation in details will be given later. Now, we consider two cases on the symmetric rays in the complex plane.

In the first example we put $M = 3$ equidistant distributed rays $z_s = e^{i2\pi(s-1)/3}$, $s = 1, 2, 3$. Taking the Legendre weight on the each rays ($\alpha_s = \beta_s = 0$, $s = 1, 2, 3$) we get a complete symmetric case. In that case, all the zeros are on the rays and symmetrically distributed. The case $N = 16$ is presented in Figure 4.1 (left). We note that five zeros are located on the each of the rays and one zero is at origin.

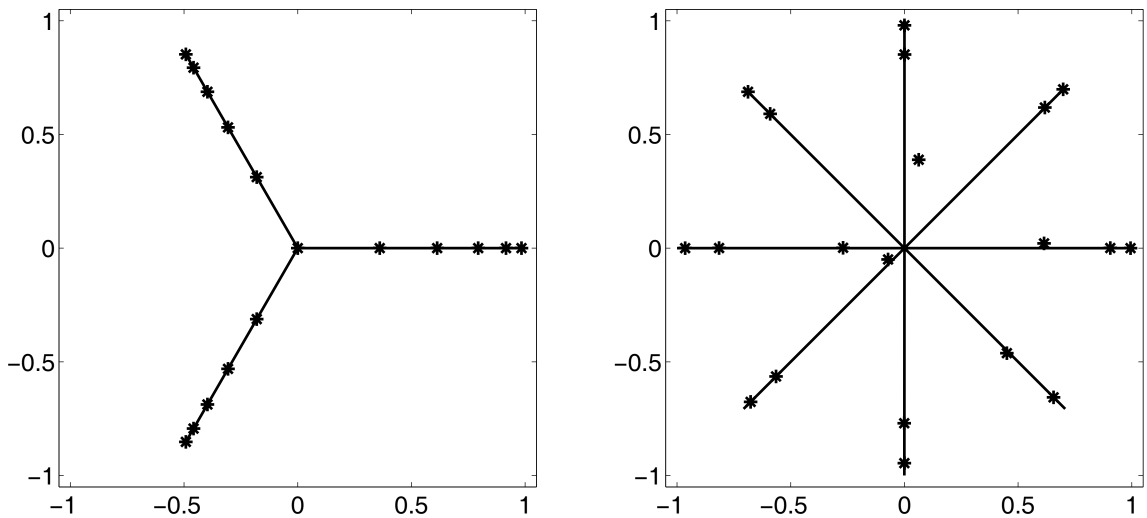

FIG. 4.1: Zero distribution for $M = 3$, $N = 16$ (left) and $M = 8$, $N = 20$ (right)

In the second example we take $M = 8$ rays symmetrically distributed in the complex plane, such that $z_s = e^{i\theta_s}$, where $\theta_s = (s-1)\pi/4$, $s = 1, 2, \dots, 8$. On the other side, we take different Jacobi weights on the rays, with parameters given in Table 4.1.

TABLE 4.1: The parameters α_s and β_s in the Jacobi weights on the rays ℓ_s

s	1	2	3	4	5	6	7	8
α_s	$-3/4$	$-1/2$	$-1/4$	0	$1/4$	$1/2$	$3/4$	1
β_s	$-1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$

The zeros of the corresponding polynomials $\pi_{20}(z)$ orthogonal on these rays with respect to the given Jacobi weights are displayed in Figure 4.1 (right). As we can see there exists a non-symmetricity in the zero distribution, which is influenced by different Jacobi weights.

5. Complete Symmetric Cases

This section is devoted to orthogonal polynomials on the equidistant distributed rays in the complex plane, with equal lengths and the same weights on the rays. Thus, we consider the cases when $a_s = r$, $\theta_s = 2\pi(s-1)/M$, and

$$|w_s(z)| = |w_s(x\varepsilon_s)| = \omega(x) \qquad (z = x\varepsilon_s \in \ell_s)$$

for each $s = 1, 2, \ldots, M$. Then, the inner product (1.2) reduces to

$$(f,g) = \int_0^r \left(\sum_{s=1}^{M} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) \omega(x)\,dx. \tag{5.1}$$

In particular, two cases are interesting: $r = 1$ and $r = +\infty$.

The first consideration of such orthogonality was given in [11], with an even number of rays ($M = 2m$). We investigated the existence and uniqueness of the (monic) polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product (5.1). Similar considerations when M is an arbitrary number of rays were investigated in [13, 14].

Since $\varepsilon_s^M = 1$ for each $s = 1, 2, \ldots, M$, according to Lemma 3.1 we see that for $\rho = M$ the required condition in this lemma is always satisfied, so that the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation (3.6) (see Theorem 3.2). Furthermore, if M is an even number, for example $M = 2m$, then there exists a smaller integer ρ which satisfies Lemma 3.1: $\rho = M/2 = m$.

5.1. Recurrence relation

At first we give some auxiliary results. We note that the single moments $\mu_p^{(s)}$, which are appeared in (2.1), not depend on s in this case, so that $\mu_p^{(s)} = \mu_p$ and $\mu_{p,q} = \mu_{p+q}\left(\sum_{s=1}^{M} \varepsilon_s^{p-q}\right)$. The last sum can be found in an explicit form.

Lemma 5.1. *For $p \in \mathbb{N}$ let $n = [p/M]$ and $\nu = p - Mn$. Then*

$$\sum_{s=1}^{M} \varepsilon_s^p = \sum_{s=1}^{M} \varepsilon_s^\nu = \begin{cases} M, & \text{if } \ \nu = 0 \\ 0, & \text{if } \ 1 \le \nu \le M-1. \end{cases}$$

Lemma 5.2. *The polynomials $\pi_N(z)$, $N = 0, 1, \ldots$, orthogonal with respect to the inner product* (5.1) *satisfy*

$$\pi_N(z\varepsilon_s) = \varepsilon_s^N \pi_N(z), \qquad s = 1, 2, \ldots, M.$$

Lemma 5.3. *For* $0 \le \nu < N \le M-1$, *we have* $(z^N, \pi_\nu) = 0$.

Lemma 5.4. *For* $N = 0, 1, \ldots, M-1$ *we have* $\pi_N(z) = z^N$.

The proofs of these results are very easy and can be found in [11, 13]. Now, we give the main result:

Theorem 5.5. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *satisfy the recurrence relation*

$$\pi_{N+M}(z) = (z^M - \alpha_N)\pi_N(z) - \beta_N \pi_{N-M}(z), \quad N \ge 0, \tag{5.2}$$

with

$$\pi_N(z) = z^N, \quad N = 0, 1, \ldots, M-1,$$

where the recursion coefficients are given by

$$\alpha_N = \frac{(z^M \pi_N, \pi_N)}{(\pi_N, \pi_N)} \quad (N \ge 0), \qquad \beta_N = \frac{(\pi_N, \pi_N)}{(\pi_{N-M}, \pi_{N-M})} \quad (N \ge M) \tag{5.3}$$

and $\beta_N = 0$ *for* $N < M$.

Proof. According to $(z^M f, g) = (f, z^M g)$ and Theorem 3.2, the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation (3.6), with recursion coefficients $\alpha_j^{(N)} = (z^\rho \pi_N, \pi_j)/(\pi_j, \pi_j)$ for $N - \rho \le j \le N + \rho - 1$, where $\rho = M$. Since

$$\begin{aligned}
(z^M \pi_N, \pi_j) &= \int_0^r \left(\sum_{s=1}^M (x\varepsilon_s)^M \pi_N(x\varepsilon_s) \overline{\pi_j(x\varepsilon_s)} \right) \omega(x)\, dx \\
&= \int_0^r \left(\sum_{s=1}^M x^M \varepsilon_s^N \pi_N(x) \overline{\varepsilon_s^j \pi_j(x)} \right) \omega(x)\, dx \\
&= \left(\sum_{s=1}^M \varepsilon_s^{N-j} \right) \int_0^r x^M \pi_N(x) \overline{\pi_j(x)} \omega(x)\, dx,
\end{aligned}$$

using Lemma 5.1, we conclude that $(z^M \pi_N, \pi_j)$ is different from zero only for $N - j = 0$ and $N - j = M$. Thus,

$$\pi_{N+M}(z) = z^M \pi_N(z) - \alpha_N^{(N)} \pi_N(z) - \alpha_{N-M}^{(N)} \pi_{N-M}(z).$$

Since

$$\alpha_{N-M}^{(N)} = \frac{(z^M \pi_N, \pi_{N-M})}{(\pi_{N-M}, \pi_{N-M})} = \frac{(\pi_N, z^M \pi_{N-M})}{(\pi_{N-M}, \pi_{N-M})},$$

we get the recursion coefficients in (5.2) as $\alpha_N = \alpha_N^{(N)}$ and $\beta_N = \alpha_{N-M}^{(N)}$. $\square$

Using (5.2), it is very easy to get the corresponding recurrence relation for orthonormal polynomials $\pi_N^*(z)$ $(= \pi_N(z)/\|\pi_N\|)$

$$z^M \pi_N^*(z) = \sqrt{\beta_N}\,\pi_{N-M}^*(z) + \alpha_N \pi_N^*(z) + \sqrt{\beta_{N+M}}\,\pi_{N+M}^*(z). \tag{5.4}$$

Let $N = Mk + \nu$, where $k = [N/M]$ and $\nu \in \{0, 1, \ldots, M-1\}$. Taking the recurrence relation (5.4) for $k = 0, 1, \ldots, n-1$ and for a fixed ν, we obtain the following matrix relation

$$z^M \mathbf{q}_{n,\nu}^{(M)} = J_n^{(\nu,M)} \mathbf{q}_{n,\nu}^{(M)} + \sqrt{\beta_{nM+\nu}}\,\pi_{nM+\nu}^*(z)\,\mathbf{e}_n,$$

i.e.,

$$(z^M I_n - J_n^{(\nu,M)})\mathbf{q}_{n,\nu}^{(M)} = \sqrt{\beta_{nM+\nu}}\,\pi_{nM+\nu}^*(z)\,\mathbf{e}_n, \tag{5.5}$$

where $\mathbf{q}_{n,\nu}^{(M)} = [\pi_\nu^*(z) \quad \pi_{M+\nu}^*(z) \quad \cdots \quad \pi_{(n-1)M+\nu}^*(z)]^T$, I_n is the identity matrix of the order n, and $J_n^{(\nu,M)}$ is the symmetric tridiagonal Jacobi matrix determined by

$$J_n^{(\nu,M)} = \begin{bmatrix} \alpha_\nu & \sqrt{\beta_{M+\nu}} & & & \mathrm{O} \\ \sqrt{\beta_{M+\nu}} & \alpha_{M+\nu} & \sqrt{\beta_{2M+\nu}} & & \\ & \sqrt{\beta_{2M+\nu}} & \alpha_{2M+\nu} & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{(n-1)M+\nu}} \\ \mathrm{O} & & & \sqrt{\beta_{(n-1)M+\nu}} & \alpha_{(n-1)M+\nu} \end{bmatrix}.$$

According to (5.5) one can obtain the following determinant representation of the monic orthogonal polynomials

$$\pi_N(z) = \pi_{nM+\nu}(z) = z^\nu \det(z^M I_n - J_n^{(\nu,M)}).$$

For fixed $\nu \in \{0, 1, \ldots, M-1\}$ and M we define a sequence of real polynomials $\{p_n^{(\nu,M)}(t)\}_{n=0}^{+\infty}$ such that

$$p_n^{(\nu,M)}(t) = \det(tI_n - J_n^{(\nu,M)}). \tag{5.6}$$

Regarding to the previous determinant representation of the monic orthogonal polynomials $\pi_N(z)$ we have

$$\pi_N(z) = \pi_{Mn+\nu}(z) = z^\nu p_n^{(\nu,M)}(z^M), \tag{5.7}$$

where $N = Mn + \nu$ and $n = [N/M]$.

Putting (5.7), with $N = Mn + \nu$, in (5.2) we obtain:

Theorem 5.6. *The monic polynomials* $p_n^{(\nu,M)}(t)$ $(\nu \in \{0, 1, \ldots, M-1\})$ *satisfy the three-term recurrence relation*

$$\begin{aligned} p_{n+1}^{(\nu,M)}(t) &= (t - \alpha_{Mn+\nu})p_n^{(\nu,M)}(t) - \beta_{Mn+\nu}p_{n-1}^{(\nu,M)}(t), \quad n = 0, 1, \ldots, \\ p_{-1}^{(\nu,M)}(t) &= 0, \quad p_0^{(\nu,M)}(t) = 1, \end{aligned}$$

with recurrence coefficients as in (5.3).

5.2. Orthogonality of polynomials $p_n^{(\nu,M)}(t)$

The following orthogonality holds:

Theorem 5.7. *Let* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *be a sequence of orthogonal polynomials with respect to the inner product defined on* M *rays in the complex plane by* (5.1) *and let* $p_n^{(\nu,M)}(t)$ *be polynomials determined by* (5.6). *For any* $\nu \in \{0, 1, \ldots, M-1\}$, *the sequence of polynomials* $\{p_n^{(\nu,M)}(t)\}_{n=0}^{+\infty}$ *is orthogonal on* $(0, r^M)$ *with respect to the weight function* $t \mapsto \Omega_{\nu,M}(t)$ *defined by*

$$\Omega_{\nu,M}(t) = t^{(2\nu-M+1)/M}\omega(t^{1/M}). \tag{5.8}$$

Proof. Let $N = Mn + \nu$, $n = [N/M]$, and $K = Mk + \nu$, $k = [K/M]$. Consider the inner product

$$(\pi_N, \pi_K) = \int_0^r \left(\sum_{s=1}^{M} \pi_N(x\varepsilon_s)\overline{\pi_K(x\varepsilon_s)} \right) \omega(x)\, dx,$$

which can be reduced to

$$\begin{aligned} (\pi_N, \pi_K) &= \int_0^r \left(\sum_{s=1}^{M} \varepsilon_s^N \pi_N(x)\varepsilon_s^{-K}\overline{\pi_K(x)} \right) \omega(x)\, dx \\ &= \left(\sum_{s=1}^{M} \varepsilon_s^{N-K} \right) \int_0^r \pi_N(x)\overline{\pi_K(x)}\, \omega(x)\, dx \\ &= M \int_0^r x^{2\nu} p_n^{(\nu,M)}(x^M) p_k^{(\nu,M)}(x^M)\, \omega(x)\, dx, \end{aligned}$$

using Lemmas 5.1 and 5.2, as well as (5.7). Changing variable $x^M = t$ in the last integral, we conclude that

$$(\pi_N, \pi_K) = (\pi_{Mn+\nu}, \pi_{Mk+\nu}) = \int_0^{r^M} p_n^{(\nu,M)}(t) p_k^{(\nu,M)}(t) \Omega_{\nu,M}(t)\, dt = 0, \quad n \neq k,$$

where $\Omega_{\nu,M}(t)$ is given by (5.8). □

As we can see, the question of the existence of the polynomials $\pi_N(z)$ is reduced to the existence of polynomials $p_n^{(\nu,M)}(t)$, orthogonal on $(0, r^M)$ with respect to the weight function $\Omega_{\nu,M}(t)$, for every $\nu = 0, 1, \ldots, M-1$.

5.3. The case $M = 2m$

Let the number of the rays be even, $M = 2m$. Then, as we mentioned on the beginning of this section, the inner product (5.1) has the property $(z^m f, g) = (f, z^m g)$ ($\rho = m$ in Lemma 3.1). Starting from Theorem 3.2 we can prove the following result (see [11]):

Theorem 5.8. *Let $M = 2m$ and the inner product $(\cdot\,,\cdot)$ be given by (5.1). The corresponding monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation*

$$\begin{aligned} \pi_{N+m}(z) &= z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \\ \pi_N(z) &= z^N, \quad N = 0, 1, \ldots, 2m-1, \end{aligned} \tag{5.9}$$

where

$$b_N = \frac{(\pi_N, z^m \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = \frac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}\,. \tag{5.10}$$

There is a connection between this relation and (5.2). Namely, using (5.9) one can get the recurrence relation (5.2), where

$$\alpha_N = b_N + b_{N+m}, \qquad \beta_N = b_N b_{N-m}.$$

We consider now a simple case when $M = 4$, $r = 1$, and $\omega(x) = 1$. The inner product (5.1) then becomes

$$(f, g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)} \Big]\, dx.$$

Using the moments determinants (see Section 2), we can calculate directly the coefficient b_N, given by (5.10),

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 0,1, \\[2ex] \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 2,3. \end{cases} \tag{5.11}$$

Now, applying the recurrence relation (5.9), i.e.,

$$\begin{aligned} \pi_{N+2}(z) &= z^2\pi_N(z) - b_N\pi_{N-2}(z), \quad N = 2,3,\ldots, \\ \pi_0(z) &= 1,\ \pi_1(z) = z,\ \pi_2(z) = z^2,\ \pi_3(z) = z^3, \end{aligned}$$

with (5.11), we obtain the corresponding sequence of orthogonal polynomials:

$$1,\ z,\ z^2,\ z^3,\ z^4 - \frac{1}{5},\ z^5 - \frac{3}{7}z,\ z^6 - \frac{5}{9}z^2,\ z^7 - \frac{7}{11}z^3,\ z^8 - \frac{10}{13}z^4 + \frac{5}{117},\ \ldots\ .$$

5.4. Differential equation

Using (5.7) we can prove (see [13]):

Theorem 5.9. *If the polynomial* $p_n^{(\nu,M)}(t)$ $(\nu \in \{0,1,\ldots,M-1\})$*, defined by* (5.6)*, satisfies a differential equation of the form*

$$a^{(\nu,M)}(t)y'' + b^{(\nu,M)}(t)y' + c^{(\nu,M)}(t,n)y = 0,$$

then the polynomial $\pi_{Mn+\nu}(z)$ *satisfies the following linear differential equation of the second order*

$$A^{(\nu,M)}(z)Y'' + B^{(\nu,M)}(z)Y' + C^{(\nu,M)}(z,N)Y = 0,$$

where

$$\begin{aligned} A^{(\nu,M)}(z) &= a^{(\nu,M)}(z^M)z^2, \\ B^{(\nu,M)}(z) &= \left[Mb^{(\nu,M)}(z^M)z^M - (M+2\nu-1)a^{(\nu,M)}(z^M)\right]z, \\ C^{(\nu,M)}(z,N) &= M^2c^{(\nu,M)}\left(z^M,(N-\nu)/M\right)z^{2M} - \nu Mb^{(\nu,M)}(z^M)z^M \\ &\quad +\nu(\nu+M)a^{(\nu,M)}(z^M). \end{aligned}$$

5.5. Zeros

Theorem 5.10. *Let $N = Mn + \nu$, $n = [N/M]$, $\nu \in \{0, 1, \ldots, M-1\}$ and let the inner product be given by* (5.1). *Then, all zeros of the polynomial $\pi_N(z)$ are simple and located symmetrically on the radial rays ℓ_s, $s = 0, 1, \ldots, M-1$, with the possible exception of a multiple zero of order ν at the origin $z = 0$.*

The proof is based on (5.7). Let $\tau_k^{(n,\nu)}$, $k = 1, \ldots, n$, denote the zeros of $p_n^{(\nu,M)}(t)$ in an increasing order, $\tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)}$. It is clear that each zero $\tau_k^{(n,\nu)}$ generates M zeros

$$z_{k,s}^{(n,\nu)} = \sqrt[M]{\tau_k^{(n,\nu)}}\, e^{i2\pi(s-1)/M}, \qquad s = 1, 2, \ldots, M,$$

of the polynomial $\pi_N(z)$. Thus, on every ray we have

$$|z_{1,s}^{(n,\nu)}| < |z_{2,s}^{(n,\nu)}| < \cdots < |z_{n,s}^{(n,\nu)}|, \quad s = 1, 2, \ldots, M.$$

As an example see Figure 4.1 (left).

6. Analogue of Some Classical Polynomials

In this section we consider two cases of polynomials orthogonal on the radial rays with respect to the inner product (5.1), which are connected with some classical orthogonal polynomials. Namely, for polynomials $p_n^{(\nu,M)}(t)$ in the representation (5.7) we take the Jacobi and the generalized Laguerre polynomials. In such a way we obtain two classes of polynomials named as the *M-generalized Gegenbauer polynomials* and the *M-generalized Hermite polynomials.*

6.1. The M-generalized Gegenbauer polynomials

Let $r = 1$ and the weight function $x \mapsto \omega(x)$ in (5.1) be given by

$$\omega(x) = (1 - x^M)^\alpha x^{M\gamma}, \qquad \alpha > -1,\ \gamma > -\frac{1}{M}\,. \tag{6.1}$$

Then for any $\nu \in \{0, 1, \ldots, M-1\}$, according to Theorem 5.7, the sequence of polynomials $\{p_n^{(\nu,M)}(t)\}_{n=0}^{+\infty}$ is orthogonal on $(0,1)$ with respect to the weight function $t \mapsto \Omega_{\nu,M}(t) = (1-t)^\alpha t^{\gamma+(2\nu-M+1)/M}$ (the Jacobi weight transformed to $(0,1)$). Thus, we have the following result:

Theorem 6.1. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (5.1), *with the weight function* (6.1), *can be expressed in the form*

$$\pi_N(z) = 2^{-n} z^\nu \hat{P}_n^{(\alpha,\beta_\nu)}(2z^M - 1), \quad N = Mn + \nu,\ n = [N/M], \tag{6.2}$$

where $\nu \in \{0, 1, \ldots, M-1\}$, $\beta_\nu = \gamma + (2\nu - M + 1)/M$, *and* $\hat{P}_n^{(\alpha,\beta)}(x)$ *denotes the monic Jacobi polynomial orthogonal on* $(-1,1)$ *with respect to the weight* $x \mapsto (1-x)^\alpha(1+x)^\beta$.

For $M = 2$ the weight (6.1) reduces to the generalized Gegenbauer weight $\omega(x) = (1-x^2)^\alpha |x|^{2\gamma}$, taking the integral in the inner product over $(-1,1)$. It is a reason to call the polynomials $\pi_N(z)$ as the *M-generalized Gegenbauer polynomials.*

Starting from the Jacobi differential equation and Theorem 5.9 we get (see [12]):

Theorem 6.2. *Let* $N = Mn + \nu$, $n = [N/M]$, $\nu \in \{0, 1, \ldots, M-1\}$. *The polynomial* $\pi_N(z)$ *orthogonal with respect to the inner product* (5.1), *with* $r = 1$ *and the weight function* (6.1), *satisfy a second order linear homogeneous differential equation of the form*

$$(1 - z^M) z^2 Y'' + C(z) z Y' + (Az^M - B) Y = 0, \tag{6.3}$$

where $A = N[N + M(\alpha+\gamma) + 1]$, $B = \nu[\nu + M(\gamma - 1) + 1]$, *and*

$$C(z) = M(\gamma - 1) + 2 - (M(\alpha+\gamma) + 2) z^M.$$

If we take $M = 2m$ the polynomials $\pi_N(z)$ satisfy the recurrence relation (5.9), with recursion coefficients (see [11])

$$b_{2mn+\nu} = \begin{cases} \dfrac{n(n+\alpha)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } 0 \le \nu \le m-1, \\[2ex] \dfrac{(n+\beta_\nu)(n+\alpha+\beta_\nu)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } m \le \nu \le 2m-1. \end{cases} \tag{6.4}$$

6.2. The M-generalized Hermite polynomials

We take now $r = +\infty$ and

$$\omega(x) = x^{M\gamma}\exp(-x^M), \qquad \gamma > -1/M. \tag{6.5}$$

In the same way as before, we can state the following result:

Theorem 6.3. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (5.1)*, with the weight function* (6.5) *on* $(0, +\infty)$*, can be expressed in the form*

$$\pi_N(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^M), \quad N = Mn + \nu,\ n = [N/M],$$

where $\nu \in \{0, 1, \ldots, M-1\}$*,* $\alpha_\nu = \gamma + (2\nu - M + 1)/M$*, and* $\hat{L}_n^{(s)}(t)$ *denotes the monic generalized Laguerre polynomial orthogonal with respect to* $t \mapsto t^s e^{-t}$ *on* $(0, +\infty)$*.*

In the case $M = 2$, the corresponding weight is the generalized Hermite weight on $(-\infty, +\infty)$, i.e., $\omega(x) = |x|^{2\gamma}\exp(-x^2)$. As in the previous subsection we can get the corresponding differential equation, as well as the recurrence relation when M is an even number (see [10, 11, 12]).

7. Remarks on Some Applications

At first we mention a physical problem connected to a non-linear diffusion equation. The equations for the dispersion of a buoyant contaminant can be approximated by the Erdogan-Chatwin equation

$$\partial_t c = \partial_y \Big\{ \Big[D_0 + (\partial_y c)^2 D_2 \Big] \partial_y c \Big\}, \tag{7.1}$$

where D_0 is the dispersion coefficient appropriate for neutrally-buoyant contaminants, and the D_2 term represents the increased rate of dispersion associated with the buoyancy-driven currents (cf. [16]). The same equation was derived also by other authors for other physical contexts.

Smith [16] obtained analytic expressions for the similarity solutions of the equation (7.1) in the limit of strong non-linearity ($D_0 = 0$), i.e.,

$$\partial_t c = D_2 \partial_y \Big[(\partial_y c)^3 \Big],$$

both for a concentration jump and for a finite discharge. He also investigated the asymptotic stability of these solutions. It is interesting that the stability analysis for the finite discharge involves a family of orthogonal polynomials $Y_N(z)$, such that

$$(1 - z^4) Y_N'' - 6z^3 Y_N' + N(N+5) z^2 Y_N = 0. \tag{7.2}$$

The degree N is restricted to the values $0, 1, 4, 5, 8, 9, \ldots$, so that the first few (monic) polynomials are:

$$1,\ z,\ z^4 - \frac{1}{3},\ z^5 - \frac{5}{11}\, z,\ z^8 - \frac{14}{17}\, z^4 + \frac{21}{221},\ z^9 - \frac{18}{19}\, z^5 + \frac{3}{19}\, z,\ \ldots\ . \tag{7.3}$$

It is easy to see that these polynomials are a special case of polynomials orthogonal on the radial rays in the complex plane. Namely, if we take four symmetric rays ($M = 4$), the inner product (5.1), and the weight function (6.1), with $\alpha = \gamma = 1/2$, i.e., $\omega(x) = (1 - x^4)^{1/2} x^2$, using Theorem 6.1 or (6.4), for $m = 2$, we get the following sequence of orthogonal polynomials:

$$1,\ z,\ z^2,\ z^3,\ z^4 - \frac{1}{3},\ z^5 - \frac{5}{11}\, z,\ z^6 - \frac{7}{13}\, z^2,\ z^7 - \frac{3}{5}\, z^3,\ z^8 - \frac{14}{17}\, z^4 + \frac{21}{221},\ \ldots\ .$$

As we can see, this polynomial sequence contains the sequence (7.3), i.e., the polynomials $Y_N(z)$ are just our polynomials $\pi_N(z)$ for the particular values $\alpha = \gamma = 1/2$. Notice that our sequence of polynomials is complete. Also, we see that our differential equation (6.3), in this special case, becomes

$$(1 - z^4) Y'' - 6z^3 Y' + [N(N+5) z^2 - \nu(\nu - 1) z^{-2}] Y = 0,$$

where $N = 4n + \nu$, $\nu \in \{0, 1, 2, 3\}$. Evidently, for $N = 4n$ and $N = 4n + 1$ ($n \in \mathbb{N}_0$), this equation reduces to equation (7.2) derived by Smith [16]. Some similar differential equations with polynomial solutions were also obtained by Smith [18].

As a second example we give an electrostatic interpretation of the zeros of polynomials $\pi_N(z)$ on the symmetric radial rays in the complex plane. The first electrostatic interpretation of the zeros of Jacobi polynomials was given by Stieltjes in 1885 (see [19, 20] for details). Stieltjes considered an electrostatic problem with particles of charge p and q ($p, q > 0$) fixed at $x = 1$ and $x = -1$, respectively, and n unit charges confined to the interval $[-1, 1]$ at points $x_1, x_2, \ldots, x_n$. Assuming a logarithmic potential, he proved that the electrostatic equilibrium arises when x_k are zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(x)$. In that case, the Hamiltonian

$$H(x_1, x_2, \ldots, x_n) = -\sum_{k=1}^{n}\Big(\log(1 - x_k)^p + \log(1 + x_k)^q\Big) - \sum_{1\le k<j\le n} \log|x_k - x_j|$$

becomes a minimum. This minimum is indeed the unique global minimum (see Szegő [21, p. 140]). Obviously, $H(x_1, x_2, \ldots, x_n)$ can be interpreted as the energy of the previous electrostatic system.

In the last time it was appeared several papers in this direction. Forrester and Rogers [3] gave an interpretation of zeros of the classical polynomials as the equilibrium positions of two-dimensional electrostatic problems. Also, Hendriksen and Rossum [6] considered an electrostatic interpretation of zeros of classical orthogonal polynomials, including Bessel polynomials, as well as some polynomials introduced by Smith [16], [18]. We mention also the papers of Grunbaum [5], Dimitrov and Van Assche [2], and Ismail [8, 9]. Recently, the electrostatic interpretation of the zeros was also exploited to obtain interpolation points suitable for approximation of smooth functions defined on a simplex (see Hesthaven [7]).

We consider now a symmetric electrostatic problem with M positive point charges all of strength q which are placed at the fixed points

$$\xi_s = \exp\Big(\frac{2(s-1)\pi i}{M}\Big) \qquad (s = 1, 2, \ldots, M) \tag{7.4}$$

and a charge of strength p ($> -(M-1)/2$) at the origin $z = 0$. Also we have N positive free unit charges, positioned at z_1, z_2, …, z_N. Assuming a logarithmic potential, it is interesting to find these points in electrostatic equilibrium.

As in [12] we are interested only in solutions with the rotational symmetry. Putting $\pi_N(z) = \prod_{k=1}^{N}(z - z_k)$ and $N = Mn + \nu$, $n = [N/M]$, we can get the following differential equation for this polynomial,

$$z^2(1 - z^M)\,\pi_N''(z) + 2\left[p - (Mq + p)z^M\right] z\pi_N'(z) \\ + \left\{N\left[N - 1 + 2(Mq + p)\right] z^M - \nu(\nu + 2p - 1)\right\} \pi_n(z) = 0.$$

Comparing this equation with (6.3) (Theorem 6.2) we obtain:

Theorem 7.1. *An electrostatic system of M positive point charges all of strength q, which are placed at the fixed points ξ_s given by* (7.4)*, and a charge of strength p $(> -(M-1)/2)$ at the origin $z = 0$, as well as N positive free unit charges, positioned at $z_1, z_2, \ldots, z_N$, is in electrostatic equilibrium if these points z_k are zeros of the polynomial $\pi_N(z)$ orthogonal with respect to the inner product* (5.1)*, with the weight function $w(x) = (1 - x^M)^{2q-1}x^{M+2(p-1)}$. This polynomial can be expressed in terms of the monic Jacobi polynomials*

$$\pi_N(z) = 2^{-n} z^\nu \hat{P}_n^{(2q-1,(2p+2\nu-1)/M)}(2z^M - 1),$$

where $N = Mn + \nu$, $n = [N/M]$.

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