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Discrete Inequalities of Wirtinger's Type for Higher Differences

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Discrete version of Wirtinger's type inequality for higher differences,

$$A_{n,m}\sum_{k=1}^{n}x_k^2\le\sum_{k=l_m}^{u_m}(\Delta^m x_k)^2\le B_{n,m}\sum_{k=1}^{n}x_k^2,$$

where $l_m=1-[m/2]$, $u_m=n-[m/2]$ and

$$\Delta^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+m-i},$$

is considered. Under some conditions, the best constants $A_{n,m}$ and $B_{n,m}$ are determined.

Keywords: Discrete inequality; difference of higher order; eigenvalue; eigenvector.

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1 INTRODUCTION AND PRELIMINARIES

In [1] (see also [2]) we presented a general method for finding the best possible constants A_n and B_n in inequalities of the form

$$A_n\sum_{k=1}^{n}p_kx_k^2\le\sum_{k=0}^{n}r_k(x_k-x_{k+1})^2\le B_n\sum_{k=1}^{n}p_kx_k^2,\qquad(1.1)$$

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where $\boldsymbol{p} = (p_k)$ and $\boldsymbol{r} = (r_k)$ are given weight sequences and $\boldsymbol{x} = (x_k)$ is an arbitrary sequence of the real numbers. The basic discrete inequalities of the form (1.1) for $p_k = r_k = 1$ were given by K. Fan, O. Taussky, and J. Todd [3]. Here, we mention some references in this direction [4–8].

The first results for the second difference were proved by Fan, Taussky and Todd [3]:

THEOREM 1.1 *If* $x_0 \,(= 0),\ x_1, x_2, \ldots, x_n,\ x_{n+1} \,(= 0)$ *are given real numbers, then*

$$\sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 \geq 16\sin^4\frac{\pi}{2(n+1)}\sum_{k=1}^{n} x_k^2, \tag{1.2}$$

with equality in (1.2) if and only if $x_k = A\sin\dfrac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$, *where* A *is an arbitrary constant.*

THEOREM 1.2 *If* $x_0, x_1, \ldots, x_n, x_{n+1}$ *are given real numbers such that* $x_0 = x_1$, $x_{n+1} = x_n$ *and*

$$\sum_{k=1}^{n} x_k = 0, \tag{1.3}$$

then

$$\sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 \geq 16\sin^4\frac{\pi}{2n}\sum_{k=1}^{n} x_k^2. \tag{1.4}$$

The equality in (1.4) is attained if and only if

$$x_k = A\cos\frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n,$$

where A *is an arbitrary constant.*

A converse inequality of (1.2) was proved by Lunter [9], Yin [10] and Chen [11] (see also Agarwal [8]).

THEOREM 1.3 *If* $x_0 \,(= 0),\ x_1, x_2, \ldots, x_n,\ x_{n+1} \,(= 0)$ *are given real numbers, then*

$$\sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 \leq 16\cos^4\frac{\pi}{2(n+1)}\sum_{k=1}^{n} x_k^2, \tag{1.5}$$

with equality in (1.5) if and only if $x_k = A(-1)^k\sin\dfrac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$, *where* A *is an arbitrary constant.*

Chen [11] also proved the following result:

THEOREM 1.4 *If $x_0, x_1, \ldots, x_n, x_{n+1}$ are given real numbers such that $x_0 = x_1$ and $x_{n+1} = x_n$, then*

$$\sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 \leq 16\cos^4\frac{\pi}{2n}\sum_{k=1}^{n}x_k^2,$$

with equality holding if and only if

$$x_k = A(-1)^k \sin\frac{(2k-1)\pi}{n}, \qquad k = 1, 2, \ldots, n,$$

where A is an arbitrary constant.

In this case, the $n \times n$ symmetric matrix corresponding to the quadratic form

$$F_2 = \sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 = (H_{n,2}\boldsymbol{x}, \boldsymbol{x})$$

is

$$H_{n,2} = \begin{bmatrix} 2 & -3 & 1 & & & & & \\ -3 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -3 \\ & & & & 1 & -3 & 2 \end{bmatrix}.$$

This matrix is the square of the $n \times n$ matrix

$$H_n = H_{n,1} = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}. \tag{1.6}$$

The eigenvalues of H_n are

$$\lambda_\nu = \lambda_\nu(H_n) = 4\cos^2\frac{(n-\nu+1)\pi}{2n}, \qquad \nu = 1, \ldots, n,$$

and therefore, the largest eigenvalue of H_n is

$$\lambda_n(H_n) = 4\cos^2\frac{\pi}{2n} > \lambda_{n-1}(H_n).$$

The corresponding eigenvector is $\boldsymbol{x}^n = [\, x_{1n} \quad x_{2n} \quad \dots \quad x_{nn} \,]^T$, where

$$x_{\nu n} = (-1)^\nu \sin \frac{(2\nu - 1)\pi}{2n}, \qquad \nu = 1, 2, \dots, n.$$

Thus, the largest eigenvalue of $H_{n,2}$ is

$$\lambda_n(H_{n,2}) = 16 \cos^4 \frac{\pi}{2n} > \lambda_{n-1}(H_{n,2}),$$

and the associated eigenvector is $\boldsymbol{x}^n$.

Notice that the minimal eigenvalue of the matrix H_n (and also $H_{n,2}$) is $\lambda_1 = 0$. Therefore, the condition (1.3) must be included in Theorem 1.2 (see Agarwal [8, Ch. 11]) and the best constant is the square of the relevant eigenvalue

$$\lambda_2 = 4 \cos^2 \frac{(n-1)\pi}{2n} = 4 \sin^2 \frac{\pi}{2n}.$$

For any n-dimensional vector $\boldsymbol{x} = [\, x_1 \quad x_2 \quad \dots \quad x_n \,]^T$, Pfeffer [12] introduced a periodically extended n-vector by setting $x_{i+rn} = x_i$ for $i = 1, 2, \dots, n$ and $r \in \mathbb{N}$, and used the mth difference of $\boldsymbol{x}$ given by $\boldsymbol{x}^{(m)} = [\, \Delta^m x_1 \quad \Delta^m x_2 \quad \dots \quad \Delta^m x_n \,]^T$, where

$$\Delta^m x_i = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} x_{i-[m/2]+r}, \qquad 1 \le i \le n,$$

in order to prove the following result:

THEOREM 1.5 *If $\boldsymbol{x}$ is a periodically extended n-vector and (1.3) holds, then*

$$(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m)}) \ge \left(4 \sin^2 \frac{\pi}{n}\right)^m (\boldsymbol{x}, \boldsymbol{x}),$$

with equality case if and only if $\boldsymbol{x}$ is the periodic extension of a vector of the form $C_1 \boldsymbol{u} + C_2 \boldsymbol{v}$, where

$$\boldsymbol{u} = [\, u_1 \quad u_2 \quad \dots \quad u_n \,]^T \qquad \text{and} \qquad \boldsymbol{v} = [\, v_1 \quad v_2 \quad \dots \quad v_n \,]^T$$

have the following components

$$u_k = \cos \frac{2k\pi}{n}, \qquad v_k = \sin \frac{2k\pi}{n}, \qquad k = 1, \dots, n,$$

and C_1 and C_2 are arbitrary real constants.

2 MAIN RESULTS

In this paper we consider inequalities of the form

$$A_{n,m}\sum_{k=1}^{n} x_k^2 \le \sum_{k=l_m}^{u_m} \left(\Delta^m x_k\right)^2 \le B_{n,m}\sum_{k=1}^{n} x_k^2, \tag{2.1}$$

where $l_m = 1 - [m/2]$, $u_m = n - [m/2]$ and

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+m-i}.$$

The quadratic form $F_m = \sum\limits_{k=l_m}^{u_m} (\Delta^m x_k)^2$ for $m = 1$ reduces to

$$F_1 = x_1^2 + \sum_{k=2}^{n-1} 2x_k^2 + x_n^2 - 2\sum_{k=1}^{n-1} x_k x_{k+1},$$

with corresponding tridiagonal symmetric matrix $H_n = H_{n,1}$ given by (1.6).

We consider inequalities (2.1) under conditions

$$x_s = x_{1-s}, \qquad x_{n+1-s} = x_{n+s} \qquad (l_m \le s \le 0) \tag{2.2}$$

and define

$$\boldsymbol{x}^{(j)} = \begin{bmatrix} \Delta^j x_{1-[j/2]} \\ \Delta^j x_{2-[j/2]} \\ \vdots \\ \Delta^j x_{n-[j/2]} \end{bmatrix}. \tag{2.3}$$

The quadratic form F_m can be expressed then in the following form

$$F_m = F_m(\boldsymbol{x}) = \sum_{k=l_m}^{u_m} \left(\Delta^m x_k\right)^2 = (\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m)}), \tag{2.4}$$

where

$$\boldsymbol{x} = \boldsymbol{x}^{(0)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

At the begining we prove three auxiliary results:

LEMMA 2.1 *If j is an even integer, under conditions (2.2), we have that*

$$\Delta^{j+1}x_{-[j/2]} = 0 \quad \textit{and} \quad \Delta^{j+1}x_{n-[j/2]} = 0. \tag{2.5}$$

Proof Let $q = 0$ or $q = n$. Putting $j = 2p$ we have

$$\begin{aligned}
\Delta^{j+1}x_{q-[j/2]} = \Delta^{2p+1}x_{q-p} &= \sum_{i=0}^{2p+1}(-1)^i\binom{2p+1}{i}x_{q+p+1-i} \\
&= \sum_{i=0}^{p}(-1)^i\binom{2p+1}{i}x_{q+p+1-i} + \sum_{i=p+1}^{2p+1}(-1)^i\binom{2p+1}{i}x_{q+p+1-i} \\
&= \sum_{i=0}^{p}(-1)^i\binom{2p+1}{i}x_{q+p+1-i} - \sum_{i=0}^{p}(-1)^i\binom{2p+1}{i}x_{q-p+i} \\
&= \sum_{i=0}^{p}(-1)^i\binom{2p+1}{i}(x_{q+p+1-i} - x_{q-p+i}) = 0
\end{aligned}$$

because of the conditions (2.2). □

LEMMA 2.2 *If j is an even integer, under conditions (2.2), we have that*

$$H_n\boldsymbol{x}^{(j)} = -\boldsymbol{x}^{(j+2)},$$

where the matrix H_n is given by (1.6).

Proof We have

$$H_n\boldsymbol{x}^{(j)} = \begin{bmatrix} \Delta^j x_{1-[j/2]} - \Delta^j x_{2-[j/2]} \\ -\Delta^{j+2}x_{1-[j/2]} \\ \vdots \\ -\Delta^{j+2}x_{n-2-[j/2]} \\ -\Delta^j x_{n-1-[j/2]} + \Delta^j x_{n-[j/2]} \end{bmatrix}. \tag{2.6}$$

Since

$$\Delta^{j+2}x_{-[j/2]} = \Delta^{j+1}x_{1-[j/2]} - \Delta^{j+1}x_{-[j/2]}$$

and

$$\Delta^{j+2}x_{n-1-[j/2]} = \Delta^{j+1}x_{n-[j/2]} - \Delta^{j+1}x_{n-1-[j/2]},$$

because of Lemma 2.1, we conclude that

$$\Delta^{j+2}x_{-[j/2]} = \Delta^{j+1}x_{1-[j/2]} \quad \text{and} \quad \Delta^{j+2}x_{n-1-[j/2]} = -\Delta^{j+1}x_{n-1-[j/2]},$$

respectively. Therefore,

$$\Delta^j x_{1-[j/2]} - \Delta^j x_{2-[j/2]} = -\Delta^{j+1}x_{1-[j/2]} = -\Delta^{j+2}x_{-[j/2]}$$

and

$$-\Delta^j x_{n-1-[j/2]} + \Delta^j x_{n-[j/2]} = \Delta^{j+1}x_{n-1-[j/2]} = -\Delta^{j+2}x_{n-1-[j/2]}.$$

Then (2.6) becomes

$$H_n\boldsymbol{x}^{(j)} = -\begin{bmatrix} \Delta^{j+2}x_{-[j/2]} \\ \Delta^{j+2}x_{1-[j/2]} \\ \vdots \\ \Delta^{j+2}x_{n-2-[j/2]} \\ \Delta^{j+2}x_{n-1-[j/2]} \end{bmatrix} = -\begin{bmatrix} \Delta^{j+2}x_{1-[(j+2)/2]} \\ \Delta^{j+2}x_{2-[(j+2)/2]} \\ \vdots \\ \Delta^{j+2}x_{n-1-[(j+2)/2]} \\ \Delta^{j+2}x_{n-[(j+2)/2]} \end{bmatrix} = -\boldsymbol{x}^{(j+2)}.$$

□

LEMMA 2.3 *If j is an even integer, under conditions (2.2), we have that*

$$(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(j+2)}) = -(\boldsymbol{x}^{(j+1)}, \boldsymbol{x}^{(j+1)}).$$

Proof Let j is an even integer. Using (2.3) we have

$$\begin{aligned}
(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(j+2)}) &= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\Delta^{j+2}x_{k-1-[j/2]} \\
&= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\big(\Delta^j x_{k-1-[j/2]} - 2\Delta^j x_{k-[j/2]} + \Delta^j x_{k+1-[j/2]}\big) \\
&= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\big(\Delta^j x_{k+1-[j/2]} - \Delta^j x_{k-[j/2]}\big) \\
&\qquad - \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\big(\Delta^j x_{k-[j/2]} - \Delta^j x_{k-1-[j/2]}\big) \\
&= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\Delta^{j+1}x_{k-[j/2]} - \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\Delta^{j+1}x_{k-1-[j/2]} \\
&= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]}\Delta^{j+1}x_{k-[j/2]} - \sum_{k=0}^{n-1} \Delta^j x_{k+1-[j/2]}\Delta^{j+1}x_{k-[j/2]}.
\end{aligned}$$

Because of (2.5) we can write

$$
\begin{aligned}
(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(j+2)}) &= \sum_{k=1}^{n} \Delta^j x_{k-[j/2]} \Delta^{j+1} x_{k-[j/2]} - \sum_{k=1}^{n} \Delta^j x_{k+1-[j/2]} \Delta^{j+1} x_{k-[j/2]} \\
&= -\sum_{k=1}^{n} \left(\Delta^{j+1} x_{k-[j/2]}\right)^2 .
\end{aligned}
$$

Since j is an even integer we have that

$$
(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(j+2)}) = -\sum_{k=1}^{n} \left(\Delta^{j+1} x_{k-[(j+1)/2]}\right)^2 = -(\boldsymbol{x}^{(j+1)}, \boldsymbol{x}^{(j+1)}).
$$

□

Now, we give the main result:

THEOREM 2.4 *If $x_1, x_2, \ldots, x_n$ are given real numbers and conditions (2.2) are satisfied, then*

$$
\sum_{k=l_m}^{u_m} \left(\Delta^m x_k\right)^2 \leq 4^m \cos^{2m} \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2, \tag{2.7}
$$

where $l_m = 1 - [m/2]$ and $u_m = n - [m/2]$. The equality in (2.7) is attained if and only if

$$
x_k = A(-1)^k \sin \frac{(2k-1)\pi}{n}, \qquad k = 1, 2, \ldots, n,
$$

where A is an arbitrary constant.

Proof We will prove that the corresponding matrix of the quadratic form (2.4) is exactly the mth power of the matrix $H_n = H_{n,1}$ so that the best constant in the right inequality (2.1), i.e., (2.7), is given by

$$
B_{n,m} = 4^m \cos^{2m} \frac{\pi}{2n} .
$$

Evidently, $A_{n,m} = 0$.

Let m be an even integer. Then, using Lemma 2.2, we find

$$
F_m = \left(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m)}\right) = \left(H_n \boldsymbol{x}^{(m-2)}, H_n \boldsymbol{x}^{(m-2)}\right),
$$

i.e.,

$$F_m = \left(H_n^{m/2}\boldsymbol{x}^{(0)}, H_n^{m/2}\boldsymbol{x}^{(0)}\right) = \left(H_n^m\boldsymbol{x}, \boldsymbol{x}\right).$$

Similarly, for an odd m, using Lemmas 2.3 and 2.4, we obtain

$$F_m = \left(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m)}\right) = -\left(\boldsymbol{x}^{(m-1)}, \boldsymbol{x}^{(m+1)}\right) = \left(\boldsymbol{x}^{(m-1)}, H_n\boldsymbol{x}^{(m-1)}\right).$$

Now, using Lemma 2.2 again, we find

$$F_m = \left(H_n^{(m-1)/2}\boldsymbol{x}^{(0)}, H_n^{(m+1)/2}\boldsymbol{x}^{(0)}\right) = \left(H_n^m\boldsymbol{x}, \boldsymbol{x}\right).$$

□

By restriction (1.3), we can obtain the following result:

THEOREM 2.5 *If $x_1, x_2, \ldots, x_n$ are given real numbers and conditions (2.2) and (1.3) are satisfied, then*

$$4^m \sin^{2m}\frac{\pi}{2n}\sum_{k=1}^{n} x_k^2 \leq \sum_{k=l_m}^{u_m}\left(\Delta^m x_k\right)^2, \tag{2.8}$$

where $l_m = 1 - [m/2]$ and $u_m = n - [m/2]$. The equality in (2.8) is attained if and only if

$$x_k = A\cos\frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n,$$

where A is an arbitrary constant.

For other generalizations of discrete Wirtinger's inequalities see [13–15]. There are also generalizations for multidimensional sequences and partial differences (see [16] and [17]).

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