INTEGRAL SPLINE OPERATORS IN CAGD

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Abstract. An application of integral Schoenberg splines in free-form curve modelling is studied. First, we introduce the λ-family of integral spline operators which, for $\lambda = 0$ reduce to the Schoenberg variation diminishing spline operator, or on integral spline operator introduced by M. W. Müller, for $\lambda = 1$. This approach permits introducing a parametric B-spline curve model that depends on a real number λ as a shape parameter. Namely, λ directly influences the shape of the corresponding B-spline curve. The properties of the λ-family are investigated.

Keywords. Free form curves, B-spline curves, Integral Schoenberg spline operator, Shape parameter.

1. Introduction

The variation diminishing splines are known to be an important class of splines introduced and mainly investigated by Schoenberg, Greville and Marsden (see, e.g., [9] and [7]), primarily as a tool of approximation theory. This kind of splines possess the properties that make them attractive for Computer Aided Geometric Design purposes [1]. Namely, variation diminishing splines can be used to produce a nice curve/surface model called B-spline curve/surface model. In this paper we deal with a curve models. The similar investigations for surfaces are in working.

In this section we recall the variation diminishing spline operator and the B-spline curve model. Also we list the properties relevant for CAGD and the way of application in geometric modelling. Then, in the next section, we define the curve model so as to be generated by integral Schoenberg spline operator [8], and investigate its properties. In Section 3 we use the integral spline model to introduce a shape parameter into B-spline curve. Changing this parameter changes the form of the curve as to help designer in choosing the final shape of the curve he works with.

Let us now recall the variation diminishing splines and the corresponding B-spline curve model. Following [9], we associate with the vector of *knots* $\mathbf{t} = (t_i)_{i=-k}^{m}$, $m = n + k$, $n, k \in \mathbb{N}$ so that

$$0 = t_{-k} = \cdots = t_0 < t_1 < \cdots < t_n < t_{n+1} = \cdots = t_{m+1} = 1\,,$$

the sequence of *normalized B-splines* of degree k,

$$B_i^k(t) = B_i^k(t; t_{i-k}, \dots, t_{i+1}), \quad i = 0, \dots, m$$

given by

$$B_i^k(t) = (t_{i+1} - t_{i-k})[t_{i-k}, \dots, t_{i+1}](\cdot - t)_+^k, \quad i = 0, \dots, m.$$

Then, the variation diminishing spline operator S_m is defined by

$$(S_m f)(t) = \sum_{i=0}^{m} f(\xi_i^k) B_i^k(t), \quad 0 \le t \le 1, \tag{1}$$

where $f : [0,1] \to \mathbb{R}$ is any bounded function and $\Xi = (\xi_i^k)_{i=0}^m$ is the sequence of *nodes* from $[0,1]$. As the consequence of proposed property that S_m preserves affine function, Greville established ([9, supplement]) that the nodes and knots are connected through the relation

$$\xi_i^k = \frac{1}{k}(t_{i-k+1} + \cdots + t_i), \quad i = 0, \dots, m. \tag{2}$$

In this case, the following identities take place

$$\sum_{i=0}^{m} B_i^k(t) = 1, \tag{3}$$

$$\sum_{i=0}^{m} \xi_i^k B_i^k(t) = t. \tag{4}$$

Note that from (2) immediately follows

$$0 = \xi_0^k < \xi_1^k < \cdots < \xi_m^k = 1; \quad \xi_{i+1}^k - \xi_i^k = \frac{1}{k}(t_{i+1} - t_{i-k+1}). \tag{5}$$

The variation diminishing splines are explored in CAGD in order to construct free-form curves and surfaces. The curve/surface is said to have *free–form* if it is possible to alter its shape by changing one or a few simple parameters with *a priori* knowledge how this changing will affect the shape of the curve/surface.

It is usual that the curve is defined by the set of so called *control points* $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_m$ – the vertices of the *control polygon* $\mathbf{P} = (\mathbf{P}_0, \dots, \mathbf{P}_m)^T$. Generally speaking, $\mathbf{P}_i \in \mathbb{R}^d$, although in practical applications d rarely exceeds 3. If replace $f(\xi_i^k)$ in (1) by $\mathbf{P}_i \in \mathbf{P}$, we get the B-spline curve model

$$(\mathbf{S}_m \mathbf{P})(t) = \sum_{i=0}^{m} \mathbf{P}_i B_i^k(t), \quad 0 \le t \le 1, \tag{6}$$

as a parametrically defined vector-valued curve. We say that the curve model (6) is *generated* by the variation diminishing spline operator S_m.

The B-spline curves obey several important properties making them very attractive for geometric modelling. Let us list these properties.

1° *Affine invariance.* B-splines $\mathbf{b}_m = (B_0^k, \dots, B_m^k)^T$ can be regarded as a partition of unit (the equation (3)). This ensures that the curve $\mathbf{S}_m\mathbf{P}$ is invariant under scaling and rotation.

2° *Convex hull property.* Each B-spline is a nonnegative function

$$B_i^k(t) \geq 0, \quad i = 0, \dots, m, \tag{7}$$

which, together with (3) makes any point of B-spline curve $(\mathbf{S}_m\mathbf{P})(t)$ to be a convex combination of its control points. Consequently, the whole curve lies inside the convex hull of the control polygon.

3° *End-points interpolation.* For the set of knots $\mathbf{t}$ as specified, the B-splines satisfy

$$B_i^k(0) = \delta_{i0}, \quad B_i^k(1) = \delta_{i1} \quad (\delta_{ij} - \text{ Kronecker's delta}),$$

which yields

$$(\mathbf{S}_m\mathbf{P})(0) = \mathbf{P}_0, \qquad (\mathbf{S}_m\mathbf{P})(1) = \mathbf{P}_n,$$

i.e., the B-spline curve begins in $\mathbf{P}_0$ and terminates in $\mathbf{P}_m$.

4° *Symmetry.* Suppose the set of knots $\mathbf{t}$ is symmetric with respect to the point $t = 1/2$, i.e.,

$$t_i + t_{n-i+1} = 1, \quad i = 1, \dots, n. \tag{9}$$

Then (see [6]),

$$B_i^k(t) = B_{m-i}^k(1-t), \quad 0 \leq t \leq 1. \tag{10}$$

If denote $\mathbf{P'} = (\mathbf{P}_m, \dots, \mathbf{P}_0)^T$, we see that

$$(\mathbf{S}_m\mathbf{P})(t) = (\mathbf{S}_m\mathbf{P'})(1-t), \quad 0 \leq t \leq 1,$$

i.e., the reversal numeration of control points does not change the curve.

5° *Reproduction of points and lines.* Due to the condition (3), $\mathbf{S}_m\mathbf{P}$ exactly reproduces control points; if $\mathbf{P}_i = \mathbf{P}_0$, $i = 1, \dots, m$, then $\mathbf{S}_m\mathbf{P} \equiv \mathbf{P}_0$. Further, if the control polygon is collinear, i.e., $\mathbf{P}_i = \xi_i^k\mathbf{a} + \mathbf{b}$, ($\mathbf{a}$,$\mathbf{b}$ are constant vectors), we have $(\mathbf{S}_m\mathbf{P})(t) = \mathbf{a}t + \mathbf{b}$, $0 \leq t \leq 1$, where we used (3) and (4),which means that $\mathbf{S}_m\mathbf{P}$ reproduces straight line, the property highly desirable in CAGD.

6° *Oscillation diminution.* The B-spline curve (6) crosses an arbitrary plane from $\mathbb{R}^d$ no more often then does the control polygon. So, $\mathbf{S}_m\mathbf{P}$ diminish the oscillation of the control polygon. This is the consequence of variation diminishing property of the operator S_m

$$v\left\{\sum_{i=0}^{m} c_i B_i^k(t)\right\} \le S^-\{\mathbf{c}\},$$

where $S^-\{\mathbf{c}\}$ is an usual notation for the number of strict sign changes in the sequence $\mathbf{c}$ and $v\{f\}$ is a variation of function f. If we consider any plane $\mathbf{a}\mathbf{x}+b=0$, $\mathbf{a}$ (constant vector), $\mathbf{x}\in\mathbb{R}^d$, $b\in\mathbb{R}$, we have

$$v\{\mathbf{a}(\mathbf{S}_m\mathbf{P})+b\} = v\left\{\sum_{i=0}^{m}(\mathbf{a}\mathbf{P}_i+b)B_i^k\right\} \le S^-\{(\mathbf{a}\mathbf{P}_i+b)_{i=0}^m\}.$$

7° *Uniqueness.* Every B-spline curve is uniquely determined by its control polygon and no two polygons produce the same curve, i.e., $\mathbf{S}_m\mathbf{P} = \mathbf{S}_m\mathbf{P}' \Leftrightarrow \mathbf{P} = \mathbf{P}'$. This property is the consequence of $\mathbf{b}_m$ being a basis in the space of splines with given knots.

8° *Local control.* The minimal support property of B-splines

$$B_i^k(t) = 0, \quad t \notin [t_{i-k}, t_{i+1}], \tag{11}$$

permits to derive the B-spline curve $\{(\mathbf{S}_m\mathbf{P})(t),\ 0\le t\le 1\}$ as a collection of B-spline segments $\{(\mathbf{S}_m\mathbf{P})(t),\ t_i \le t \le t_{i+1}\}$ so as each of them is affected *only* by the vertices $\mathbf{P}_i,\dots,\mathbf{P}_{i+k}$.

9° *Stable numerical algorithm.* For calculation of B-splines one's use de Boor-Cox algorithm [2]

$$B_i^0 = \begin{cases} 1, & t_i \le t < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_i^k(t) = \frac{t-t_{i-k}}{t_{i-1}-t_{i-k}} B_i^{k-1}(t) + \frac{t_i - t}{t_i - t_{i-k+1}} B_{i+1}^{k-1}(t).$$

This recursion algorithm allows to compute B-spline curve $(\mathbf{S}_m\mathbf{P})(t)$ at an arbitrary point $t_i \le t < t_{i+1}$ in the stable and rapid way (see [2])

$$\begin{aligned} &\mathbf{P}_i^0(t) = \mathbf{P}_i, \quad i = 0,\dots,m, \\ &\mathbf{P}_i^r(t) = \frac{t_{i-1}-t}{t_{i-1}-t_{i-k-r+1}}\mathbf{P}_{i-1}^{r-1}(t) + \frac{t-t_{i-k}}{t_{i-r}-t_{i-k}}\mathbf{P}_i^{r-1}(t), \quad i = 0,\dots,m-r, \end{aligned} \tag{12}$$

end thus $(\mathbf{S}_m\mathbf{P})(t) = \mathbf{P}_0^m(t)$. This algorithm is known as de Boor algorithm [3].

10° *Smoothness.* The B-spline curve is smooth enough. If ν knots coincide, e.g., $t_i = \dots = t_{i+p+1}$ the curve has $k-\nu-1$ continuous derivatives at t_i ([2], [3]). Smoothnes is an important property of a free-form curve. Only smooth curve can pretend to be aesthetically pleasant.

11° *Tensor product surfaces.* The B-spline curve can be used to form a surface as a mosaic of rectangular patches. This technique is known as a tensor product of curves. Also, the B-spline surface can be built up by the triangular patches [3].

We shall end this section with a matrix form of (6) which we need in the next section

$$(\mathbf{S}_m\mathbf{P})(t) = \mathbf{b}_m^T(t)\mathbf{P}, \quad 0 \le t \le 1.$$

2. Integral variation diminishing splines and related curve model

For fixed k, the spline $S_m f$, given by (1), converges towards f for any continuous f. In [8], Müller has extended Schoenberg's approximation method, so to be applicable on any function $f \in L_p[0,1],\ 1 \le p \le \infty$, by replacing $f(\xi_i^k)$ in (1) with its integral mean over the interval $I_i = (\xi_i^{k+1}, \xi_{i+1}^{k+1})$

$$\mu_i f = \frac{1}{\Delta_i^{k+1}} \int_{\xi_i^{k+1}}^{\xi_{i+1}^{k+1}} f(u)\, du, \qquad i = 0, \dots, m, \tag{13}$$

where $\Delta_i^{k+1} = \xi_{i+1}^{k+1} - \xi_i^{k+1}$. It is easy to see that $\xi_i^k \in I_i$. Namely, by (2) we have

$$\xi_i^{k+1} = \frac{t_{i-k} + \ldots + t_i}{k+1}, \qquad i = 0, \dots, m+1,$$

so that

$$\delta_i^r = \xi_{i+1}^{k+1} - \xi_i^k = \frac{t_{i+1} - \xi_i^k}{k+1}, \quad \delta_i^l = \xi_i^k - \xi_i^{k+1} = \frac{\xi_i^k - t_{i-k}}{k+1}, \tag{14}$$

which, due to the set of obvious inequalities $t_{i-k} < \xi_i^k < t_{i+1},\ i = 0, \dots, m$, yields

$$\xi_i^{k+1} < \xi_i^k < \xi_{i+1}^{k+1}, \quad i = 0, \dots, m.$$

This means that the node ξ_i^k divide the interval I_i into two subintervals, the lengths of which are δ_i^l (left) and δ_i^r (right).

In this manner we get the Schoenberg integral spline operator [8]

$$(T_m f)(t) = \sum_{i=0}^{m} \mu_i f\, B_i^k(t), \quad 0 \le t \le 1, \tag{15}$$

which is defined for every $f \in L_p[0,1]$. Since the operator T_m has the same basis function as S_m, it shares a great deal of good properties with S_m.

We saw in the preceding section that the operator S_m generates the curve model $\mathbf{S}_m\mathbf{P}$ for a given control polygon $\mathbf{P}$. Can we use the operator T_m, given by (15), to produce any curve model? What properties will it have?

To construct the curve model $\mathbf{T}_m\mathbf{P}$ generated by T_m we use the following technique. First, we note that the value of the parameter $t \in [0,1]$ corresponding to the control point P_i is $t = \xi_i^k$ ([3]). So, the vector-valued parametric equation of the control polygon $\mathbf{P}_0\mathbf{P}_1 \ldots \mathbf{P}_m$, with $\mathbf{P}_i = (x_1^i, \ldots, x_d^i)^T \in \mathbb{R}^d$, is $\mathbf{r}(t) = \Phi(t)$, where

$$\Phi(t) = (\varphi_1(t), \ldots, \varphi_d(t))^T, \quad 0 \le t \le 1,$$

and $\varphi_j(t)$ is a piecewise-linear interpolant at the points $(\xi_\nu^k, x_j^\nu)_{\nu=0}^m$ and $j = 1, \ldots, d$. Now, it is obvious that

$$(\mathbf{S}_m\Phi)(t) = (\mathbf{S}_m\mathbf{P})(t), \quad 0 \le t \le 1,$$

i.e.,

$$S_m\varphi_j = (\mathbf{S}_m\mathbf{P})_j, \quad j = 1, \ldots, d, \tag{16}$$

where $(\mathbf{a})_j$ denotes the j-th coordinate of a vector $\mathbf{a}$. The model $\mathbf{T}_m\mathbf{P}$ is proposed to satisfy the condition analogous to (16), i.e.,

$$(\mathbf{T}_m\mathbf{P})_j = T_m\varphi_j, \quad j = 1, \ldots, d. \tag{17}$$

The direct calculation gives

$$\mu_i\varphi_j = \frac{\int_{\xi_i^{k+1}}^{\xi_{i+1}^{k+1}} \varphi_j(u)\, du}{\xi_{i+1}^{k+1} - \xi_i^{k+1}} = \begin{cases} \beta_0 x_j^0 + \gamma_0 x_j^1, \\ \alpha_i x_j^{i-1} + \beta_i x_j^i + \gamma_i x_j^{i+1}, \quad i = 1, \ldots, m-1, \\ \alpha_m x_j^{m-1} + \beta_m x_j^m, \end{cases} \tag{18}$$

where

$$\begin{cases} \alpha_0 = 0, \quad \alpha_i = \frac{(\delta_i^l)^2}{2\Delta_{i-1}^k \Delta_i^{k+1}}, \quad i = 1, \ldots, m; \\ \gamma_i \ = \frac{(\delta_i^r)^2}{2\Delta_i^k \Delta_i^{k+1}}, \quad i = 0, \ldots, m-1, \ \gamma_m = 0; \\ \beta_i \ = 1 - \alpha_i - \gamma_i, \quad i = 0, \ldots, m, \end{cases} \tag{19}$$

where $\Delta_i^k = \xi_{i+1}^k - \xi_i^k$ and δ^l, δ^r are given by (14). Therefore, (17) and (18) yields $(\mathbf{T}_m\mathbf{P})_j = \sum_{i=0}^m B_i^k(\cdot)(M\mathbf{P})_j$, or simply

$$(\mathbf{T}_m\mathbf{P})(t) = \mathbf{b}_m^T(t)(M\mathbf{P}), \quad 0 \le t \le 1, \tag{20}$$

where M is an $(m+1) \times (m+1)$ three-diagonal transformation matrix that depends on the knot vector $\mathbf{t}$,

$$M = M(\mathbf{t}) = \begin{bmatrix} \beta_0 & \gamma_0 & & & \mathrm{O} \\ \alpha_1 & \beta_1 & \gamma_1 & & \\ & \alpha_2 & \beta_2 & \ddots & \\ & & \ddots & \ddots & \gamma_{m-1} \\ \mathrm{O} & & & \alpha_m & \beta_m \end{bmatrix}, \tag{21}$$

and with the entries given by (19). In this manner, the curve model $\mathbf{T}_m\mathbf{P}$ can be regarded as the B-spline curve model $\mathbf{S}_m\mathbf{Q}$ produced by the new control polygon $\mathbf{Q}$ that gives rise by transforming $\mathbf{P}$. This transformation furnishes via the transformation matrix M, i.e., in the *global* way

$$\mathbf{Q} = M\mathbf{P}, \tag{22}$$

so that $\mathbf{T}_m\mathbf{P} = \mathbf{S}_m\mathbf{Q}$. We shall say that $\mathbf{T}_m\mathbf{P}$ is an integral mean modification of the source free-form curve model $\mathbf{S}_m\mathbf{P}$.

In the papers [4] and [5] Goldman studied the properties of the curve model that obtains from some beginning model by the global transformation of type (22). Our terminology and notations intend to follow these references. The following Lemma describes main properties of the matrix M given by (21).

Lemma 1 (Markov chain property)**.** *For an arbitrary knot vector* $\mathbf{t}$*, the transformation matrix* $M(\mathbf{t}) = (M_{ij}),\ i,j = 0,\dots,m$*, satisfies*

$$M_{ij} \geq 0, \quad i,j = 0,\dots,m, \tag{23}$$

and

$$\sum_j M_{ij} = 1, \quad j = 0,\dots,m. \tag{24}$$

Proof. Due to monotonicity of nodes (5), we have $\Delta_i^k > 0$, $\Delta_i^{k+1} > 0$ for all i, so according to (17), $\alpha_i \geq 0,\ \gamma_i \geq 0,\ i = 0,\dots,m$. Further, note that $\Delta_i^{k+1} = \delta_i^l + \delta_i^r$ and $\delta_i^l \leq \Delta_{i-1}^k,\ \delta_i^r \leq \Delta_i^k$, as the consequence of $\xi_i^k \in I_i$. This gives,

$$(\delta_i^l)^2 \leq \Delta_i^{k+1}\Delta_{i-1}^k, \qquad (\delta_i^r)^2 \leq \Delta_i^{k+1}\Delta_i^k\ ,$$

and, consequently,

$$\Delta_{i-1}^k\left[\Delta_i^{k+1}\Delta_i^k - (\delta_i^r)^2\right] + \Delta_i^k\left[\Delta_i^{k+1}\Delta_{i-1}^k - (\delta_i^l)^2\right] \geq 0.$$

Dividing both sides of the last inequality by $\Delta_i^{k+1}\Delta_{i-1}^k\Delta_i^k > 0$, yields $\beta_i \geq 0$. So, (23) is valid. From (19) we have $\sum_j M_{ij} = \alpha_j + \beta_j + \gamma_j = 1,\ j = 0,\dots,m.$ □

The square matrix with properties (23) and (24) is a Markov chain. For more details concerning the role of Markov chain transforms in the curve modelling see [4] and [5].

Lemma 2. *Let the vector of knots* $\mathbf{t}$ *is symmetric with respect to the point* $t = 1/2$*, i.e.,*

$$t_i + t_{n-i+1} = 1, \quad i = 1,\dots,n. \tag{25}$$

Then the matrix $M = M(\mathbf{t})$ *is central symmetric, namely*

$$M_{m-i,m-j} = M_{ij}, \quad i,j = 0,\dots,m. \tag{26}$$

Proof. Since $[M_{ij}]$ is three-diagonal it is enough to prove that

$$\alpha_i = \gamma_{m-i}, \quad \beta_i = \beta_{m-i}, \qquad i = 0, \dots, m. \tag{27}$$

First, note that the symmetry of knot vector $\mathbf{t}$, described by (25), implies the symmetry of nodes. Namely,

$$\begin{aligned} \xi_i^k &= \frac{t_{i-k+1} + \dots + t_i}{k} = \frac{1 - t_{n+k-i} - \dots - t_{n+1-i}}{k} \\ &= \frac{k - t_{n+k-i} - \dots - t_{(n+k)-k-i+1}}{k}, \end{aligned}$$

or, if we use $m = n + k$,

$$\xi_i^k = 1 - \frac{t_{m-i-k+1} + \dots + t_{m-i}}{k} = 1 - \xi_{m-i}^k,$$

for $i = 0, \dots, m$. In the similar manner we get $\xi_i^{k+1} = 1 - \xi_{m+1-i}^{k+1}$, $i = 0, \dots, m$. Now, we have

$$\delta_i^l = \xi_i^k - \xi_i^{k+1} = \xi_{m-i+1}^{k+1} - \xi_{m-i}^k = \delta_{m-i}^r, \quad i = 0, \dots, m,$$

$$\Delta_i^{k+1} = \xi_{i+1}^{k+1} - \xi_i^{k+1} = \xi_{m-i+1}^{k+1} - \xi_{m-1}^{k+1} = \Delta_{m-i}^{k+1}, \qquad i = 1, \dots, m,$$

and

$$\Delta_{i-1}^k = \xi_i^k - \xi_{i-1}^k = \Delta_{m-i}^k, \qquad i = 1, \dots, m,$$

which yield

$$\alpha_i = \frac{(\delta_i^l)^2}{2\Delta_{i-1}^k \Delta_i^{k+1}} = \frac{(\delta_{m-i}^r)^2}{2\Delta_{m-i}^k \Delta_{m-i}^{k+1}} = \gamma_{m-i}, \qquad i = 1, \dots, m,$$

and, if we use the fact that $\alpha_0 = \gamma_m$, we obtain the first equality in (27). The second one is the simple consequence of the relation $\beta_i = 1 - \alpha_i - \gamma_i$ and the equality just proved. □

Lemma 3. *The transformation matrix $M(\mathbf{t})$ is a Decartes matrix for any knot vector $\mathbf{t}$.*

Proof. Recall that $[M_{ij}]$ is a Decartes matrix if

i) for any ν the minors $M\begin{pmatrix} j_0 & \dots & j_\nu \\ k_0 & \dots & k_\nu \end{pmatrix}$ are of the same sign;

ii) for each $j_0 \dots j_\nu$ $(k_0 \dots k_\nu)$ there exists $k_0 \dots k_\nu$ $(j_0 \dots j_\nu)$ so that

$$M\begin{pmatrix} j_0 & \dots & j_\nu \\ k_0 & \dots & k_\nu \end{pmatrix} \neq 0,$$

where for $j_0 < j_1 < \ldots < j_\nu,\ k_0 < k_1 < \ldots < k_\nu$,

$$M\begin{pmatrix} j_0 & \ldots & j_\nu \\ k_0 & \ldots & k_\nu \end{pmatrix} = \begin{vmatrix} M_{j_0k_0} & \ldots & M_{j_0k_\nu} \\ \vdots & & \\ M_{j_\nu k_0} & \ldots & M_{j_\nu k_\nu} \end{vmatrix}.$$

Since $M(t)$ is a three-diagonal matrix with nonnegative entries, it is enough to prove that it has a dominant diagonal, i.e., that

$$\beta_i \ > \ \alpha_i \ + \ \gamma_i, \quad i = 0, \ldots, m. \tag{28}$$

In virtue of $\beta_i = 1 - \alpha_i - \gamma_i$, (28) becomes

$$\alpha_i \ + \ \gamma_i \ < \ \frac{1}{2}, \quad i = 0, \ldots, m. \tag{29}$$

To prove (29), let us note that $\delta_i^l < \Delta_{i-1}^k$ and $\delta_i^r < \Delta_i^k$ are valid and imply

$$\frac{(\delta_i^l)^2}{\Delta_{i-1}^k} < \delta_i^l \qquad \text{and} \qquad \frac{(\delta_i^r)^2}{\Delta_i^k} < \delta_i^r,$$

wherefrom we get $(\delta_i^l)^2/\Delta_{i-1}^k + (\delta_i^r)^2/\Delta_i^k \ < \ \delta_i^l + \delta_i^r$. Taking into account that $\delta_i^l + \delta_i^r = \Delta_i^{k+1}$, we have

$$\frac{(\delta_i^l)^2}{\Delta_{i-1}^k\Delta_i^{k+1}} + \frac{(\delta_i^r)^2}{\Delta_i^k\Delta_i^{k+1}} \ < \ 1,$$

which is equivalent to (29) for $i = 1, \ldots, m-1$. For $i = 0$ and $i = m$ ones get

$$\gamma_0 = \frac{(\delta_0^r)^2}{2\Delta_0^k\Delta_0^{k+1}} = \frac{(\delta_0^r)^2}{2\Delta_0^k\delta_0^r} = \frac{\delta_0^r}{2\Delta_0^k} = \frac{1}{2}\cdot\frac{\xi_1^{k+1} - \xi_0^k}{\xi_1^k - \xi_0^k} < \frac{1}{2},$$

$$\alpha_m = \frac{(\delta_m^l)^2}{2\Delta_{m-1}^k\Delta_m^{k+1}} = \frac{(\delta_m^l)^2}{2\Delta_{m-1}^k\delta_m^l} = \frac{\delta_m^l}{2\Delta_{m-1}^k} = \frac{1}{2}\cdot\frac{\xi_m^k - \xi_m^{k+1}}{\xi_m^k - \xi_{m-1}^k} < \frac{1}{2},$$

Therefore (29) is valid for $i = 0, \ldots, m$, i.e., (28) holds.

By positivity of α_i, β_i and γ_i (Lemma 1) we have all 1×1 minors of $M(\mathbf{t})$ positive. Also, each 2×2 "diagonal" minor

$$\begin{vmatrix} \beta_i & \gamma_i \\ \alpha_{i+1} & \beta_{i+1} \end{vmatrix}, \quad i = 1, \ldots, m-1,$$

is positive. This is the consequence of (28). Namely, from (28) we get $\beta_i > \gamma_i$ and $\beta_{i+1} > \alpha_{i+1}$ and $\beta_i\beta_{i+1} > \alpha_{i+1}\gamma_i,\ i = 1, \ldots, m-1$. Positivity of subdeterminant of higher order is provided by the diagonal dominance property (28). $\square$

Corollary 1. $\det M(\mathbf{t}) > 0$, *for any* $\mathbf{t}$.

Now, we are ready to prove the following result:

Theorem 1. *The curve model* $\mathbf{T}_m\mathbf{P}$ *given by* (20) *has the following properties:*

(1) *it is invariant under affine transformations of the coordinate system,*
(2) $\mathbf{T}_m\mathbf{P} \subset$ *convex hull* $(\mathbf{P})$,
(3) *it is symmetric,*
(4) $\mathbf{T}_m\mathbf{P}$ *is variation diminishing,*
(5) $\mathbf{T}_m\mathbf{P} = \mathbf{T}_m\mathbf{R} \Leftrightarrow \mathbf{P} = \mathbf{R}$.

Proof. Use the Lemmata 1–3 and compare Table IV in [4].

Remark 1. Note that $\mathbf{T}_m\mathbf{P}$ does not interpolate end points $\mathbf{P}_0$ and $\mathbf{P}_m$ Indeed, we have

$$(\mathbf{T}_m\mathbf{P})(0) = \beta_0\mathbf{P}_0 + \gamma_0\mathbf{P}_1 = (1-\gamma_0)\mathbf{P}_0 + \gamma_0\mathbf{P}_1,$$
$$(\mathbf{T}_m\mathbf{P})(1) = \alpha_m\mathbf{P}_{m-1} + \beta_m\mathbf{P}_m = \alpha_m\mathbf{P}_{m-1} + (1-\alpha_m)\mathbf{P}_m.$$

Thus, $\mathbf{T}_m\mathbf{P}$ interpolates $\mathbf{P}_0$ and $\mathbf{P}_m$ if and only if $\mathbf{P}_0 = \mathbf{P}_1$ and $\mathbf{P}_{m-1} = \mathbf{P}_m$. This technique is known as the doubling of end vertices and is commonly used (see [1]).

Remark 2. The model $\mathbf{T}_m\mathbf{P}$ reproduces straight line but in general it does not preserve the disposition of the control points along this line. Reproducing of straight lines is the consequence of variation diminishing property.

Remark 3. The calculation of the point $(\mathbf{T}_m\mathbf{P})(t)$ for any t is not much complicated then calculation of the basic model (see property 9°). First, we evaluate $\mathbf{Q}$ by (20) and then apply de Boor algorithm (12).

3. λ-family of curves

Except for changing of control polygon $\mathbf{P}$ we can alter the shape of the curve $\mathbf{S}_m\mathbf{P}$, and therefore the curve $\mathbf{T}_m\mathbf{P}$, by changing the knot vector $\mathbf{t}$. The main disadvantage of doing that through changing $\mathbf{P}$ is the local character of this action [4]. On the other hand, changing the curve shape via changing the knot vector can not be controlled intuitively [1]. The most convenient method is to introduce a parameter (or more parameters) into the curve model so that it influences the form of the curve in a predictable and easy way. Further, the parameter being involved must not disturb all good properties of the beginning curve, and finally, the computation of the altered curve should not be too much complicated then calculation of the source model. Such models already exist. For example, various splines under tension, Polya curves, β-splines (see [1]) and so on.

Here, we use the natural and easy way to introduce the parameter into the integral Schoenberg operator T_m, given by (15). Namely, we can make the integral mean (13) dependent on the length of the interval of integration. In this sense we define

$$\mu_i^\lambda f = \frac{1}{\eta_i - \zeta_i}\int_{\zeta_i}^{\eta_i} f(u)\,du, \qquad i = 0,\dots,m, \tag{30}$$

where $\zeta_i = (1-\lambda)\xi_i^k + \lambda\xi_i^{k+1}$, $\eta_i = (1-\lambda)\xi_i^k + \lambda\xi_{i+1}^{k+1}$, and $\lambda \in (0,1]$ is a parameter. If we accept the usual convention that for any integrable function

$$\lim_{\epsilon\to 0+} \int_{a-\epsilon}^{a+\epsilon} f(u)\,du = f(a), \tag{31}$$

we can define a λ-family of integral Schoenberg splines

$$(T_m^\lambda f)(t) = \sum_{i=0}^{m} \mu_i^\lambda f\; B_i^k(t), \quad 0 \le t \le 1, \tag{32}$$

with the parameter λ runs over $[0,1]$. Then, by (32) we have $\mu_i^0 f = f(\xi_i^k)$ and $\mu_i^1 f = \mu_i f$, where $\mu_i f$ is given by (13). In such manner, we have

$$T_m^0 f = S_m f, \quad T_m^1 f = T_m f.$$

The transformation matrix now depends on λ if $\mathbf{t}$ is fixed. Some calculation will give

$$M(\lambda) = M(\mathbf{t},\lambda) = \begin{bmatrix} \beta_0^\lambda & \gamma_0^\lambda & & & \mathrm{O} \\ \alpha_1^\lambda & \beta_1^\lambda & \gamma_1^\lambda & & \\ & \alpha_2^\lambda & \beta_2^\lambda & \ddots & \\ & & \ddots & \ddots & \gamma_{m-1}^\lambda \\ \mathrm{O} & & & \alpha_m^\lambda & \beta_m^\lambda \end{bmatrix}, \tag{33}$$

where

$$\alpha_i^\lambda = \lambda\alpha_i, \quad \gamma_i^\lambda = \lambda\gamma_i, \quad \beta_i^\lambda = 1 - \lambda(\alpha_i + \gamma_i), \quad i = 0, \dots, m, \tag{34}$$

where α_i and γ_i are given by (19). The related family of curve models is given by

$$(\mathbf{T}_m^\lambda \mathbf{P})(t) = \mathbf{b}_m^T(t)(M(\lambda)\mathbf{P}), \quad 0 \le t \le 1,\ 0 \le \lambda \le 1. \tag{35}$$

It is clear that

$$\mathbf{T}_m^0\mathbf{P} = \mathbf{S}_m\mathbf{P}, \quad \mathbf{T}_m^1\mathbf{P} = \mathbf{T}_m\mathbf{P},$$

i.e., for extreme values of λ we get B-spline curve or its integral modification.

The modified control polygon will depend on λ, i.e.,

$$\mathbf{Q}^\lambda = M(\lambda)\mathbf{P}. \tag{36}$$

The transformation matrix, being a λ-matrix of first degree, permits the following decomposition

$$M(\lambda) = (1-\lambda)I + \lambda M(1), \quad 0 \le \lambda \le 1, \tag{37}$$

as the consequence of the obvious relation $\beta_i^\lambda = 1 - \lambda + \lambda\beta_i,\ i = 0, \ldots, m$, where β_i is given by (19). Note that $M(0) = I$, an identity matrix. From (36) and (37), we get

$$\mathbf{Q}^\lambda = (1-\lambda)\mathbf{P} + \lambda M(1)\mathbf{P},$$

or, if we recall that $\mathbf{Q}^1 = \mathbf{Q} = M(1)\mathbf{P}$ and $\mathbf{P} = M(0)\mathbf{P} = \mathbf{Q}^0$ we have

$$\mathbf{Q}^\lambda = (1-\lambda)\mathbf{Q}^0 + \lambda\mathbf{Q}^1, \quad \lambda \in [0,1]. \tag{38}$$

The equation (38) can be interpreted geometrically. Let us restrict ourselves on i-th coordinate of the vector $\mathbf{Q}$. From (22) we have $Q_i = \alpha_i P_{i-1} + \beta_i P_i + \gamma_i P_{i+1}$, with $\alpha_i, \beta_i, \gamma_i \geq 0$ and $\alpha_i + \beta_i + \gamma_i = 1$. Thus, α_i, β_i and γ_i are the barycentric coordinates of the point Q_i^1 with respect to the triangle $P_{i-1}P_iP_{i+1}$. According to (38), the point Q_i^λ divides the segment $Q_i^0 Q_i^1$ in the ratio $\lambda : (1-\lambda)$ (see Fig. 1).

Fig. 1

Therefore, $\{\mathbf{Q}^\lambda\}_{\lambda\in[0,1]}$ is an one-parameter family of control polygons producing a family of B-spline curves. Each curve can be considered as a blend of two extreme curves in the family, the B-spline curve and its integral modification

$$\mathbf{T}_m^\lambda\mathbf{P} = \mathbf{b}_m(t)[(1-\lambda)\mathbf{P} + \lambda\mathbf{Q}^1] = (1-\lambda)(\mathbf{S}_m\mathbf{P}) + \lambda(\mathbf{T}_m\mathbf{P}). \tag{39}$$

The following theorem describes the properties of the model $\mathbf{T}_m^\lambda\mathbf{P}$.

Theorem 2. *For all* $\lambda \in [0,1]$ *and any knot vector* $\mathbf{t}$, $\mathbf{T}_m^\lambda\mathbf{P}$ *preserves all properties of the curve model* $\mathbf{T}_m\mathbf{P}$ *from Theorem* 1.

Proof. The affine invariance and the convex hull property follow from the fact that $M(\mathbf{t}, \lambda)$ is a Markov chain for all $\lambda \in [0,1]$ and $\mathbf{t}$. This is the consequence of

(37) and the fact that $I = [\delta_{ij}]$ and $M(\mathbf{t}, 1) = M = [M_{ij}]$ given by (21) are Markov chains (Lemma 1). Namely, if we put $M(\mathbf{t}, \lambda) = [M_{ij}^\lambda]$, we shall have

$$\sum_j M_{ij}^\lambda = (1-\lambda)\sum_j \delta_{ij} + \lambda \sum_j M_{ij} = (1-\lambda) + \lambda = 1,$$

and

$$\delta_{ij} \geq 0,\ M_{ij} \geq 0 \Rightarrow M_{ij}^\lambda = (1-\lambda)\delta_{ij} + \lambda M_{ij} \geq 0.$$

The symmetric property follows from Lemma 2 and the fact that $\delta_{m-i,m-j} = \delta_{ij}$ wherefrom, for every λ, $\mathbf{t}$

$$M_{m-i,m-j}^\lambda = (1-\lambda)\delta_{m-i,m-j} + \lambda M_{m-i,m-j} = (1-\lambda)\delta_{ij} + \lambda M_{ij} = M_{ij}^\lambda.$$

To prove the variation diminishing property, note that Lemma 3 asserts that $M(\mathbf{t})$ is a Decartes matrix for any $\mathbf{t}$. The key relation that justifies this assertion is the diagonal dominance (28). From (33) and (37) we have $\beta_i^\lambda = 1 - \lambda + \lambda\beta_i > (1-\lambda) + \lambda(\alpha_i + \gamma_i)$ and so, for any $\lambda \in [0,1]$, we get $\beta_i^\lambda > \lambda(\alpha_i+\gamma_i) = \alpha_i^\lambda + \gamma_i^\lambda$. Thus, the matrix $M(\mathbf{t}, \lambda)$ is a three-diagonal with a dominant diagonal for all $\lambda \in [0,1]$ and any knot vector $\mathbf{t}$. According to [4], $\mathbf{T}_m^\lambda \mathbf{P}$ diminishes the variation of the control polygon $\mathbf{P}$. This also means that transformed control polygon $\mathbf{Q}^\lambda$ oscillates less than $\mathbf{P}$ for any $\lambda \in (0,1]$.

From the same argument the uniqueness property follows: $\det M(\mathbf{t}, \lambda) \neq 0$, implies that

$$\mathbf{T}_m^\lambda \mathbf{P} = \mathbf{T}_m^\lambda \mathbf{R} \Leftrightarrow \mathbf{P} = \mathbf{R},$$

for any $\lambda \in [0,1]$ and $\mathbf{t}$. $\square$

We call the collection of curves $\{\mathbf{T}_m^\lambda \mathbf{P}\}_{\lambda\in[0,1]}$ *a pencil* of B-spline curves defined by the control polygon $\mathbf{P}$ and denote by $\pi(\mathbf{P})$.

Remark 4. From (39) follows that $\mathbf{Q}^\lambda = (1-\lambda)\mathbf{P} + \lambda\mathbf{Q}$ (where $\mathbf{Q} = \mathbf{Q}^1$). This means that the curve $\mathbf{T}_m^\lambda \mathbf{P}$ interpolates the points

$$\mathbf{Q}_0^\lambda = (1-\lambda)\mathbf{P}_0 + \lambda\mathbf{Q}_0, \qquad \mathbf{Q}_m^\lambda = (1-\lambda)\mathbf{P}_m + \lambda\mathbf{Q}_m,$$

or

$$\mathbf{Q}_0^\lambda = (1-\lambda\gamma_0)\mathbf{P}_0 + \lambda\gamma_0\mathbf{P}_1, \qquad \mathbf{Q}_m^\lambda = \lambda\alpha_m\mathbf{P}_{m-1} + (1-\lambda\alpha_m)\mathbf{P}_m,$$

so, by doubling endpoints of the beginning control polygon $\mathbf{P}$, we provide interpolation of endpoints $\mathbf{P}_0$ and $\mathbf{P}_m$ by any curve from the pencil $\pi(\mathbf{P})$.

Remark 5. The curve $\mathbf{T}_m^\lambda \mathbf{P}$ reproduces straight line for any $\lambda \in [0,1]$ but not the arrangement of the points on it (see Remark 2).

Remark 6. The evaluation of the point $(\mathbf{T}_m^\lambda \mathbf{P})(t)$ for some fixed t, λ and $\mathbf{t}$, furnishes in the same way as $(\mathbf{T}_m^1 \mathbf{P})(t)$. First, we calculate $\mathbf{Q}^\lambda = M(\lambda)\mathbf{P}$ and then use the recursion algorithm (12).

Instead of calculating $\mathbf{Q}^\lambda$ we can calculate the new set of basis functions

$$\mathbf{d}_m^\lambda = (D_0^{m,\lambda}, \dots , D_m^{m,\lambda})^T,$$

which is determined by

$$\begin{aligned}\mathbf{T}_m^\lambda \mathbf{P} &= \mathbf{b}_m^T (M(\lambda)\mathbf{P}) = (\mathbf{b}_m^T M(\lambda))\mathbf{P}\\ &= (M^T(\lambda)\mathbf{b}_m)^T \mathbf{P} = (\mathbf{d}_m^\lambda)^T \mathbf{P},\end{aligned}$$

so

$$\mathbf{d}_m^\lambda = M^T(\lambda)\mathbf{b}_m,$$

i.e.,

$$\begin{aligned}&D_0^{m,\lambda}(t) = \beta_0^\lambda B_0^k(t) + \alpha_1^\lambda B_1^k(t),\\ &D_i^{m,\lambda}(t) = \gamma_{i-1}^\lambda B_{i-1}^k + \beta_i^\lambda B_i^k(t) + \alpha_{i+1}^\lambda B_{i+1}^k(t), \quad i = 1, \dots , m-1,\\ &D_m^{m,\lambda}(t) = \gamma_{m-1}^\lambda B_{m-1}^k(t) + \beta_m^\lambda B_m^k(t).\end{aligned}$$

Finally, we want to mention an important feature of the curve pencil $\pi(\mathbf{P})$, the existence of fixed points.

We call the point $\mathbf{F}$ *fixed point* of the pencil $\pi(\mathbf{P})$ if all curves from $\pi(\mathbf{P})$ pass through F. We saw in the Remark 4 that the end points $\mathbf{P}_0$ and $\mathbf{P}_m$ might be the fixed points if $\mathbf{P}_0 = \mathbf{P}_1$ and $\mathbf{P}_{m-1} = \mathbf{P}_m$. The following theorem takes place:

Theorem 3. *Suppose that two curves $\mathbf{T}^{\lambda_1}$ and $\mathbf{T}^{\lambda_2}$, $\lambda_1 < \lambda_2$ from $\pi(\mathbf{P})$ intersect each other at the point $\mathbf{F} = \mathbf{T}^{\lambda_1}(t_F) = \mathbf{T}^{\lambda_2}(t_F)$. Than, every curve $\mathbf{T}^\lambda \in \pi(\mathbf{P})$ passes through $\mathbf{F}$.*

Proof. First, we need the representation of $\mathbf{T}^\lambda$ via $\mathbf{T}^{\lambda_1}$ and $\mathbf{T}^{\lambda_2}$. From (39) we get

$$\mathbf{T}^{\lambda_1} = (1-\lambda_1)\mathbf{T}^0 + \lambda_1 \mathbf{T}^1, \quad \mathbf{T}^{\lambda_2} = (1-\lambda_2)\mathbf{T}^0 + \lambda_2 \mathbf{T}^1$$

and

$$\mathbf{T}^\lambda = (1-\lambda)\mathbf{T}^0 + \lambda \mathbf{T}^1.$$

Elimination of $\mathbf{T}^0$ and $\mathbf{T}^1$ from the above equation yields

$$\mathbf{T}^\lambda = \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1}\mathbf{T}^{\lambda_1} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}\mathbf{T}^{\lambda_2},$$

or

$$\mathbf{T}^\lambda = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\mathbf{T}^{\lambda_1} - \frac{\lambda_1}{\lambda_2 - \lambda_1}\mathbf{T}^{\lambda_2}\right) + \frac{\lambda}{\lambda_2 - \lambda_1}\left(\mathbf{T}^{\lambda_2} - \mathbf{T}^{\lambda_1}\right).$$

Calculation of the last expression for $t = t_F$ gives

$$\mathbf{T}^\lambda(t_F) = \frac{\lambda_2}{\lambda_2 - \lambda_1}\mathbf{F} - \frac{\lambda_1}{\lambda_2 - \lambda_1}\mathbf{F} = \mathbf{F}. \quad \square$$

Remark 7. Note that we can not change the position of the point $\mathbf{T}^\lambda(t_F)$ despite of changing λ. That is why we call $\mathbf{F}$ a fixed point.

Let the pencil $\pi(\mathbf{P})$ have $q \geq 1$ fixed points, $\mathbf{F}_1, \ldots, \mathbf{F}_q$. Knowing the number and location of these points may be very helpful to designer. The curve model $\mathbf{T}_m^\lambda \mathbf{P}$ interpolates these points. Also, note that relation (39) implies that if all control points lie in the plane, the curve $\mathbf{T}^\lambda$ lies in the plane region bordered by the line segments $\mathbf{P}_0\mathbf{Q}_0^1$, $\mathbf{P}_m\mathbf{Q}_m^1$ and two extreme members of the pencil $\pi(\mathbf{P})$, $\mathbf{T}^0$ and $\mathbf{T}^1$.

4. Examples

In the following examples we use integral modification of B-spline model with the cubic B-splines.

Example 1. Figure 2a shows a family of control polygons $\mathbf{Q}^\lambda = M(\lambda)\mathbf{P}$. The beginning polygon $\mathbf{P}$ is plotted by heavy line, and modified polygons $\mathbf{Q}^\lambda$, for $\lambda = 0(0.1)0.4$, are dotted. The corresponding family of curves $\{\mathbf{T}_m^\lambda \mathbf{P}\}$ is displayed in Fig. 2b.

Fig. 2a

Fig. 2b

Example 2. The pencil of curves with tripled endpoints is given in Fig. 3a, while Fig. 3b shows the pencil with simple endpoints. In both cases we can notice a fixed point in the middle.

Fig. 3a

Fig. 3b

Example 3. We can apply the described method also for closed curves. A few examples are displayed in Fig. 4. We can remark an interesting behavior of the shape of curves involved. Namely, for λ outside of $[0, 1]$ some unstability is appeared.

Fig. 4

Example 4. Figure 5 shows outlines of two plant leafs (*ceasalpina japonica*). The left one is modelled as a composite cubic B-spline curve (17 segments are used). The shape parameter is zero for all segments. The better form is achieved (Fig. 5, right), when we adjust these parameters on some segments.

Fig. 5

Example 5. The same effect of the shape parameter selecting for the outline of a Raphael's female head is given in Fig. 6.

Fig. 6

REFERENCES

1. BARTELS, R. H., BEATTY, J. C., BARSKY, B. A.: *An Introduction to Splines for use in Computer Graphics and Geometric Modeling.* Morgan Kaufman, Los Altos, CA, 1987.
2. DE BOOR, C.: *On Calculating with B-splines.* J. Approx. Theory **6**(1972), 50–62.
3. BÖHM, W., FARIN, G., KAHMANN, J.: *A survey of curve and surface methods in CAGD.* Comput. Aided Geom. Design **1**(1984), 1–60.
4. GOLDMAN, R. N.: *Markov chains and computer–aided geometric design: Part I -Problems and constraints.* ACM Trans. Graph. **3**(1984), no. 3, 204–222.
5. GOLDMAN, R. N.: *Markov chains and computer–aided geometric design: Part II – Examples and subdivision matrices.* ACM Trans. Graph. **4**(1985), no. 1, 12–40.
6. GOLDMAN, R. N.: *Urn models and B-splines.* Constr. Approx. **4**(1988), 265–288.
7. MARSDEN, M. J.: *An Identity for Spline Functions with Applications to Variation – Diminishing Spline Approximation.* J. Approx. Theory **3**(1970), 7–49.
8. MÜLLER, M. J.: *Degree of L_p-Approximation by Integral Schoenberg Splines.* J. Approx. Theory **21**(1977), 385–393.
9. SCHOENBERG, I. J.: *On Spline Functions.* In: "Inequalities", Symposium at Wright – Patterson Air Force Base (O. Shisha, ed.), pp. 255–291, Academic Press, New York, 1967.