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COMPLEX ORTHOGONAL POLYNOMIALS WITH THE HERMITE WEIGHT

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

In this paper we connect complex orthogonal polynomials of the Gegenbauer type on a semicircle with the orthogonal polynomials of the Hermite type $X_n(z)$. For the last of them, we give the three-term recurrence relation and their relationship with the classical Hermite polynomials. Also, we study a zero distribution of such polynomials and obtain a linear second-order differential equation for $X_n(z)$. Some applications in numerical integration are included.

1. INTRODUCTION

In 1983 during a joint visit to Henri Poincare Institute at Paris, the first author of this paper announced Professor Mitrinović an idea on orthogonal polynomials on the semicircle with a non-Hermitian complex inner product. Since he liked this idea, he was always interested about progress in that direction. Furthermore, he asked Milovanović to prepare a survey about that (see [**10**]) for his and Kečkić's monograph *"The Cauchy Method of Residues, Vol. 2 – Theory and Applications"* published by Kluwer in 1993 (and previously in Serbian by Naučna Knjiga, Belgrade 1991). Such polynomials orthogonal on the semicircle

$$\gamma = \{z \in \mathbf{C} \mid z = e^{i\theta},\ 0 \le \theta \le \pi\}$$

have been introduced by Gautschi and Milovanović [**3–4**]. The inner product is given by

$$(f,g) = \int_\gamma f(z)g(z)(iz)^{-1}\,\mathrm{d}z = \int_0^\pi f(e^{i\theta})g(e^{i\theta})\,\mathrm{d}\theta.$$

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This inner product is not Hermitian, but the corresponding (monic) orthogonal polynomials $\{\pi_k\}$ exist uniquely and satisfy a three-term recurrence relation of the form

$$\begin{aligned}\pi_{k+1}(z) &= (z - i\alpha_k)\pi_k(z) - \beta_k \pi_{k-1}(z), \qquad k = 0, 1, 2, \ldots, \\ \pi_{-1}(z) &= 0, \quad \pi_0(z) = 1.\end{aligned}$$

Notice that the inner product possesses the property $(zf, g) = (f, zg)$.

The general case of complex polynomials orthogonal with respect to a *complex weight function* was considered by Gautschi, Landau and Milovanović [**2**]. Namely, let $w : (-1, 1) \mapsto \mathbf{R}_+$ be a weight function which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbf{C} \mid |z| < 1, \operatorname{Im} z > 0\}$, and

$$(f, g) = \int_\gamma f(z)g(z)w(z)(iz)^{-1}\,\mathrm{d}z = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,\mathrm{d}\theta.$$

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$(\pi_k, \pi_m) = 0 \quad \text{for} \quad k \neq m \qquad \text{and} \qquad (\pi_k, \pi_m) > 0 \quad \text{for} \quad k = m \qquad (k, m \in \mathbf{N}_0),$$

where we assume that π_k is monic of degree k. The existence of the orthogonal polynomials $\{\pi_k\}$ can be established assuming only that

$$\operatorname{Re}(1, 1) = \operatorname{Re} \int_0^\pi w(e^{i\theta})\,\mathrm{d}\theta \neq 0.$$

Some applications of such polynomials, especially with the Gegenbauer weight, were given in [**9**] (see also [**8**] and [**11**–**14**).

In this paper we consider an orthogonality on a growing semicircle $\gamma_{(\sqrt{\lambda})}$ with radius $\lambda > 1$, and especially a limit case when λ tends to infinity (Sections 2 and 3). In Section 4 we discuss applications in numerical integration.

2. COMPLEX POLYNOMIALS ORTHOGONAL ON A GROWING SEMICIRCLE

Let $w_\lambda(z)$ be the Gegenbauer weight function,

$$w_\lambda(z) = (1 - z^2)^{\lambda - 1/2} \qquad (\lambda > 0),$$

and D_+ the half disc $D_+ = \big\{z \in \mathbf{C} \mid |z| \leq 1,\ \operatorname{Im} z > 0\big\}$, bounded by the semicircle γ and the interval $[-1, 1]$.

In [**2**] it was defined an inner product on the semicircle γ with respect to the weight function $w(z)$ which is not Hermitian. Namely,

$$(f, g)_\lambda = \int_\gamma f(z)g(z)w_\lambda(z)(iz)^{-1}\,\mathrm{d}z.$$

But, it was proved that there exists an unique sequence of monic polynomials $\{\pi_n^\lambda(z)\}$ such that is

$$(\pi_m^\lambda, \pi_n^\lambda) = \delta_{mn}\|\pi_n^\lambda\|_\lambda^2 \qquad (m, n = 0, 1, \ldots).$$

A connection with the monic GEGENBAUER polynomials $\widehat{C}_n^\lambda(z)$ was also found,

$$\pi_n^\lambda(z) = \widehat{C}_n^\lambda(z) - i\theta_{n-1}^\lambda \widehat{C}_{n-1}^\lambda(z),$$

where

$$\theta_{-1}^\lambda = \pi, \quad \theta_0^\lambda = \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\,\Gamma(\lambda+1)}, \quad \theta_k^\lambda = \frac{1}{\lambda+k}\frac{\Gamma((k+2)/2)\Gamma(\lambda+(k+1)/2)}{\Gamma((k+1)/2)\Gamma(\lambda+k/2)}.$$

The norm of such polynomials is given by

$$\|\pi_n^\lambda\|_\lambda^2 = \pi(\theta_0^\lambda\theta_1^\lambda\cdots\theta_{n-1}^\lambda)^2 > 0 \qquad (n \ge 1).$$

Introduce now a new variable u by $u = z\sqrt{\lambda}$. Then the semicircle γ becomes a new one (see Fig. 2.1)

$$\gamma_{(\sqrt{\lambda})} = \{u \in \mathbf{C} \mid |u| \le \sqrt{\lambda},\ \mathrm{Im}\, u > 0\}.$$

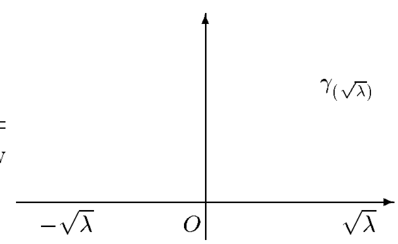

Fig. 2.1

The orthogonality condition becomes

$$\int_{\gamma_{(\sqrt{\lambda})}} \pi_m^\lambda(u/\sqrt{\lambda})\pi_n^\lambda(u/\sqrt{\lambda})\left(1-u^2/\lambda\right)^{\lambda-1/2}\frac{\mathrm{d}u}{iu} = \delta_{mn}\|\pi_n^\lambda\|_\lambda^2.$$

We can define a new sequence of polynomials $\{X_n^\lambda(u)\}$ by

$$X_n^\lambda(u) = \sqrt{\lambda^n}\,\pi_n^\lambda(u/\sqrt{\lambda}),$$

which are orthogonal with respect to the weight

$$\widetilde{w}_\lambda(u) = \left(1 - \frac{u^2}{\lambda}\right)^{\lambda-1/2}$$

on $\gamma_{(\sqrt{\lambda})}$. Thus,

$$\langle X_m^\lambda, X_n^\lambda\rangle_\lambda = \int_{\gamma_{(\sqrt{\lambda})}} X_m^\lambda(u)X_n^\lambda(u)\widetilde{w}_\lambda(u)\,\frac{\mathrm{d}u}{iu} = \delta_{mn}\|X_n^\lambda\|^2. \tag{2.1}$$

Putting $P_n^\lambda(t) = \sqrt{\lambda^n}\,\widehat{C}_n^\lambda(t/\sqrt{\lambda})$ $(n \in \mathbf{N}_0)$, we see that this sequence satisfies the following three-term recurrence relation

$$P_{k+1}^\lambda(t) = tP_k^\lambda(t) - b_k^\lambda P_{k-1}^\lambda(t), \quad P_{-1}^\lambda(t) = 0, \quad P_0^\lambda(t) = 1,$$

where

$$b_0^\lambda = \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} \left(1 - \frac{t^2}{\lambda}\right)^{\lambda-1/2} \mathbf{d}t = \sqrt{\lambda\pi}\,\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}$$

and

$$b_k^\lambda = \frac{k\lambda(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}, \qquad k \ge 1.$$

These polynomials are orthogonal on $(-\sqrt{\lambda}, \sqrt{\lambda})$ with respect to $\widetilde{w}_\lambda(t)$. Then, we yield

$$X_n^\lambda(u) = P_n^\lambda(u) - i\vartheta_{n-1}^\lambda P_{n-1}^\lambda(u), \tag{2.2}$$

where

$$\vartheta_{-1}^\lambda = \theta_{-1}^\lambda, \qquad \vartheta_{n-1}^\lambda = \theta_{n-1}^\lambda\sqrt{\lambda} \qquad (n \ge 1).$$

Similarly as in [**2**–**3**] we can prove that they satisfy three-term recurrence relation and a second order differential equation. Their zeros lie in the region bounded by $\gamma_{(\sqrt{\lambda})}$ and $[-\sqrt{\lambda}, \sqrt{\lambda}]$.

Using Cauchy's theorem the inner product $\langle \cdot\,,\cdot\rangle_\lambda$ can be expressed in the form

$$\langle f, g\rangle_\lambda = \pi f(0)g(0) + i\ \text{v.p.}\int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} f(x)g(x)\frac{w_\lambda(x)}{x}\,\mathrm{d}x. \tag{2.3}$$

3. COMPLEX POLYNOMIALS ORTHOGONAL WITH THE HERMITE WEIGHT

It is known (see Szegő [**17**, p. 107]) that is

$$\lim_{\lambda\to+\infty} P_k^\lambda(t) = \widehat{H}_k(t). \tag{3.1}$$

where $\widehat{H}_k(t)$, $k = 0, 1, \ldots$, are the monic Hermite polynomials which satisfy

$$\widehat{H}_{k+1}(t) = t\widehat{H}_k(t) - b_k\widehat{H}_{k-1}(t), \quad \widehat{H}_{-1}(t) = 0, \quad \widehat{H}_0(t) = 1,$$

with $b_0 = \sqrt{\pi}$ and $b_k = k/2$ $(k \ge 1)$. Defining $\vartheta_k = \lim\limits_{\lambda\to+\infty} \theta_k^{(\lambda)}$, we find

$$\vartheta_{-1} = \pi, \qquad \vartheta_n = \frac{\Gamma((n+2)/2)}{\Gamma((n+1)/2)} \qquad (n = 0, 1, \ldots).$$

Knowing that (cf. [**16**])

$$\lim_{x\to+\infty} \frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1 \qquad (a > 0),$$

we conclude that $\lim\limits_{n\to+\infty} \vartheta_n = \lim\limits_{n\to+\infty} \sqrt{(n+1)/2} = +\infty$.

Now, we can define a sequence of complex polynomials $\{X_n(z)\}$ by

$$X_n(z) = \lim_{\lambda\to+\infty} X_n^\lambda(z).$$

Using (3.1) and (2.2) we conclude that these polynomials satisfy the following recurrence relation

$$X_n(z) = \widehat{H}_n(z) - i\vartheta_{n-1}\widehat{H}_{n-1}(z), \quad X_{-1}(z) = 0, \quad X_0(z) = 1.$$

Recently, Notaris [**15**, Lemma 2.1] proved that

$$\lim_{\lambda\to+\infty} \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} q_k^{(\lambda)}(t)\widetilde{w}_\lambda(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} q_k(t)e^{-t^2}\,\mathrm{d}t$$

for any monic polynomial $q_m^{(\lambda)}$ of degree m, whose coefficients depend on a parameter λ, such that $\lim\limits_{\lambda\to+\infty} q_m^{(\lambda)}(t) = q_m(t)$, where q_m is a monic polynomial of degree m. Therefore, from (2.1) and (2.3) we obtain the inner product

$$\langle f, g\rangle = \pi f(0)g(0) + i\ \text{v.p.}\int_{-\infty}^{+\infty} f(x)g(x)\frac{e^{-x^2}}{x}\,\mathrm{d}x. \tag{3.2}$$

Theorem 3.1 *The sequence of polynomials* $\{X_n(z)\}$ *is orthogonal with respect the inner product* (3.2), *i.e.,*

$$\langle X_m, X_n\rangle = \delta_{mn}\|X_n\|^2 \qquad (m, n = 0, 1, \ldots).$$

Remark 3.1 The sequence $\{X_n(z)\}$ can be introduced using two functionals (see P. Maroni [**7**]): L for a real polynomial sequence and u for the corresponding complex polynomial sequence. If L is the Hermite functional, then the functional

$$u = \delta_c + \Lambda(x-c)^{-1}L$$

generates the sequence $\{X_n(z)\}$.

Theorem 3.2. *The sequence* $\{X_n(z)\}$ *satisfies the three-term recurrence relation*

$$X_{n+1}(z) = (z - i\alpha_n)X_n(z) - \beta_n X_{n-1}(z), \qquad X_{-1}(z) = 0, \quad X_0(z) = 1,$$

where

$$\alpha_0 = \vartheta_0, \quad \beta_0 = \vartheta_{-1}, \quad \alpha_n = \vartheta_n - \vartheta_{n-1}, \quad \beta_n = \vartheta_{n-1}^2 \quad (n \ge 1).$$

The norm of polynomials is given by

$$\|X_n\|^2 = \langle X_n, X_n\rangle = \beta_0\beta_1\cdots\beta_n = \Gamma\Big(\frac{n+1}{2}\Big)^2 \qquad (n \ge 0).$$

Example 3.1 A few values of ϑ_n and $X_n(z)$ are given by

$$\begin{aligned}
\vartheta_0 &= \frac{1}{\sqrt{\pi}}, & X_0(z) &= 1,\\
\vartheta_1 &= \frac{\sqrt{\pi}}{2}, & X_1(z) &= z - i\frac{1}{\sqrt{\pi}},\\
\vartheta_2 &= \frac{2}{\sqrt{\pi}}, & X_2(z) &= z^2 - i\frac{\sqrt{\pi}}{2}z - \frac{1}{2},\\
\vartheta_3 &= \frac{3\sqrt{\pi}}{4}, & X_3(z) &= z^3 - i\frac{2}{\sqrt{\pi}}z^2 - \frac{3}{2}z + i\frac{1}{\sqrt{\pi}},\\
\vartheta_4 &= \frac{8}{3\sqrt{\pi}}, & X_4(z) &= z^4 - i\frac{3\sqrt{\pi}}{4}z^3 - 3z^2 + i\frac{9\sqrt{\pi}}{8}z + \frac{3}{4}.
\end{aligned}$$

Notice that $\vartheta_{-1} = \pi$.

Remark 3.2 Introducing

$$\langle f, g\rangle_\lambda = \pi f(0)g(0) + i \text{ v.p.} \int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} f(x)g(x)|x|^s \Big(1 - \frac{x^2}{\lambda}\Big)^{\lambda-1/2} \mathrm{d}x,$$

when $\lambda \to +\infty$, we obtain the inner product whose corresponding orthogonal polynomials are known as *the generalized Hermite polynomials*.

Since zeros of the monic Hermite polynomials $\widehat{H}_n(z)$ $(n = 1, 2, \ldots)$ are real, simple and satisfy the separation theorem, for the polynomials $X_n(z)$ we have:

Theorem 3.3. *All zeros of $X_n(z)$ are contained in the rectangle*

$$R_+ = \{z \in \mathbf{C} \mid -\xi_n \le \operatorname{Re} z \le \xi_n,\ 0 < \operatorname{Im} z < \vartheta_{n-1}/2\},$$

where ξ_n is the largest zero of $\widehat{H}_n(z)$.

Proof. At first, we note that $X_n(z)$ cannot have any real zero. Indeed, if we suppose that exists a real ζ such that $X_n(\zeta) = 0$, i.e.,

$$\widehat{H}_n(\zeta) - i\vartheta_{n-1}\widehat{H}_{n-1}(\zeta) = 0,$$

then it must be $\widehat{H}_n(\zeta) = \widehat{H}_{n-1}(\zeta) = 0$, because $\widehat{H}_n(\zeta)$, $\widehat{H}_{n-1}(\zeta)$, and ϑ_{n-1} are real. However, this is a contradiction with the separation theorem.

According to a result of Giroux [**5**, Corollary 3] (see also [**12**, p. 269]) all zeros of $X_n(z)$ either lie in the half strip

$$S_+ = \{z \in \mathbf{C} \mid -\xi_n \le \operatorname{Re} z \le \xi_n,\ 0 < \operatorname{Im} z\},$$

or in the conjugate half strip. Since $X_n(z) = z^n - i\vartheta_{n-1}z^{n-1} - \cdots$, using the Viète rule, we find

$$\sum_{j=1}^{n} \zeta_j = i\vartheta_{n-1},$$

from which we conclude that

$$\operatorname{Im}\Big\{\sum_{j=1}^{n}\zeta_j\Big\} = \vartheta_{n-1} > 0,$$

i.e., $\operatorname{Im}\zeta_j > 0$, $j = 1,\dots,n$. Like in [**2**], we can prove that all zeros of $X_n(z)$ are located symmetrically with respect to imaginary axis because of the symmetric weight. It also gives bounds for zeros. If n is odd, then $X_n(z)$ has one purely imaginary zero. □

Remark 3.3. An interesting result on the zero distribution for polynomials orthogonal on the semicircle can be found in [**1**].

Theorem 3.4. *The polynomial $X_n(z)$ satisfies the differential equation*

$$P(z)X_n''(z) - 2\big(zP(z) + i\vartheta_{n-1}\big)X_n'(z) + 2\big(nP(z) - 2\vartheta_{n-1}^2\big)X_n(z) = 0, \tag{3.3}$$

where $P(z) = 2i\vartheta_{n-1}z + 2\vartheta_{n-1}^2 - n$.

Proof. We can prove it starting with the function

$$\Omega(z) = e^{-z^2+2i\vartheta_{n-1}z},$$

which satisfies

$$\big(\Omega(z)\widehat{H}_{n-1}(z)\big)' = A\,\Omega(z)X_n(z), \quad A = \text{const},$$

by the same procedure as in [**2**]. A general way for finding such differential equations was given in [**7**]. □

Dividing (3.3) by $P(z)$ we obtain the equation

$$X_n''(z) - 2\left(z + \frac{i\vartheta_{n-1}}{P(z)}\right)X_n'(z) + 2\left(n - \frac{2\vartheta_{n-1}^2}{P(z)}\right)X_n(z) = 0,$$

which is more similar to the Hermite equation.

For $\vartheta_{n-1} = 0$, the polynomial $X_n(z)$ becomes the polynomial $\widehat{H}_n(z)$ and the differential equation is the corresponding one.

4. QUADRATURES OF GAUSSIAN TYPE

In this section we construct a Gaussian quadrature formula

$$L(f) = \sum_{\nu=1}^{n}\sigma_\nu f(\zeta_\nu) + R_n(f), \qquad R_n(\mathcal{P}_{2n-1}) = 0, \tag{4.1}$$

for the functional

$$L(f) = \pi f(0) + i\ \text{v.p.}\int_{-\infty}^{+\infty}\frac{f(t)}{t}e^{-t^2}\,\mathrm{d}t. \tag{4.2}$$

Here $\mathcal{P}_m$ denotes the set of all polynomials of degree at most m.

Taking

$$X_n(z) = \prod_{k=1}^{n} (z - \zeta_k),$$

we obtain an interpolatory quadrature (4.1) if

$$\sigma_\nu = \frac{1}{X_n'(\zeta_\nu)} L\big(X_n(\cdot)/(\cdot - \zeta_\nu)\big), \qquad \nu = 1, \ldots, n.$$

This formula will be of Gaussian type if and only if the node polynomial X_n is chosen in such a way to be orthogonal to $\mathcal{P}_{n-1}$ with respect to the functional L, i.e., to the inner product (3.2). Thus, X_n must be given as in Theorem 3.1.

It is easy to find the nodes ζ_ν and weights σ_ν in an analytic form when $n = 1$ and $n = 2$. Namely, for $n = 1$ we have $\zeta_1 = i/\sqrt{\pi}$ and $\sigma_1 = \pi$, and for $n = 2$ the parameters are

$$\zeta_{1,2} = \mp \frac{\sqrt{8-\pi}}{4} + i\frac{\sqrt{\pi}}{4} \approx \mp 0.5510448794 + 0.4431134627\, i$$

and

$$\sigma_{1,2} = \frac{\pi}{2} \mp i\frac{(4-\pi)\sqrt{\pi}}{2\sqrt{8-\pi}} \approx \; 1.5707963268 \mp 0.3451369080\, i.$$

For $f(z) \equiv 1$, from (4.1) and (4.2) we obtain that

$$\sum_{\nu=1}^{n} \sigma_\nu = \pi.$$

Letting $\widetilde{X}_k(z) = X_k(z)/\|X_k\|$ denote the normalized orthogonal polynomials and

$$\mathbf{X}(z) = \big[\widetilde{X}_0(z), \widetilde{X}_1(z), \ldots, \widetilde{X}_{n-1}(z)\big]^T$$

the vector of the first of them, it is easily seen that

$$z\widetilde{X}_k(z) = \vartheta_{k-1}\widetilde{X}_{k-1}(z) + i\alpha_k\widetilde{X}_k(z) + \vartheta_k\widetilde{X}_{k+1}(z), \quad k = 0, 1, \ldots,$$

and

$$J_n\mathbf{X}(\zeta_\nu) = \zeta_\nu\mathbf{X}(\zeta_\nu),$$

where

$$J_n = \begin{bmatrix} i\alpha_0 & \vartheta_0 & & & \mathrm{O} \\ \vartheta_0 & i\alpha_1 & \vartheta_1 & & \\ & \vartheta_1 & i\alpha_2 & \ddots & \\ & & \ddots & \ddots & \vartheta_{n-2} \\ \mathrm{O} & & & \vartheta_{n-2} & i\vartheta_{n-1} \end{bmatrix}.$$

The nodes ζ_ν are therefore the eigenvalues of the Jacobi matrix J_n and $\mathbf{X}(\zeta_\nu)$ the corresponding eigenvalues. By an adaptation of the procedure of Golub and Welsch [**6**] as in [**4**] and [**9**] and using the EISPACK routine HQR2 and the LINPACK routines CGECO and CGESL we can compute the parameters of the Gaussian quadrature (4.1). In Table 4.1 we display these parameters (to 8 decimals only, to save space) for $n = 5,\ 10,\ 20$ (numbers in parentheses denote decimal exponents).

Table 4.1. Gaussian formula for $n = 5,\ 10,\ 20$

n	ν	ζ_ν		σ_ν	
5	1,2	±1.85901878	$+0.21826725\,i$	$0.17689235(-1)$	$\pm0.11513438(-1)\,i$
	3,4	±0.80848233	$+0.32121144\,i$	0.53995718	$\pm0.31710987\,i$
	5		$0.42554818\,i$	2.02629982	
10	1,2	±3.31975547	$+0.13515758\,i$	$0.42561000(-5)$	$\pm0.29610519(-5)\,i$
	3,4	±2.40909579	$+0.16286583\,i$	$0.82451753(-3)$	$\pm0.66849399(-3)\,i$
	5,6	±1.62845184	$+0.19933977\,i$	$0.24576916(-1)$	$\pm0.22489105(-1)\,i$
	7,8	±0.91319867	$+0.25497858\,i$	0.25135294	$\pm0.22500063\,i$
	9,10	±0.27441118	$+0.33813279\,i$	1.29403770	$\pm0.49500070\,i$
20	1,2	±5.30573927	$+0.088616419\,i$	$0.83944621(-13)$	$\pm0.57817620(-13)\,i$
	3,4	±4.51840887	$+0.097901493\,i$	$0.17095758(-9)$	$\pm0.13031066(-9)\,i$
	5,6	±3.85631387	$+0.10752170\,i$	$0.43738933(-7)$	$\pm0.36658071(-7)\,i$
	7,8	±3.25640496	$+0.11824584\,i$	$0.32839496(-5)$	$\pm0.30258528(-5)\,i$
	9,10	±2.69454220	$+0.13075122\,i$	$0.10186797(-3)$	$\pm0.10337127(-3)\,i$
	11,12	±2.15824427	$+0.14595677\,i$	$0.15724549(-2)$	$\pm0.17577769(-2)\,i$
	13,14	±1.64017858	$+0.16537297\,i$	$0.13707030(-1)$	$\pm0.16722936(-1)\,i$
	15,16	±1.13642167	$+0.19184148\,i$	$0.76184193(-1)$	$\pm0.96815143(-1)\,i$
	17,18	±0.64866565	$+0.23107063\,i$	0.32528096	$\pm0.35394326\,i$
	19,20	±0.19977194	$+0.28422720\,i$	1.15394649	$\pm0.47469771\,i$

We notice that σ_ν is real if ζ_ν is purely imaginary; and that is $\sigma_{\nu+1} = \overline{\sigma}_\nu$ if $\zeta_{\nu+1} = -\overline{\zeta}_\nu$.

An interesting application of Gaussian formulae (4.1) could be to Cauchy principal value integrals.

Let $z \mapsto f(z)$ be a holomorphic function in $\operatorname{Im} z \geq 0$. Then we have

$$\text{v.p.} \int_{-\infty}^{+\infty} \frac{f(t)}{t} e^{-t^2}\,\mathrm{d}t \approx i\Big\{\pi f(0) - \sum_{\nu=1}^{n} \sigma_\nu f(\zeta_\nu)\Big\}.$$

In particular, if $f(z)$ is real for real z, then

$$\text{v.p.} \int_{-\infty}^{+\infty} \frac{f(t)}{t} e^{-t^2}\,\mathbf{d}t \approx \operatorname{Im} \sum_{\nu=1}^{n} \sigma_\nu f(\zeta_\nu). \tag{4.3}$$

Example 4.1. We apply (4.3) to Cauchy principal value integral

$$I = \text{v.p.} \int_{-\infty}^{+\infty} \frac{e^t}{t}\, e^{-t^2}\,\mathrm{d}t = 1.93192\,89830\,08213\,74957\,02\ldots\,.$$

TABLE 4.2. Gaussian approximation of CAUCHY principal value integral I and relative errors

n	Approximation	Rel. error
2	$1.9\underline{1}7996$	$7.21(-3)$
3	$1.931\underline{4}077$	$2.70(-4)$
4	$1.9319\underline{1}41$	$7.70(-6)$
5	$1.931928\underline{6}39$	$1.78(-7)$
6	$1.9319289\underline{7}63$	$3.47(-9)$
7	$1.93192898\underline{2}895$	$5.84(-11)$
8	$1.93192898300\underline{6}54$	$8.65(-13)$
9	$1.931928983008\underline{1}92$	$1.13(-14)$
10	$1.93192898300821\underline{4}$	$1.15(-16)$

The obtained results for $n = 2(1)10$, with relative errors, are given in Table 4.2. In each entry the first digit in error is underlined.

REFERENCES

1. W. GAUTSCHI: *On the zeros of polynomials orthogonal on the semicircle.* SIAM J. Math. Anal. **20** (1989), 738–743.
2. W. GAUTSCHI, H. J. LANDAU, G. V. MILOVANOVIĆ: *Polynomials orthogonal on the semicircle, II.* Constr. Approx. **3** (1987), 389–404.
3. W. GAUTSCHI, G. V. MILOVANOVIĆ: *Polynomials orthogonal on the semicircle.* Rend. Sem. Mat. Univ. Politec. Torino (Special Functions: Theory and Computation), 1985, 179–185.
4. W. GAUTSCHI, G. V. MILOVANOVIĆ: *Polynomials orthogonal on the semicircle.* J. Approx. Theory **46** (1986), 230–250.
5. A. GIROUX: *Estimates for the imaginary parts of the zeros of a polynomials.* Proc. Amer. Math. Soc. **44** (1974), 61–67.
6. G. H. GOLUB, J. H. WELSCH: *Calculation of Gauss quadrature rules.* Math. Comp. **23** (1969), 221–230.
7. P. MARONI: *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x - c)^{-1}L$. Period. Math. Hung. **21** (1990), 223–248.
8. G. V. MILOVANOVIĆ: *Some applications of the polynomials orthogonal on the semicircle.* In: Numerical Methods (Miskolc, 1986), Colloq. Math. Soc. János Bolyai, Vol. **50**, North-Holland, Amsterdam – New York 1987, 625–634.
9. G. V. MILOVANOVIĆ: Complex orthogonality on the semicircle with respect to Gegenbauer weight: Theory and applications. In: Topics in Mathematical Analysis (Th.M. Rassias, ed.), World Scientific Publ., Singapore 1989, 695–722.

10. G. V. Milovanović: *Complex polynomials orthogonal on the semicircle.* In: The Cauchy Method of Residues, Vol. 2 – Theory and Applications (D. S. Mitrinović and J. D. Kečkić), Kluwer, Dordrecht 1993, 147–161.
11. G. V. Milovanović: *On polynomials orthogonal on the semicircle and applications.* J. Comput. Appl. Math. **49** (1993), 193–199.
12. G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias: *Topics in Polynomials: Extremal Problems, Inequalities, Zeros.* World Scientific, Singapore – New Jersey – London – Hong Kong 1994.
13. G. V. Milovanović, P. M. Rajković: *Geronimus concept of orthogonality for polynomials orthogonal on a circular arc.* Rend. di Matematica, Serie VII (Roma) **10** (1990), 383–390.
14. G. V. Milovanović, P. M. Rajković: *On polynomials orthogonal on a circular arc.* J. Comput. Appl. Math. **51** (1994), 1–13.
15. S. E. Notaris: *Some new formulae for the Stieltjes polynomials relative to classical weight functions.* SIAM J. Numer. Anal. **27** (1991), 1196–1206.
16. J. G. Wendel: *Note on the gamma function.* Amer. Math. Monthly **55** (1948), 563–564.
17. G. Szegő: *Orthogonal Polynomials.* Amer. Math. Soc. Colloq. Publ., vol. 23, 4th ed., Amer. Math. Soc., Providence, R. I. 1975.

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