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ORTHOGONAL POLYNOMIALS ON THE RADIAL RAYS AND AN ELECTROSTATIC INTERPRETATION OF ZEROS

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Abstract. For polynomials orthogonal on the radial rays in the complex plane, which were introduced in [**12**], we give first a short account, and then we develop two interesting classes of orthogonal polynomials: (1) the generalized Hermite polynomials; (2) the generalized Gegenbauer polynomials. For such polynomials we obtain the corresponding linear differential equations of the second order. Assuming a logarithmic potential, we give an electrostatic interpretation of the zeros of the generalized Gegenbauer polynomials.

1. Introduction

In [**13**] we considered a class of Humbert's polynomials $\{p_{n,m}^{\lambda}(x)\}_{n=0}^{\infty}$ defined by the generating function (see also [**14**])

$$G_m^{\lambda}(x,t) = (1-2xt+t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n, \tag{1.1}$$

where $m \in \mathbb{N}$ and $\lambda > -1/2$. These polynomials reduce to Horadam polynomials [**6**], Gegenbauer polynomials [**3**], and Horadam-Pethe polynomials [**7**], for $m = 1$, $m = 2$, an $m = 3$, respectively. The explicit form of this incomplete polynomials $p_{n,m}^{\lambda}(x)$ is

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n-mk)!}\,(2x)^{n-mk}, \tag{1.2}$$

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where $(\lambda)_0 = 1,\ (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1),\ n \in \mathbb{N}$, and a recurrence relation is given by

$$np_{n,m}^{\lambda}(x) = (\lambda+n-1)2xp_{n-1,m}^{\lambda}(x) - (n+m(\lambda-1))p_{n-m,m}^{\lambda}(x),$$

where $n \ge m \ge 1$. For corresponding monic polynomials $\hat{p}_{n,m}^{\lambda}(x)$, we have

$$\begin{aligned} &\hat{p}_{n,m}^{\lambda}(x) = x\hat{p}_{n-1,m}^{\lambda}(x) - b_n\hat{p}_{n-m,m}^{\lambda}(x), \quad n \ge m \ge 1, \\ &\hat{p}_{n,m}^{\lambda}(x) = x^n, \quad 0 \le n \le m-1, \end{aligned}$$

where

$$b_n = \frac{(n-1)!}{(m-1)!} \cdot \frac{n+m(\lambda-1)}{2^m(\lambda+n-m)_m}\,.$$

One interesting question can be stated: *Is it possible to find an inner product* $(\,\cdot\,,\,\cdot\,)$ *such that the polynomials* $p_{n,m}^{\lambda}(x)$ *be orthogonal with respect to* $(\,\cdot\,,\,\cdot\,)$*?* Working on this subject, we have not solved the problem, but recently we introduced a new class of polynomials orthogonal on some radial rays in the complex plane and investigated their existence and uniqueness. A recurrence relation for these polynomials, a representation and the connection with standard polynomials orthogonal on $(0,1)$ were derived in [**12**]. It was shown that their zeros are simple and distributed symmetrically on the radial rays, with the possible exception of a multiple zero at the origin. An analogue of the Jacobi polynomials and the corresponding problem with the generalized Laguerre polynomials, were also treated.

In this paper we give first a short account on polynomials orthogonal on the radial rays in the complex plane, and then we develop two interesting classes of polynomials:

- the generalized Hermite polynomials;
- the generalized Gegenbauer polynomials.

Also, we obtain the corresponding linear differential equations of the second order for such orthogonal polynomials and give an electrostatic interpretation of zeros of the generalized Gegenbauer polynomials.

The paper is organized as follows. In Section 2 we give a preliminary material on polynomials orthogonal on the radial rays in the complex plane. Sections 3 and 4 are dedicated to the generalized Hermite and generalized Gegenbauer orthogonal polynomials, respectively. Assuming a logarithmic potential, an electrostatic interpretation of zeros of the generalized Gegenbauer orthogonal polynomials is discussed in Section 5.

2. Preliminaries

We start with a type of nonstandard orthogonality on the radial rays in the complex plane. Suppose that we have M points in the complex plane, $z_s = a_s e^{i\varphi_s} \in \mathbb{C},\ s = 0,1,\ldots,M-1$, with different arguments φ_s. Some of a_s (or all) can be ∞.

The case $M = 6$ is shown in Fig. 2.1. We can define an inner product on these radial rays ℓ_s in the complex plane which connect the origin $z = 0$ and the points z_s, $s = 0, 1, \ldots, M-1$. Namely,

$$(f, g) = \sum_{s=0}^{M-1} e^{-i\varphi_s} \int_{\ell_s} f(z)\overline{g(z)}\,|w(z)|\,dz,$$

where $z \mapsto w(z)$ is a suitable complex weight function. This product can be expressed in the form

$$(f, g) = \sum_{s=0}^{M-1} \int_0^{a_s} f\big(xe^{i\varphi_s}\big)\overline{g\big(xe^{i\varphi_s}\big)}\,\big|w\big(xe^{i\varphi_s}\big)\big|\,dx \tag{2.1}$$

and we can see that $\|f\|^2 = (f, f) > 0$, except when $f(z) = 0$.

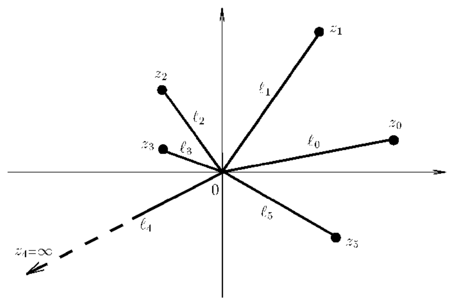

Fig. 2.1

The case when M is an even number ($M = 2m$), $a_s = 1$, $\varphi_s = \pi s/m$, $s = 0, 1, \ldots, 2m-1$, and $z \mapsto w(z)$ is a holomorphic function such that

$$|w(x\varepsilon_s)| = w(x), \quad s = 0, 1, \ldots, 2m-1,$$

where $\varepsilon_s = \exp(i\varphi_s) = \exp(i\pi s/m)$ and $x \mapsto w(x)$ is a weight function on $(0, 1)$ (nonnegative on $(0, 1)$ and $\int_0^1 w(x)\,dx > 0$), was considered in [**12**]. In this symmetric case, the inner product (2.1) reduces to

$$(f, g) = \int_0^1 \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\right) w(x)\,dx. \tag{2.2}$$

In the case $m = 1$, (2.2) becomes

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx,$$

so we have the standard case of polynomials orthogonal on $(-1,1)$ with respect to the even weight function $x \mapsto w(x)$.

Several cases when M is an arbitrary number of rays were investigated in [**11**] and [**17–18**].

In [**12**] we proved an existence result for the (monic) orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (2.2). It is well known that an orthogonal sequence of polynomials satisfies a three-term recurrence relation if the inner product has the property $(zf,g) = (f,zg)$. However, in our case the corresponding property is given by $(z^m f, g) = (f, z^m g)$. Then the following result holds:

THEOREM 2.1. *Let the inner product* $(\cdot,\cdot)$ *be given by* (2.2) *and let the corresponding monic orthogonal polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *exist. They satisfy the recurrence relation*

$$\begin{gathered}\pi_{N+m}(z) = z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \\ \pi_N(z) = z^N, \quad N = 0,1,\ldots,2m-1,\end{gathered} \tag{2.3}$$

where

$$b_N = \frac{(\pi_N, z^m \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = \frac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}\,. \tag{2.4}$$

In a simple case when $m = 2$ and $w(x) = 1$, i.e., when the inner product $(\cdot,\cdot)$ is given by

$$(f,g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)}\Big]\,dx,$$

we can calculate directly the coefficient b_N in the recurrence relation (2.3). Namely, then we have

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \ \nu = 0,1, \\[2ex] \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \ \nu = 2,3. \end{cases} \tag{2.5}$$

Notice that

$$b_N \to \frac{1}{4} \qquad \text{as } \ N \to +\infty,$$

just like in Szegő's theory for orthogonal polynomials on the interval $(-1,1)$.

A more general case with the Jacobi weight will be considered in Section 4. In that case we obtain the generalized Gegenbauer polynomials.

3. Generalized Hermite polynomials

Now, we study orthogonal polynomials relative to the inner product

$$(f,g)=\int_0^{+\infty}\left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\right)w(x)\,dx, \tag{3.1}$$

where

$$w(x)=x^{2m\gamma}\exp\left(-x^{2m}\right),\quad \gamma>-\frac{1}{2m}\,. \tag{3.2}$$

Here we have $2m$ radial rays in the complex plane ℓ_s, $s=0,1,\ldots,2m-1$, which connect the origin $z=0$ and $z=\infty$ with $2m$ different angles $\varphi_s=\pi s/m$, $s=0,1,\ldots,2m-1$. As before, ε_0, ε_1, $\ldots$, ε_{2m-1} are $(2m)$th roots of unity, i.e., $\varepsilon_s=\exp(i\pi s/m)$, $s=0,1,\ldots,2m-1$.

In the case $m=1$, (3.1) becomes

$$(f,g)=\int_{-\infty}^{+\infty} f(x)\overline{g(x)}w(x)\,dx, \tag{3.3}$$

so we have the standard case of polynomials orthogonal on the real line with respect to the weight function $x\mapsto w(x)=|x|^{2\gamma}\exp\left(-x^2\right)$. For $\gamma=0$, (3.3) reduces to the inner product, which gives the Hermite polynomials $\{H_n(x)\}_{n=0}^{+\infty}$. It is well known that these polynomials can be expressed in terms of the generalized Laguerre polynomials $\{L_n^{(s)}(x)\}_{n=0}^{+\infty}$, which are orthogonal on the half line with respect to the inner product

$$(f,g)=\int_0^{+\infty} f(t)\overline{g(t)}\,t^s e^{-t}\,dt. \tag{3.4}$$

Namely, we have (cf. [**10**, p. 120 and p. 147])

$$H_{2k}(x)=c_k L_k^{(-1/2)}(x^2)\quad\text{and}\quad H_{2k+1}(x)=d_k x L_k^{(1/2)}(x^2), \tag{3.5}$$

where c_k and d_k are constants. The following theorem gives the generalized Hermite polynomials.

Theorem 3.1. *Let the inner product $(\cdot,\cdot)$ be given by (3.1) and $w(x)$ by (3.2). The corresponding monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation*

$$\begin{aligned}\pi_{N+m}(z)&=z^m\pi_N(z)-b_N\pi_{N-m}(z),\quad N\ge m,\\ \pi_N(z)&=z^N,\quad N=0,1,\ldots,2m-1,\end{aligned}$$

where

$$b_{2mn+\nu} = \begin{cases} n+1+\alpha_\nu & \text{if } 0 \le \nu \le m-1, \\ n & \text{if } m \le \nu \le 2m-1. \end{cases}$$

Also, they can be expressed in the form

$$\pi_N(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^{2m}), \quad N = 2mn+\nu,\ n = [N/2m], \tag{3.6}$$

where $\nu \in \{0,1,\dots,2m-1\}$, $\alpha_\nu = \gamma + (2\nu+1-2m)/(2m)$, *and* $\hat{L}_n^{(s)}(t)$ *denotes the monic generalized Laguerre polynomial orthogonal with respect to the inner product* (3.4).

Proof. Using [**12**, Sect. 6] we conclude that

$$\pi_{2mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^{2m}), \qquad \nu = 0,1,\dots,2m-1;\ n = 0,1,\dots,$$

where $q_n^{(\nu)}(t)$ are monic polynomials orthogonal on $[0,+\infty)$ with respect to the weight

$$t \mapsto w_\nu(t) = t^{(2\nu+1-2m)/2m} w(t^{1/2m}) = t^{\gamma+(2\nu+1-2m)/2m} e^{-t}.$$

In fact, this is the generalized Laguerre weight with the parameter $s = \alpha_\nu = \gamma + (2\nu+1-2m)/(2m)$. Thus, the representation (3.6) holds.

Let $\hat{L}_n^{(s)}(t)$ be the monic generalized Laguerre polynomials orthogonal with respect to the weight $t \mapsto t^s e^{-t}$ on $(0,+\infty)$. They satisfy the three-term recurrence relation (cf. [**16**, p. 46])

$$\hat{L}_{n+1}^{(s)}(t) = \big(t-(2n+s+1)\big)\hat{L}_n^{(s)}(t) - n(n+s)\hat{L}_{n-1}^{(s)}(t),$$

as well as the following relations (see [**25**])

$$t\hat{L}_{n-1}^{(s+1)}(t) = \hat{L}_n^{(s)}(t) + (n+s)\hat{L}_{n-1}^{(s)}(t), \quad \hat{L}_n^{(s)}(t) = \hat{L}_n^{(s-1)}(t) - n\hat{L}_{n-1}^{(s)}(t).$$

In order to determine b_N in the recurrence relation for $\pi_N(z)$ we combine (3.6) and the previous relations for $L_n^{(s)}(t)$, taking $t = z^{2m}$ and $s = \alpha_\nu$. $\square$

Remark 3.1. For $m = 1$ we can see that formulae (3.6) reduce to (3.5).

The zero distribution of the polynomials $\pi_N(z)$ immediately follows from (3.6):

THEOREM 3.2. *Let* $N = 2mn+\nu$, $n = [N/2m]$, $\nu \in \{0,1,\dots,2m-1\}$. *All zeros of the polynomial* $\pi_N(z)$ *are simple and located symmetrically on the radial rays* l_s, $s = 0,1,\dots,2m-1$, *with the possible exception of a multiple zero of order* ν *at the origin* $z = 0$.

Like the generalized Laguerre polynomial $L_n^{(s)}(t)$, the polynomial $\pi_N(z)$ satisfies a second order linear homogeneous differential equation.

THEOREM 3.3. *The polynomials $\pi_N(z)$ orthogonal with respect to the inner product* (3.1) *satisfy a second order linear homogeneous differential equation of the form*

$$z^2y'' + B(z)y' + C(z)y = 0, \tag{3.7}$$

where

$$B(z) = 2z\big[1 + m(\gamma - 1 - z^{2m})\big], \quad C(z) = 2mNz^{2m} - \nu(\nu + 2m(\gamma - 1) + 1), \tag{3.8}$$

and $N = 2mn + \nu$, $n = [N/(2m)]$, $\nu \in \{0, 1, \dots, 2m - 1\}$.

Proof. Let $N = 2mn + \nu$, $n = [N/(2m)]$, $\nu \in \{0, 1, \dots, 2m - 1\}$. Starting from the representation of the orthogonal polynomial $\pi_N(z)$ given by (3.6), where $\alpha_\nu = \gamma + (2\nu + 1 - 2m)/(2m)$, $\gamma > -1/(2m)$, we find

$$\begin{aligned} 2mt\Delta\hat{L}_n^{(\alpha_\nu)}(t) &= z^{-\nu}\left[z\pi_N'(z) - \nu\pi_N(z)\right], \\ 4m^2t^2\Delta^2\hat{L}_n^{(\alpha_\nu)}(t) &= z^{-\nu}\left[z^2\pi_N''(z) - (2\nu + 2m - 1)z\pi_N'(z) + \nu(\nu + 2m)\pi_N(z)\right], \end{aligned}$$

where $t = z^{2m}$ and Δ is the standard differentiation operator $\Delta = \dfrac{d}{dt}$.

Now, using the generalized Laguerre differential equation

$$t\Delta^2\hat{L}_n^{(\alpha_\nu)}(t) + (\alpha_\nu + 1 - t)\hat{L}_n^{(\alpha_\nu)}(t) + n\hat{L}_n^{(\alpha_\nu)}(t) = 0,$$

we obtain

$$z^2y'' + 2z\big[1 + m(\gamma - 1 - z^{2m})\big]y' + \big[2mNz^{2m} - \nu(\nu + 2m(\gamma - 1) + 1)\big]y = 0,$$

i.e., (3.7), where $y = \pi_N(z)$. $\square$

Remark 3.2. For $m = 1$, the equation (3.7) reduces to the Hermite equation

$$y'' - 2zy' + 2Ny = 0.$$

Remark 3.3. A simple case could be if we choose the parameter γ in the weight function (3.2) such that the coefficient $B(z)$ in (3.8) reduces to a monomial of degree $2m + 1$. Namely, if $\gamma = (m - 1)/m$, the equation (3.7) reduces to

$$y'' - 2mz^{2m-1}y' + \left[2mNz^{2m-2} - \frac{\nu(\nu - 1)}{z^2}\right]y = 0.$$

Using (3.6), we put

$$\pi_{2mn}^{(\gamma)}(z) = \pi_{2mn}(z) = \hat{L}_n^{(\alpha_0)}(z^{2m}), \qquad \alpha_0 = \gamma - 1 + \frac{1}{2m}\,.$$

Since

$$\pi_{2mn+\nu}^{(\gamma)}(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^{2m}) \quad \text{and} \quad \alpha_\nu = \gamma - 1 + \frac{1}{2m} + \frac{\nu}{m}\,,$$

we conclude that

$$\pi_{2mn+\nu}^{(\gamma)}(z) = z^\nu \pi_{2mn}^{(\gamma+\nu/m)}(z) \qquad (\nu = 1, \dots, 2m-1).$$

Thus, it is enough to consider differential equation (3.7) only for $N = 2mn$, i.e.,

$$zy'' + 2\bigl[1 + m(\gamma - 1 - z^{2m})\bigr]y' + 2mNz^{2m-1}y = 0.$$

4. Generalized Gegenbauer polynomials

As we mentioned in Section 1, the Humbert's polynomials $p_{n,m}^\lambda(x)$ defined by (1.1), i.e. (1.2), are a generalization of the well-known Gegenbauer polynomials. Their generating function (1.1) reduces for $m = 2$ to the corresponding one of the polynomials $C_n^\lambda(x)$. Regarding to the orthogonality there are the generalized Gegenbauer polynomials $W_n^{(\alpha,\beta)}(x)$ introduced by Lascenov [**9**] (see also [**1**, pp. 155–156]). These polynomials are orthogonal on $[-1,1]$ with respect to the weight function $x \mapsto |x|^\mu(1-x^2)^\alpha$, where $\mu, \alpha > -1$ and $\beta = (\mu-1)/2$. It is interesting to say that these polynomials have been "rediscovered" in 1980 (see [**19**]). We used these polynomials in the least squares approximation with constraints [**15**] and in quadrature processes [**8**].

The relations between generalized monic Gegenbauer polynomials $W_N^{(\alpha,\beta)}(x)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are given by

$$\begin{aligned} W_{2n}^{(\alpha,\beta)}(x) &= \frac{n!}{(n+\alpha+\beta+1)_n}\, P_n^{(\alpha,\beta)}(2x^2-1), \\ W_{2n+1}^{(\alpha,\beta)}(x) &= \frac{n!}{(n+\alpha+\beta+2)_n}\, xP_n^{(\alpha,\beta+1)}(2x^2-1). \end{aligned} \tag{4.1}$$

An important relation relation for generalized polynomials is the following

$$W_{2n+1}^{(\alpha,\beta)}(x) = xW_{2n}^{(\alpha,\beta+1)}(x). \tag{4.2}$$

The corresponding three-term recurrence relation is given by

$$\begin{aligned} &W_{N+1}^{(\alpha,\beta)}(x) = xW_N^{(\alpha,\beta)}(x) - \Lambda_N W_{N-1}^{(\alpha,\beta)}(x), \qquad N = 0, 1, \dots, \\ &W_{-1}^{(\alpha,\beta)}(x) = 0, \quad W_0^{(\alpha,\beta)}(x) = 1, \end{aligned}$$

where

$$\Lambda_{2n} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\,, \quad \Lambda_{2n-1} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)}\,,$$

for $n = 1, 2, \dots$, except for $\alpha + \beta = -1$ when $\Lambda_1 = (\beta + 1)/(\alpha + \beta + 2)$.

It is easy to prove that polynomials $W_N^{(\alpha,\beta)}(x)$ $\big(N = 2n + \nu,\ \nu = 0 \ \text{ or } \ 1\big)$ satisfy a second order linear differential equation

$$(1-x^2)x^2y'' + \big[2\beta+1-(2\alpha+2\beta+3)x^2\big]xy' + \big[N\big(N+2(\alpha+\beta+1)\big)x^2 - \nu(\nu+2\beta)\big]y = 0.$$

It is enough to consider only case $\nu = 0$, i.e. $N = 2n$. Then, this equation reduces to

$$(1 - x^2)y'' + \big[2\beta + 1 - (2\alpha + 2\beta + 3)x^2\big]\frac{1}{x}\,y' + N\big(N + 2(\alpha + \beta + 1)\big)y = 0. \tag{4.3}$$

Another case ($\nu = 1,\ N = 2n + 1$) can be obtain from (4.3) putting $\beta{:} = \beta + 1$ and $N = 2n$, and using equality (4.2). This fact will be used below.

A natural extension of the generalized Gegenbauer polynomials can be given by Theorem 2.1 and using the inner product (1.2), with the weight function

$$w(x) = (1 - x^{2m})^\alpha x^{2m\gamma}, \qquad \alpha > -1,\ \gamma > -\frac{1}{2m}\,. \tag{4.4}$$

Theorem 4.1. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (2.2), *where the weight function is given by* (4.4), *can be expressed in the form*

$$\pi_N(z) = 2^{-n} z^\nu \hat{P}_n^{(\alpha,\beta_\nu)}(2z^{2m} - 1), \quad N = 2mn + \nu,\ n = [N/2m], \tag{4.5}$$

where $\nu \in \{0, 1, \dots, 2m-1\}$, $\beta_\nu = \gamma + (2\nu+1-2m)/(2m)$, *and* $\hat{P}_n^{(\alpha,\beta)}(x)$ *denotes the monic Jacobi polynomial orthogonal with respect to the weight* $x \mapsto (1-x)^\alpha(1+x)^\beta$ *on* $(-1, 1)$. *The polynomials* $\pi_N(z)$ *satisfy the recurrence relation* (2.3), *where*

$$b_{2mn+\nu} = \begin{cases} \dfrac{n(n + \alpha)}{(2n + \alpha + \beta_\nu)(2n + \alpha + \beta_\nu + 1)} & \textit{if} \quad 0 \le \nu \le m - 1, \\[2ex] \dfrac{(n + \beta_\nu)(n + \alpha + \beta_\nu)}{(2n + \alpha + \beta_\nu)(2n + \alpha + \beta_\nu + 1)} & \textit{if} \quad m \le \nu \le 2m - 1. \end{cases} \tag{4.6}$$

The proof of this theorem can be done in a similar way as for Theorem 3.1.

Also, a result on a symmetric location of zeros of $\pi_N(z)$ on the radial rays l_s, $s = 0, 1, \dots, 2m - 1$, can be proved. Namely, let $N = 2mn + \nu$, $n = [N/2m]$, $\nu \in \{0, 1, \dots, 2m - 1\}$ and let $\tau_k^{(n,\nu)}$, $k = 1, \dots, n$, denote the zeros in increasing order of the Jacobi polynomial $P_n^{(\alpha,\beta_\nu)}(x)$ $\big(\alpha > -1,\ \beta_\nu = \gamma + (2\nu + 1 - 2m)/(2m) > -1\big)$, i.e.,

$$-1 < \tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)} < 1.$$

Each zero $\tau_k^{(n,\nu)}$ generates $2m$ zeros $z_{k,s}^{(n,\nu)}$ of $\pi_N(z)$ on the radial rays l_s,

$$z_{k,s}^{(n,\nu)} = \sqrt[2m]{\frac{1}{2}\Big(1+\tau_k^{(n,\nu)}\Big)}\; e^{is\pi/m}, \qquad s = 0,1,\dots,2m-1.$$

If $\nu > 0$, there exists a zero of order ν at the origin $z = 0$. Thus, $\pi_N(z)$ has all zeros inside the unit circle.

Remark 4.1. For $m = 1$, formulae (4.5) reduce to (4.1). If $\alpha = \gamma = 0$ and $m = 2$, (4.6) reduces to (2.5).

According to Theorem 4.1 we see that for $N = 2mn + \nu$, where $\nu \in \{0,1,\dots,2m-1\}$, we have an equality (up to a multiplicative constant) of the form

$$\pi_N(z) = \pi_N^{(\alpha,\gamma)}(z) \asymp z^\nu P_n^{(\alpha,\beta_\nu)}(2z^{2m}-1),$$

where $\beta_\nu = \beta_0 + \nu/m$, $\beta_0 = \gamma - 1 + 1/(2m)$. For $N = 2mn$, i.e. $\nu = 0$, we have

$$\pi_N(z) = \pi_{2mn}^{(\alpha,\gamma)}(z) \asymp P_n^{(\alpha,\beta_0)}(2z^{2m}-1), \tag{4.7}$$

and for an arbitrary ν,

$$\pi_{2mn+\nu}^{(\alpha,\gamma)}(z) \asymp z^\nu P_n^{(\alpha,\beta_0+\nu/m)}(2z^{2m}-1),$$

i.e.,

$$\pi_{2mn+\nu}^{(\alpha,\gamma)}(z) = z^\nu \pi_{2mn}^{(\alpha,\gamma+\nu/m)}(z). \tag{4.8}$$

In order to get a linear differential equation for $\pi_N(z)$ we take $N = 2mn$ and start from the corresponding differential equation for the Jacobi polynomials $y = P_n^{(\alpha,\beta)}(x)$,

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0, \tag{4.9}$$

where $\beta = \beta_0 = \gamma - 1 + 1/(2m)$. Putting $N = 2mn$, $x = 2z^{2m} - 1$ and using (4.7), we find

$$\Delta P_n^{(\alpha,\beta)}(x) \asymp \pi_N'(z)\,\frac{dz}{dx}, \qquad \Delta^2 P_n^{(\alpha,\beta)}(x) \asymp \pi_N''(z)\Big(\frac{dz}{dx}\Big)^2 + \pi_N'(z)\,\frac{d^2z}{dx^2}.$$

Then, substituting into (4.9), we obtain the following result:

THEOREM 4.2. *The polynomials $\pi_N(z)$ orthogonal with respect to the inner product* (2.2)*, with the weight function* (4.4)*, satisfy a second order linear homogeneous differential equation of the form*

$$(1-z^{2m})z^2Y'' + C(z)zY' + Az^{2m}Y = 0, \tag{4.10}$$

where $N = 2mn$ and

$$A = N\big[N + 2m(\alpha+\gamma) + 1\big], \quad C(z) = 2\big[m(\gamma-1) + 1 - \big(m(\alpha+\gamma)+1\big)z^{2m}\big].$$

Remark 4.2. For $m = 1$, (4.10) reduces to (4.3). Notice that $\mu = 2\beta + 1 = 2\gamma$.

Now, using (4.8) and (4.10), with $\gamma := \gamma + \nu/m$, we get the corresponding differential equation for each N $(= 2mn + \nu)$.

THEOREM 4.3. *The polynomials $\pi_N(z)$ orthogonal with respect to the inner product* (2.2)*, with the weight function* (4.4)*, satisfy a second order linear homogeneous differential equation of the form*

$$(1-z^{2m})z^2Y'' + C(z)zY' + \big(Az^{2m}-B\big)\,Y = 0, \tag{4.11}$$

where $N = 2mn+\nu$, $\nu \in \{0,1,\dots,2m-1\}$, *and*

$$C(z) = 2\big[m(\gamma-1)+1-\big(m(\alpha+\gamma)+1\big)z^{2m}\big],$$
$$A = N\big[N+2m(\alpha+\gamma)+1\big], \quad B = \nu\big[\nu+2m(\gamma-1)+1\big].$$

Remark 4.3. It is interesting to mention a physical problem connected to equation (4.11). The equations for the dispersion of a buoyant contaminant can be approximated by the Erdogan-Chatwin equation

$$\partial_t c = \partial_y\Big\{\Big[D_0 + \big(\partial_y c\big)^2 D_2\Big]\partial_y c\Big\}.$$

Smith [**20**] showed that in the limit of strong non-linearity ($D_0 = 0$) there are similarity solutions for a concentration jump and for a finite discharge. A stability analysis for this problem involves a family of orthogonal polynomials $Y_N(z)$, where

$$(1-z^4)Y_N'' - 6z^3Y_N' + N(N+5)z^2Y_N = 0, \tag{4.12}$$

and the degree N is restricted to the values $0,1,4,5,8,9,\dots$. As we can see, the polynomials $Y_N(z)$ are just a special case of our polynomials $\pi_N(z)$. Namely, for $m=2$, $\alpha=\gamma=1/2$, equation (4.11) becomes

$$(1-z^4)Y'' - 6z^3Y' + \big[N(N+5)z^2 - \nu(\nu-1)z^{-2}\big]Y = 0,$$

where $N = 4n+\nu$, $\nu \in \{0,1,2,3\}$. Evidently, for $N=4n$ and $N=4n+1$ ($n \in \mathbb{N}_0$), this equation reduces to (4.12).

5. Electrostatics and the zeros of orthogonal polynomials

An electrostatic interpretation of the zeros of Jacobi polynomials was given by Stieltjes in 1885 (see [**22**]–[**24**]). Namely, he considered an electrostatic problem with particles of charge p and q ($p,q>0$) fixed at $x=1$ and $x=-1$, respectively, and n unit charges confined to the interval $[-1,1]$ at points $x_1,x_2,\dots,x_n$. Assuming a logarithmic potential, Stieltjes proved that the electrostatic equilibrium arises when x_k are zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(x)$. In that case, the Hamiltonian

$$H(x_1,x_2,\dots,x_n) = -\sum_{k=1}^{n}\Big(\log(1-x_k)^p + \log(1+x_k)^q\Big) - \sum_{1\le k<j\le n}\log|x_k-x_j|$$

becomes a minimum. This minimum is indeed the unique global minimum (cf. Szegő [**25**, p. 140]). Obviously, $H(x_1, x_2, \dots, x_n)$ can be interpreted as the energy of the previous electrostatic system. Stieltjes' approach is closely connected with the calculation of the discriminant of the classical orthogonal polynomials (cf. [**16**, pp. 65–69]).

Let $x_k = x_k(\alpha, \beta)$ denote the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Then Markov inequalities (see e.g. Szegő [**25**, p. 121])

$$\frac{\partial x_\nu}{\partial \alpha} < 0, \qquad \frac{\partial x_\nu}{\partial \beta} > 0 \qquad (\nu = 1, 2, \dots, n) \tag{5.1}$$

hold. They can be verified very easy from Stieltjes' interpretation of the zeros of $P_n^{(\alpha,\beta)}(x)$. For example, when the charge at $x = -1$ is increased, the n unit charges are repelled towards the fixed charge at $x = 1$.

The cases of Laguerre and Hermite orthogonal polynomials are also treated (see Szegő [**25**, pp. 140–142]).

Forrester and Rogers [**2**] gave an interpretation of zeros of the classical polynomials as the equilibrium positions of two-dimensional electrostatic problems. Also, Hendriksen and Rossum [**4**] considered an electrostatic interpretation of zeros of classical orthogonal polynomials, including Bessel polynomials, as well as some polynomials introduced by Smith [**20**], [**21**]. Recently, the electrostatic interpretation of the zeros was also exploited to obtain interpolation points suitable for approximation of smooth functions defined on a simplex (see Hesthaven [**5**]).

Now, we consider a symmetric electrostatic problem with $2m$ positive point charges all of strength q which are placed at the fixed points

$$\xi_k = \exp\Big(\frac{k\pi i}{m}\Big) \qquad (k = 0, 1, \dots, 2m-1) \tag{5.1}$$

and a charge of strength p $(> -m + 1/2)$ at the origin $z = 0$. Also we have N positive free unit charges, positioned at $z_1, z_2, \dots, z_N$. Assuming a logarithmic potential, we try to find these points in electrostatic equilibrium.

In order to define the conditions for equilibrium we need some basic facts. At first, we consider the simplest electrostatic problem with a point charge of strength q placed at the origin. Assuming a logarithmic potential, the field force at the point M, with the radius vector $\boldsymbol{r}$, is given by

$$\mathbf{E} = \frac{Aq}{r}\,\boldsymbol{r}_0 = \frac{Aq}{r^2}\,\boldsymbol{r},$$

where A is a constant and $\boldsymbol{r}_0 = \boldsymbol{r}/r$. In a complex notation, it can be represented as

$$\frac{Aq}{z\bar{z}}\,z = \frac{Aq}{\bar{z}}\,, \tag{5.2}$$

where $z \in \mathbb{C}$ corresponds to the point M.

Now, using (5.2) we can state the equilibrium conditions for the previous symmetric electrostatic problem. Omitting the constant A and equating to zero the resultant force on the unit charge at z_ν $(\nu = 1, 2, \ldots, N)$, we obtain

$$q\Big(\frac{1}{\bar{z}_\nu - \bar{\xi}_0} + \frac{1}{\bar{z}_\nu - \bar{\xi}_1} + \cdots + \frac{1}{\bar{z}_\nu - \bar{\xi}_{2m-1}}\Big) + p\,\frac{1}{\bar{z}_\nu}$$
$$+ \frac{1}{\bar{z}_\nu - \bar{z}_1} + \frac{1}{\bar{z}_\nu - \bar{z}_2} + \cdots + \frac{1}{\bar{z}_\nu - \bar{z}_{\nu-1}} + \frac{1}{\bar{z}_\nu - \bar{z}_{\nu+1}} + \cdots + \frac{1}{\bar{z}_\nu - \bar{z}_N} = 0.$$

Put

$$\pi_N(z) = \prod_{k=1}^{N} (z - z_k) \tag{5.3}$$

and note that

$$z^{2m} - 1 = \prod_{k=0}^{2m-1} (z - \xi_k).$$

Lemma 5.1. *We have*

$$\sum_{\substack{k=1\\k\neq\nu}}^{N} \frac{1}{z_\nu - z_k} = \frac{\pi_N''(z_\nu)}{2\pi_N'(z_\nu)}$$

and

$$\sum_{k=0}^{2m-1} \frac{1}{z_\nu - \xi_k} = \frac{2m z_\nu^{2m-1}}{z_\nu^{2m} - 1}\,, \tag{5.4}$$

where $\pi_N(z)$ *is defined by* (5.3).

Proof. Let $\pi_N(z)$ be defined by (5.3). The logarithmic derivative gives

$$\frac{\pi_N'(z)}{\pi_N(z)} = \sum_{k=1}^{N} \frac{1}{z - z_k} \qquad \big(z \notin \{z_1, z_2, \ldots, z_N\}\big).$$

Then, for $z \notin \{z_1, z_2, \ldots, z_N\}$, we have

$$R_\nu(z) \equiv \sum_{\substack{k=1\\k\neq\nu}}^{N} \frac{1}{z - z_k} = \frac{\pi_N'(z)}{\pi_N(z)} - \frac{1}{z - z_\nu} = \frac{(z - z_\nu)\pi_N'(z) - \pi_N(z)}{(z - z_\nu)\pi_N(z)}\,.$$

In the limit case when $z \to z_\nu$, we obtain

$$\sum_{\substack{k=1\\k\neq\nu}}^{N} \frac{1}{z_\nu - z_k} = \lim_{z\to z_\nu} R_\nu(z) = \lim_{z\to z_\nu} \frac{\pi_N''(z_\nu)}{\dfrac{\pi_N(z)}{z - z_\nu} + \pi_N'(z)}\,,$$

i.e.,

$$\sum_{\substack{k=1\\k\neq\nu}}^{N}\frac{1}{z_\nu-z_k}=\frac{\pi_N''(z_\nu)}{2\pi_N'(z_\nu)}\,.$$

Similarly, we prove (5.4). □

Using Lemma 5.1, the previous equilibrium conditions can be represented in the simpler form

$$\frac{2mq\bar{z}_\nu^{2m-1}}{\bar{z}_\nu^{2m}-1}+\frac{p}{\bar{z}_\nu}+\frac{\overline{\pi_N''(z_\nu)}}{2\overline{\pi_N'(z_\nu)}}=0\qquad(\nu=1,\dots,N),$$

or

$$4mqz_\nu^{2m}\pi_N'(z_\nu)+2p\bigl(z_\nu^{2m}-1\bigr)\pi_N'(z_\nu)+z_\nu\bigl(z_\nu^{2m}-1\bigr)\pi_N''(z_\nu)=0\qquad(\nu=1,\dots,N).$$

Thus, we conclude that the polynomial $Q(z)$ defined by

$$Q(z)=z^2\bigl(1-z^{2m}\bigr)\pi_N''(z)+2\bigl[p-(p+2qm)z^{2m}\bigr]z\pi_N'(z)$$

has zeros at the points $z_1,z_2,\dots,z_N$ and its degree is $N+2m$. Then we must have

$$Q(z)+\Lambda_{2m}(z)\pi_N(z)=0, \tag{5.5}$$

where

$$\Lambda_{2m}(z)=a_{2m}z^{2m}+a_{2m-1}z^{2m-1}+\cdots+a_0.$$

We are interested here only in solutions with the rotational symmetry. Using this fact we can conclude that a such solution is an incomplete polynomial

$$\pi_N(z)=\sum_{k=0}^{n}b_kz^{N-2mk}=z^\nu\sum_{k=0}^{n}b_kz^{2m(n-k)}, \tag{5.6}$$

where $N=2mn+\nu$, $n=[N/(2m)]$, $\nu\in\{0,1,\dots,2m-1\}$. In that case the polynomial $\Lambda_{2m}(z)$ reduces also to an incomplete polynomial,

$$\Lambda_{2m}(z)=a_{2m}z^{2m}+a_0. \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5) and putting $z^{2m}=t$ we get the following identity

$$\begin{aligned}(1-t)&\sum_{k=0}^{n}(2mk+\nu)(2mk+\nu-1)b_kt^k\\&2\bigl[p-(p+2qm)t\bigr]\sum_{k=0}^{n}(2mk+\nu)b_kt^k+\bigl(a_{2m}t+a_0\bigr)\sum_{k=0}^{n}b_kt^k\equiv 0.\end{aligned}$$

Now, equating the coefficients of z^{N+2m} in the last identity, we obtain

$$-N(N-1)-(2p+4qm)N+a_{2m}=0,$$

i.e.,

$$a_{2m}=N\big[N+2(p+2qm)-1\big]. \tag{5.8}$$

Also, this identity for $t=0$ yields

$$a_0=-\nu(\nu+2p-1). \tag{5.9}$$

Thus, using (5.7), (5.8), and (5.9), equation (5.5) becomes

$$\begin{aligned}(1-z^{2m})z^2\pi_N''(z)&+2\big[p-(p+2qm)z^{2m}\big]z\pi_N'(z)\\&+\big[N\big(N+2(p+2qm)-1\big)z^{2m}-\nu(\nu+2p-1)\big]\pi_N(z)=0,\end{aligned} \tag{5.10}$$

where $N=2mn+\nu$, $n=[N/(2m)]$, $\nu\in\{0,1,\ldots,2m-1\}$.

Comparing (5.10) and (4.11) we find

$$\begin{aligned}&2m(\gamma-1)+2-2\big(m(\alpha+\gamma)+1\big)z^{2m}=2p-2(p+2qm)z^{2m},\\&N\big(N+2m(\alpha+\gamma)+1\big)=N\big(N+2(p+2qm)-1\big),\\&\nu\big(\nu+2m(\gamma-1)+1\big)=\nu(\nu+2p-1),\end{aligned}$$

i.e., $\alpha=2q-1$ and $\gamma=1+(p-1)/m$.

Thus, assuming a logarithmic potential, we proved the following result:

THEOREM 5.2. *An electrostatic system of $2m$ positive point charges all of strength q, which are placed at the fixed points ξ_k given by* (5.1)*, and a charge of strength p* ($>-m+1/2$) *at the origin $z=0$, as well as N positive free unit charges, positioned at z_1, z_2, $\ldots$, z_N, is in electrostatic equilibrium if these points z_k are zeros of the polynomial $\pi_N(z)$ orthogonal with respect to the inner product* (2.2)*, with the weight function $w(x)=(1-x^{2m})^{2q-1}x^{2m+2(p-1)}$.*

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