CONSTRUCTION OF CHAKALOV-POPOVICIU'S TYPE QUADRATURE FORMULAE*

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For quadrature formulae of Chakalov-Popoviciu's type with multiple nodes

$$\int_{\mathbb{R}} f(t)\,d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f),$$

which have the maximum degree of exactness $d = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1$, two numerical procedures for computation of coefficients $A_{i,\nu}$ are presented. A procedure for nodes τ_ν was given by Gori, Lo Cascio and Milovanović [4]. Similar procedures for Gauss-Turán quadratures have been done by authors [7] and Gautschi and Milovanović [3].

1. Introduction and Preliminaries

Given a nonnegative measure $d\lambda(t)$ on the real line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_k = \int_{\mathbb{R}} t^k\,d\lambda(t), \quad k = 0, 1, \ldots,$$

exist and are finite, and $\mu_0 > 0$. A quadrature formula of the form

$$\int_{\mathbb{R}} f(t)\,d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.1}$$

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which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, was considered firstly by P. Turán [13], in the case when $d\lambda(t) = dt$ on $[-1,1]$. The case $d\lambda(t) = \omega(t)\,dt$ on $[a,b]$ has been considered by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, etc. (see [4] for references).

The nodes τ_ν in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the following integral

$$F \equiv F(a_0, a_1, \ldots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\lambda(t),$$

where

$$\pi_n(t) = t^n + a_{n-1}t^{n-1} + \ \ldots\ + a_1 t + a_0.$$

In order to minimize F we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k\, d\lambda(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{1.2}$$

Such polynomials $\pi_n(t)$, which satisfies this type of orthogonality "*power orthogonality*" are known as s-orthogonal (or s-self associated) polynomials with respect to the measure $d\lambda(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

In this paper, we consider a generalization of Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities. Such formulas were derived, independently, by Chakalov [1–2] and Popoviciu [8]. A deep theoretical progress in this subject was made by Stancu [9–11] (see also [12]).

Let $n \in \mathbb{N}$ and let $\sigma = (s_1, s_2, \ldots, s_n)$ be a sequence of nonnegative integers.

In this case, it is important to assume that the nodes τ_ν are ordered, say

$$a \le \tau_1 < \tau_2 < \cdots < \tau_n \le b, \tag{1.3}$$

with odd multiplicities

$$2s_1 + 1, 2s_2 + 2, \ldots, 2s_n + 1,$$

respectively. Then the corresponding quadrature formula

$$\int_{\mathbb{R}} f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.4}$$

has the maximum degree of exactness

$$d_{\max} = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1, \tag{1.5}$$

if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1} t^k \, d\lambda(t) = 0, \qquad k = 0, 1, \dots, n-1. \tag{1.6}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov, Popoviciu, Morelli and Verna, and existence and uniqueness subject to (1.3) by Ghizzetti and Ossicini, Gautschi (see [4] for references).

The conditions (1.6) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in N_0}$,

$$\pi_{n,\sigma}(t) = \prod_{\nu=1}^{n} (t-\tau_\nu^{(n)}), \qquad a \le \tau_1^{(n)} < \tau_2^{(n)} < \cdots < \tau_n^{(n)} \le b,$$

such that

$$\int_{\mathbb{R}} \pi_{k,\sigma}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1} \, d\lambda(t) = 0, \qquad k = 0, 1, \dots, n-1.$$

These polynomials called σ–orthogonal polynomials and they correspond to the sequence $\sigma = (s_1, s_2, \dots)$. If we have $\sigma = (s, s, \dots)$, the above polynomials reduce to the s-orthogonal polynomials.

Recently Milovanović [5] (see also [6] and [3]) gave a stable procedure for numerical construction of s-orthogonal polynomials with respect to $d\lambda(t)$ on $\mathbb{R}$, taking advantage from the following interpretation of the "orthogonality conditions" (1.2):

$$\int_{\mathbb{R}} \pi_n(t) t^k \pi_n(t)^{2s} \, d\lambda(t) = 0, \qquad k = 0, 1, \ \dots \ , n-1,$$

i.e.,

$$\int_{R} \pi_k^{s,n}(t) t^\nu \, d\mu(t) = 0, \qquad \nu = 0, 1, \ \dots \ , k-1,$$

where $\{\pi_k^{s,n}\}_{k\in N_0}$ is a sequence of monic orthogonal polynomials with respect to the new measure

$$d\mu(t) = d\mu^{s,n}(t) = (\pi_n(t))^{2s} d\lambda(t).$$

Of course, we are interested only in $\pi_n(\cdot) = \pi_n^{s,n}(\cdot)$. Thus, we can see that the sequence of polynomials $\pi_k^{s,n}$, $k = 0, 1, \ \dots$, is implicitly defined. This approach to the σ-orthogonal polynomials was extended in [4], providing an algorithm for constructing such polynomials. For a given $\sigma = (s_1, s_2, \dots, s_n)$, the "orthogonality conditions" (1.6) can be interpreted as

$$\int_{\mathbb{R}} \pi_{k,\sigma}^{(n)}(t) t^i \, d\mu(t) = 0, \qquad i = 0, 1, \ \dots \ , k-1,$$

where

$$\pi_{n,\sigma}(t) = \prod_{\nu=1}^{n} (t - \tau_\nu^{(n)})$$

and

$$d\mu(t) = \prod_{\nu=1}^{n} (t - \tau_\nu^{(n)})^{2s_\nu} \, d\lambda(t).$$

Then, we can conclude that $\pi_{k,\sigma}^{(n)}$ is a sequence of standard orthogonal polynomials with respect to the measure $d\mu(t)$. These polynomials $\pi_{k,\sigma}^{(n)}(\cdot) \equiv \pi_k(\cdot)$ satisfy a three-term recurrence relation

$$\begin{aligned}
&\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2, \ldots \,, \\
&\pi_{-1}(t) = 0, \quad \pi_0(t) = 1,
\end{aligned}$$

where, because of orthogonality,

$$\alpha_k = \alpha_k(\sigma, k) = \frac{\langle t\pi_k, \pi_k \rangle}{\langle \pi_k, \pi_k \rangle}\,, \qquad \beta_k = \beta_k(\sigma, k) = \frac{\langle \pi_k, \pi_k \rangle}{\langle \pi_{k-1}, \pi_{k-1} \rangle}$$

and

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t) g(t) \, d\mu(t).$$

The coefficient β_0 is arbitrary, but sometimes it is convenient to take $\beta_0 = \langle 1, 1 \rangle = \int_{\mathbb{R}} d\mu(t)$.

Evidently, this orthogonality is defined implicitly, because of the fact that the measure $d\mu(t)$ depends on zeros of π_n and their multiplicities.

In order to find the coefficients α_k, β_k $(k = 0, 1, \ldots, n-1)$ Gori, Lo Cascio and Milovanović [4] considered the system of nonlinear equations

$$\begin{aligned}
&f_0 \equiv \beta_0 - \int_{\mathbb{R}} d\mu(t) = 0, \\
&f_{2k+1} \equiv \int_{\mathbb{R}} (\alpha_k - t)\pi_k^2(t) \, d\mu(t) = 0 \quad (k = 0, 1, \ldots, n-1), \\
&f_{2k} \equiv \int_{\mathbb{R}} (\beta_k \pi_{k-1}^2(t) - \pi_k^2(t)) \, d\mu(t) = 0 \quad (k = 1, \ldots, n-1),
\end{aligned} \tag{1.7}$$

and used a version of the secant method for its solving. The speed of convergence is superlinear. A problem of the choice of the strting points for this iterative process was discussed in [4].

All of the integrals in (1.7) can be calculated exactly, except for rounding errors, by using a Gauss-Christoffel quadrature formula with respect to the measure $d\lambda(t)$,

$$\int_{\mathbb{R}} g(t) \, d\lambda(t) = \sum_{k=1}^{N} A_k^{(N)} g(\tau_k^{(N)}) + R_N(g), \tag{1.8}$$

taking $N = n + \sum_{\nu=1}^{n} s_\nu$ nodes. This formula is exact for all polynomials g of degree at most

$$2N - 1 = 2\left(n + \sum_{\nu=1}^{n} s_\nu\right) - 1 = 2(n-1) + 1 + 2\sum_{\nu=1}^{n} s_\nu.$$

2. Numerical Procedures for Coefficients $A_{i,\nu}$

Some methods for determining the coefficients $A_{i,\nu}$ in generalized Gauss-Turán quadrature formula, in the special case $s_1 = s_2 = \cdots = s_n = s \in \mathbb{N}_0$, were given in [7], [3], [6]. Here, we give two methods for the numerical calculation of coefficients in (1.4).

1° The first method is a generalization of one from [7]. Basing on the Hermite interpolation we can obtain that

$$A_{i,\nu} = \frac{1}{i\,!} \sum_{k=0}^{2s_\nu - i} \frac{1}{k\,!} \left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu} \int_{\mathbb{R}} \frac{\Omega(t)}{(t-\tau_\nu)^{2s_\nu - i - k + 1}}\, d\,\lambda(t),$$

where

$$\Omega(t) = (t-\tau_1)^{2s_1+1}(t-\tau_2)^{2s_2+1}\cdots(t-\tau_n)^{2s_n+1}.$$

Denote

$$\Omega_{i,\nu,k}(t) = \frac{\Omega(t)}{(t-\tau_\nu)^{2s_\nu-i-k+1}} = (t-\tau_\nu)^{i+k} \times$$
$$\times (t-\tau_1)^{2s_1+1}\cdots(t-\tau_{\nu-1})^{2s_{\nu-1}+1}(t-\tau_{\nu+1})^{2s_{\nu+1}+1}\cdots(t-\tau_n)^{2s_n+1}.$$

For the degree of $\Omega_{i,\nu,k}$ we can conclude that

$$\begin{aligned}\deg(\Omega_{i,\nu,k}) &\le (2s_1+1) + \cdots + (2s_{\nu-1}+1) + 2s_\nu + (2s_{\nu+1}+1) + \cdots + (2s_n+1)\\ &= 2\sum_{\nu=1}^{n} s_\nu + n - 1 \le 2\Big(\sum_{\nu=1}^{n} s_\nu + n\Big) - 1 = 2N - 1 = d_{\max},\end{aligned}$$

where $N = \sum_{\nu=1}^{n} s_\nu + n$ i $d_{\max}$ given by (1.5).

Hence, we have

$$A_{i,\nu} = \frac{1}{i\,!} \sum_{k=0}^{2s_\nu - i} \frac{1}{k\,!} \left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu} \int_{\mathbb{R}} \Omega_{i,\nu,k}(t)\, d\lambda(t)$$
$$i = 0, 1, \ldots, 2s_\nu; \quad \nu = 1, \ldots, n \quad \text{and} \quad \deg(\Omega_{i,\nu,k}) \le 2N - 1.$$

Using the quadrature formula (1.8) with N nodes, the integrals

$$\int_{\mathbb{R}} \Omega_{i,\nu,k}(t)\, d\lambda(t)$$
$$i = 0, 1, \ldots, 2s_\nu;\ \nu = 1, \ldots, n;\ k = 0, 1, \ldots, 2s_\nu - i,$$

can be calculated exactly, except for rounding errors. For determining the derivatives (in $t = \tau_\nu$),

(2.1)
$$\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu}$$
$$i = 0, 1, \ldots, 2s_\nu;\ \nu = 1, \ldots, n;\ k = 0, 1, \ldots, 2s_\nu - i,$$

we use a numerical procedure based on the next two results.

Lemma 2.1. *If $g \in C^{(m)}(E)$, $m \in \mathbb{N}_0$, $E \subset \mathbb{R}$, then*

$$(e^g)^{(0)} = e^g, \quad (e^g)^{(p)} = \sum_{l=1}^{p} \binom{p-1}{l-1} g^{(l)} (e^g)^{(p-l)}, \quad p = 1, \ldots, m.$$

Theorem 2.2. *Let τ_ν, $\nu = 1, \ldots, n$, are the zeros of σ-orthogonal polynomial $\pi^{(n)}_{n,\sigma}$. Then, the coefficients of the generalized Gauss-Turán quadrature formula* (1.4) *can be expressed in the form*

$$A_{i,\nu} = \frac{1}{i\,!} \sum_{k=0}^{2s_\nu - i} \frac{1}{k\,!} \left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu} \int_{\mathbb{R}} \Omega_{i,\nu,k}(t)\, d\lambda(t),$$
$$i = 0, 1, \ldots, 2s_\nu; \quad \nu = 1, \ldots, n \quad \textit{and} \quad \deg(\Omega_{i,\nu,k}) \le 2N - 1,$$

and the derivatives (2.1) *in the form*

$$\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu} = (-1)^{n-\nu} \left[e^{g(t)}\right]^{(k)}_{t=\tau_\nu},$$

where

$$g^{(p)}(t) = -\sum_{\substack{1 \le j \le n \\ j \ne \nu}} g_j^{(p)}(t), \qquad p = 0, 1, \ldots,$$

with

$$g_j^{(0)}(t) = (2s_j + 1) \log|t - \tau_j|, \qquad g_j^{(p)}(t) = (-1)^{p-1} \frac{(2s_j+1)\,(p-1)!}{(t-\tau_j)^p}, \quad p \in \mathbb{N}.$$

Lemma 2.1 was proved in [7]. The proof of Theorem 2.2 can be done in a similar way as the corresponding theorem in [7].

2° Gautschi and Milovanović [3] proposed a stable method for determining the coefficients in (1.4) in a special case, when $s_\nu = s,\ \nu = 1,\dots,n,\ s \in \mathbb{N}_0$. We present now a generalization of this method to the general case when $s_\nu \in \mathbb{N}_0,\ \nu = 1,\dots,n$.

Firstly, we define

$$\Omega_\nu(t) = \prod_{i\neq\nu}(t-\tau_i)^{2s_i+1}, \qquad \nu = 1,\dots,n. \tag{2.2}$$

and use the polynomials

$$f_{k,\nu}(t) = (t-\tau_\nu)^k\Omega_\nu(t) = (t-\tau_\nu)^k\prod_{i\neq\nu}(t-\tau_i)^{2s_i+1}, \tag{2.3}$$

$$0 \le k \le 2s_\nu, \quad 1 \le \nu \le n.$$

Since the quadrature formula (1.4) is exact for all polynomials of degree at most

$$\sum_{i=1}^n(2s_i+1)+n-1 = 2\Big(\sum_{i=1}^n s_i + n\Big) - 1$$

and

$$\deg f_{k,\nu} = \sum_{i\neq\nu}(2s_i+1)+k \le \sum_{i=1}^n(2s_i+1)-1 = 2\sum_{i=1}^n s_i + n - 1,$$

we see that the integration (1.4) is exact for the polynomials (2.3), i.e.,

$$R(f_{k,\nu}) = 0, \qquad 0 \le k \le 2s_\nu, \quad 1 \le \nu \le n.$$

Thus, we have

$$\sum_{i=0}^{2s_j}\sum_{j=1}^n A_{i,j}f_{k,\nu}^{(i)}(\tau_j) = \int_{\mathbb{R}} f_{k,\nu}(t)\,d\lambda(t),$$

that is,

$$\sum_{i=0}^{2s_\nu} A_{i,\nu}f_{k,\nu}^{(i)}(\tau_\nu) = \mu_{k,\nu}, \tag{2.4}$$

because for every $j \neq \nu$ we have $f_{k,\nu}^{(i)}(\tau_j) = 0,\ 0 \le i \le 2s_j$. Here, we have put

$$\mu_{k,\nu} = \int_{\mathbb{R}} f_{k,\nu}(t)\,d\lambda(t) = \int_{\mathbb{R}}(t-\tau_\nu)^k\prod_{i\neq\nu}(t-\tau_i)^{2s_i+1}\,d\lambda(t).$$

For each ν we have in (2.4) a system of $2s_\nu + 1$ linear equations in the same number of unknowns, $A_{i,\nu},\ i = 0,1,\dots,2s_\nu$.

Using Leibniz's formula of differentiation, one easily proves the following auxiliary result.

Lemma 2.3. *For the polynomials $f_{k,\nu}$ given by (2.3) we have*

$$f_{k,\nu}^{(i)}(\tau_\nu) = \begin{cases} 0, & i < k, \\ i^{(k)}\Omega_\nu^{(i-k)}(\tau_\nu), & i \geq k, \end{cases}$$

where $i^{(k)} = i(i-1)\dots(i-k+1)$ (with $0^{(0)} = 1$) and Ω_ν is defined in (2.2).

Lemma 2.3 shows that each system of linear equations (2.4) is upper triangular. Thus, once all zeros of the σ-orthogonal polynomial π_n, i.e., the nodes of the quadrature formula (1.4), are known, the determination of its weights $A_{i,\nu}$ is reduced to solving the n linear systems of $2s_\nu + 1$ equations

$$\begin{bmatrix} f_{0,\nu}(\tau_\nu) & f'_{0,\nu}(\tau_\nu) & \dots & f_{0,\nu}^{(2s_\nu)}(\tau_\nu) \\ & f'_{1,\nu}(\tau_\nu) & \dots & f_{1,\nu}^{(2s_\nu)}(\tau_\nu) \\ & & \ddots & \\ & & & f_{2s_\nu,\nu}^{(2s_\nu)}(\tau_\nu) \end{bmatrix} \begin{bmatrix} A_{0,\nu} \\ A_{1,\nu} \\ \vdots \\ A_{2s_\nu,\nu} \end{bmatrix} = \begin{bmatrix} \mu_{0,\nu} \\ \mu_{1,\nu} \\ \vdots \\ \mu_{2s_\nu,\nu} \end{bmatrix}.$$

Put $a_{k,k+j} = f_{k-1,\nu}^{(k-1+j)}(\tau_\nu)$, so that the matrix of the system has elements $a_{l,j}$, $1 \leq l, j \leq 2s_\nu + 1$, with $a_{l,j} = 0$ for $j < l$. Then, by Lemma 2.3,

$$a_{l,j} = (j-1)^{(l-1)}\Omega_\nu^{(j-l)}(\tau_\nu), \qquad j \geq l;\ 1 \leq l, j \leq 2s_\nu + 1. \tag{2.5}$$

Lemma 2.4. *Let $\tau_1, \dots, \tau_n$ be the zeros of the σ-orthogonal polynomial π_n. For the elements $a_{l,j}$, defined by (2.5), the following relations hold:*

$$a_{k,k} = (k-1)!\, a_{1,1}, \quad 1 \leq k \leq 2s_\nu + 1,$$

$$a_{k,k+j} = -(k+j-1)^{(k-1)} \sum_{l=1}^{j} u_l a_{l,j}, \quad 1 \leq k \leq 2s_\nu - j + 1,\ j = 1, \dots, 2s_\nu,$$

where

$$\begin{aligned} a_{1,1} &= \Omega_\nu(\tau_\nu) = \prod_{i \neq \nu} (\tau_\nu - \tau_i)^{2s_i+1}, \\ u_l &= \sum_{i \neq \nu} (2s_i + 1)(\tau_i - \tau_\nu)^{-l}, \quad l = 1, \dots, 2s_\nu. \end{aligned} \tag{2.6}$$

Proof. The first relation is an immediate consequence of the definition of $a_{k,k}$ and Lemma 2.3. To prove the second, define $v(t) = \sum_{i \neq \nu} (2s_i + 1)(t - \tau_i)^{-1}$. Since $\Omega_\nu(t) = \prod_{i \neq \nu} (t - \tau_i)^{2s_i+1}$ we have that

$$\Omega'_\nu(t) = \sum_{i \neq \nu} (2s_i + 1) \frac{\prod_{j \neq \nu} (t - \tau_j)^{2s_j+1}}{t - \tau_i} = v(t)\Omega_\nu(t)$$

and

$$\Omega_\nu^{(j)}(t) = \frac{d^{j-1}}{dt^{j-1}}\left(\Omega_\nu'(t)\right) = \frac{d^{j-1}}{dt^{j-1}}\left(v(t)\Omega_\nu(t)\right)$$
$$= \sum_{l=0}^{j-1}\binom{j-1}{l}\Omega_\nu^{(j-1-l)}(t)v^{(l)}(t).$$

Then, (2.5) becomes

$$a_{k,k+j} = (k+j-1)^{(k-1)}\sum_{l=1}^{j}\binom{j-1}{l-1}\Omega_\nu^{(j-l)}(\tau_\nu)v^{(l-1)}(\tau_\nu).$$

Since

$$v^{(l-1)}(\tau_\nu) = (-1)^{l-1}(l-1)!\sum_{i\neq\nu}(2s_i+1)(\tau_\nu-\tau_i)^{-l} = -(l-1)!\,u_l$$

and

$$\Omega_\nu^{(j-l)}(\tau_\nu) = \frac{a_{l,j}}{(j-1)^{(l-1)}} = \frac{(j-l)!}{(j-1)!}\,a_{l,j},$$

we get

$$a_{k,k+j} = -(k+j-1)^{(k-1)}\sum_{l=1}^{j}u_l a_{l,j}. \quad \square$$

Using the normalization

$$\hat{a}_{k,j} = \frac{a_{k,j}}{(j-1)!\,a_{1,1}}, \qquad 1 \le k, j \le 2s_\nu+1, \tag{2.7}$$

and putting

$$b_k = (k-1)!\,A_{k-1,\nu}, \quad 1 \le k \le 2s_\nu+1,$$

(2.8)

$$\hat{\mu}_{k,\nu} = \frac{\mu_{k,\nu}}{a_{1,1}} = \frac{\mu_{k,\nu}}{\prod_{i\neq\nu}(\tau_\nu-t_i)^{2s_i+1}} = \int_R (t-\tau_\nu)^k \prod_{i\neq\nu}\left(\frac{t-\tau_i}{\tau_\nu-\tau_i}\right)^{2s_i+1} d\lambda(t),$$

we have the following result:

Theorem 2.5. *For fixed* ν, $1 \le \nu \le n$, *the coefficients* $A_{i,\nu}$ *in the generalized Gauss-Turán quadrature formula* (1.4) *are given by*

$$\begin{aligned} & b_{2s_\nu+1} = (2s_\nu)!\, A_{2s_\nu,\nu} = \hat{\mu}_{2s_\nu,\nu}, \\ & b_k = (k-1)!\, A_{k-1,\nu} = \hat{\mu}_{k-1,\nu} - \sum_{j=k+1}^{2s_\nu+1} \hat{a}_{k,j} b_j, \quad k = 2s_\nu, \ldots, 1, \end{aligned} \tag{2.9}$$

where $\hat{\mu}_{k,\nu}$ *are given by* (2.8), *and*

$$\hat{a}_{k,k} = 1, \quad \hat{a}_{k,k+j} = -\frac{1}{j} \sum_{l=1}^{j} u_l \hat{a}_{l,j}, \tag{2.10}$$

the u_l *being defined by* (2.6).

Proof. The relations (2.10) follow directly from Lemma 2.4 and the normalization (2.7).

The coefficients b_k, $1 \le k \le 2s_\nu + 1$, are obtained from the corresponding upper triangular system of equations $\hat{A}\vec{b} = \vec{c}$, where

$$\hat{A} = [\hat{a}_{ij}], \quad \vec{b} = [b_1, \ldots, b_{2s_\nu+1}]^\top, \quad \vec{c} = [\hat{\mu}_{0,\nu}, \ldots, \hat{\mu}_{2s_\nu,\nu}]^\top. \quad \square$$

The normalized moments $\hat{\mu}_{k,\nu}$ can be computed exactly, except for rounding errors, by using the same Gauss–Christoffel formula as in the construction of σ–orthogonal polynomials, i.e., (1.7) with $N = \sum_{\nu=1}^{n} s_\nu + n$ knots.

3. Numerical Example

We consider $d\lambda(t) = \exp(-t^2)$ on $(-\infty, \infty)$. For $\sigma = (2, 3, 1, 0)$ and $n = 2, 3, 4$ we obtained nodes presented in [4, Table 3.2]. The corresponding coefficients $A_{i,\nu}$ in (1.4) are given in Table 3.1. The program was realized in double precision arithmetics in FORTRAN 77. Numbers in parentheses denote decimal exponents.

(n,ν)	i	$A_{i,\nu}$	$A_{i+1,\nu}$
$(2,1)$	0	5.5750534971103(−01)	3.4778922593627(−01)
	2	1.1456073878170(−01)	1.9324854881444(−02)
	4	1.6831045076734(−03)	
$(2,2)$	0	1.2149485011945(+00)	−9.4898082628822(−01)
	2	4.6409822415796(−01)	−1.3936562310488(−01)
	4	2.9601486207523(−02)	−3.8946444133502(−03)
	6	3.0213805488220(−04)	
$(3,1)$	0	9.8928293038098(−02)	5.6292658871948(−02)
	2	1.4901698533911(−02)	1.9926066743596(−03)
	4	1.2337522705640(−04)	
$(3,2)$	0	1.6643430091052(+00)	−3.8422572567644(−01)
	2	3.3544868474452(−01)	−4.6647400978611(−02)
	4	2.0078549180029(−02)	−1.3347959561273(−03)
	6	3.6139855170650(−04)	
$(3,3)$	0	9.1825487621984(−03)	−2.6791280976776(−03)
	2	2.7691297098298(−04)	
$(4,1)$	0	3.3814580519967(−02)	1.8234555269879(−02)
	2	4.4272482400368(−03)	5.3818002728426(−04)
	4	2.9404522795306(−05)	
$(4,2)$	0	1.6731958587618(+00)	6.4256576039810(−02)
	2	2.9158368086251(−01)	5.4969167382666(−03)
	4	1.5689106119729(−02)	1.0149676939238(−04)
	6	2.6496909868643(−04)	
$(4,3)$	0	6.5415234972435(−02)	−1.7240365858891(−02)
	2	2.2465753694907(−03)	
$(4,4)$	0	2.8176651292167(−05)	

Finally, we consider the integral,

$$I = \int_{-\infty}^{\infty} e^{-t^2} \cos t \, dt,$$

whose exact value is

$$I = \sqrt{\pi}\exp(-1/4) = 1.38038844704314\ldots\ .$$

n	I_n	R_n
2	1.38038845047992	2.5(−09)
3	1.38038844704384	5.1(−13)
4	1.38038844704314	4.8(−15)

The Gauss-Turán quadrature formula (1.4) gives the results I_n, $n = 2,3,4$, showed in Table 3.2. The corresponding relative errors $R_n = |(I_n - I)/I|$ are given in the last column of this table.

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