SOME IMPROVEMENTS IN CALCULATION OF THE NATURAL LOG OF A SQUARE MATRIX*

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A computationally efficient algorithm for estimating the state-space model of a continuous-time multivariable system from data obtained with a discrete-time model is presented. Some accelerated series for the calculation of the natural log of a square matrix are given.

1. Introduction and Preliminaries

The identification of process parameters for control purposes must often be done, using a digital computer, from samples of input and output observations. The problem may be conveniently divided into two parts. The first part consists of determining a discrete-time model for the system from sampled data. The second part of the problem is the determination of the continuous-time system model from discrete-time model. Here, we focus our attention to the latter part of the problem. It may be noted that this part of the problem is basically deterministic, since the effect of the noise has already been taken into account in estimating the parameters of the discrete-time model.

We wish to obtain the continuous-time system representation

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t),$$
$$y(t) = \boldsymbol{C}x(t)$$

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determined from the discrete-time model

$$x_{k+1} = \boldsymbol{F} x_k + \boldsymbol{G} u_k,$$
$$y_k = \boldsymbol{C} x_k,$$

where $\boldsymbol{F} = e^{\boldsymbol{A}T}$ and

$$\boldsymbol{G} = \int_0^T e^{\boldsymbol{A}t} \boldsymbol{B}\, dt.$$

The matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{F}$ and $\boldsymbol{G}$ are real constant matrices of appropriate dimensions such that $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, and $x_k = x(kT)$, where T is the sampling interval.

The algorithm derived by Lastman, Puthenpura and Sinha [4] is based on the summing series

$$\boldsymbol{A}T = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \boldsymbol{L}^k}{k}, \tag{1.1}$$

where $\boldsymbol{L} = \boldsymbol{F} - \boldsymbol{I}$. This algorithm is convergent in the case when the matrix $\boldsymbol{L}$ has a spectral radius $\varrho(\boldsymbol{L})$ less than one. If $1/2 \le \varrho(\boldsymbol{L}) < 1$, the convergence can take numerous iterations causing the algorithm to be painfully slow.

A computational improvement on the previous algorithm was given by Cooper and Bingulac [1]. Using the following property of the logarithm function, $\log \boldsymbol{F} = r \log(\boldsymbol{F}^{1/r})$, with $r = 2^j$, and a Newton-like procedure for calculating the square root of a matrix, they provided a more general use of the algorithm. Namely, if there exists such j that

$$\varrho(\boldsymbol{F}^{1/r} - \boldsymbol{I}) < \frac{1}{2},$$

the corresponding series (1.1) has a satisfactory convergence even for matrices $\boldsymbol{L}$ with the spectral radius greater than one. In this paper we will give a few potential matrix series for $\log \boldsymbol{F}$ with faster convergence than Taylor series (1.1).

2. Accelerated Series

Suppose that $z > 0$ and introduce the bilinear transformation $z = (1-y)/(1+y)$. Than $|y| < 1$ and

$$\log z = -2 \sum_{k=1}^{\infty} \frac{y^{2k}}{2k+1}.$$

The corresponding matrix analogue is given by

$$\log \boldsymbol{F} = -2 \sum_{k=1}^{\infty} \frac{\boldsymbol{L}^{2k}}{2k+1}, \tag{2.1}$$

where

$$\boldsymbol{L} = (\boldsymbol{I} + \boldsymbol{F})^{-1}(\boldsymbol{I} - \boldsymbol{F}). \tag{2.2}$$

It is easy to see that (2.1) converges if the all eigenvalues of the matrix $\boldsymbol{F}$ are in the right half plane, i.e.

$$\operatorname{Re}\lambda_i(\boldsymbol{F}) > 0, \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

where n is the order of the square matrix $\boldsymbol{F}$. This expansion was considered by Lastman and Sinha [5].

Applying the Euler-Abel transformation (cf. Milovanović [6, pp. 49–50]) we obtain the following accelerated series

$$\log z = -\frac{2y}{1-y^2}\Big(1 - 2\sum_{k=1}^{\infty} \frac{y^{2k}}{4k^2-1}\Big). \tag{2.4}$$

Using (2.2) we can get the matrix analogue of (2.4) in the form

$$\log \boldsymbol{F} = -2\boldsymbol{L}(\boldsymbol{I} - \boldsymbol{L}^2)^{-1}\Big(\boldsymbol{I} - 2\sum_{k=1}^{\infty} \frac{\boldsymbol{L}^{2k}}{4k^2-1}\Big). \tag{2.5}$$

Repeating the same procedure we can find the following series with the faster convergence

$$\log \boldsymbol{F} = -\boldsymbol{E} + \frac{1}{3}\boldsymbol{L}\boldsymbol{E}^2\Bigg(\boldsymbol{I} - 12\sum_{k=1}^{\infty} \frac{\boldsymbol{L}^{2k}}{(2k+3)(4k^2-1)}\Bigg), \tag{2.6}$$

where $\boldsymbol{E} = 2\boldsymbol{L}(\boldsymbol{I} - \boldsymbol{L}^2)^{-1}$. The series (2.5) and (2.6) converge if the conditions (2.3) are satisfied. In these series we can also use the improvement as in Cooper and Bingulac [1], so that we take such j for which $\log \boldsymbol{F} = 2^j \log \tilde{\boldsymbol{F}}$, where $\tilde{\boldsymbol{F}} = \boldsymbol{F}^{1/2^j}$ and

$$\varrho((\boldsymbol{I} + \tilde{\boldsymbol{F}})^{-1}(\boldsymbol{I} - \tilde{\boldsymbol{F}})) < \frac{1}{2}\,. \tag{2.7}$$

Since in same cases the standard Newton procedure for solving the matrix equation $\boldsymbol{X}^2 = \boldsymbol{A}$ is numerically unstable, we use an alternative procedure:

$$\begin{aligned} \boldsymbol{P}_{k+1} &= \frac{1}{2}(\boldsymbol{P}_k + \boldsymbol{Q}_k^{-1}), \\ \boldsymbol{Q}_{k+1} &= \frac{1}{2}(\boldsymbol{Q}_k + \boldsymbol{P}_k^{-1}), \end{aligned} \tag{2.8}$$

starting with $\boldsymbol{P}_0 = \boldsymbol{A}$, $\boldsymbol{Q}_0 = \boldsymbol{I}$. This procedure was derived by Denman and Beavers [2] using the matrix sign function, and later analysed by Higham [3]. Under certain conditions the sequences $(\boldsymbol{P}_k)$ and $(\boldsymbol{Q}_k)$ converge to $\boldsymbol{A}^{1/2}$ and $\boldsymbol{A}^{-1/2}$, respectively.

3. Numerical Example

In order to illustrate the convergence of the series (1.1), (2.1), (2.5) and (2.6) we take the m-th partial sums $\boldsymbol{S}_m$ in them. Previously we use the square root procedure (2.8) to reduce the spectral radius in order to achieve the condition (2.7) for some j.

The following example, given in [4], illustrates the effectiveness of our algorithm. Let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix}, \qquad T = 2.$$

Then we have

$$\boldsymbol{F} = e^{\boldsymbol{A}T} = \begin{bmatrix} 0.554615 & 0.823320 & 0.404041 \\ -0.404041 & -0.253466 & 0.015239 \\ -0.015239 & -0.434519 & -0.283944 \end{bmatrix}.$$

In order to save space, we showed the elements of $\boldsymbol{F}$ only with six decimal places.

Using the matrix norm $\| \,.\, \|_\infty$, for truncated series (1.1), (2.1), (2.5) and (2.6) we obtain the relative errors

$$R_m = \frac{\|\boldsymbol{S}_m - \log \boldsymbol{F}\|_\infty}{\|\log \boldsymbol{F}\|_\infty},$$

which are presented in Table I. Numbers in parentheses indicate decimal exponents. We note that $\log \boldsymbol{F} = \boldsymbol{A}T$ and $\|\log \boldsymbol{F}\|_\infty = 10$.

TABLE I: Relative Errors R_m for $m = 1(1)7$

m	Series (1.1)	Series (2.1)	Series (2.5)	Series (2.6)
1	8.72(−2)	7.93(−4)	1.33(−5)	5.40(−7)
2	2.66(−2)	3.14(−5)	6.70(−7)	1.12(−8)
3	9.44(−3)	2.33(−6)	1.99(−8)	3.30(−10)
4	1.96(−3)	9.12(−8)	6.95(−10)	1.91(−11)
5	5.15(−4)	3.56(−9)	5.20(−11)	5.93(−13)
6	2.31(−4)	3.35(−10)	1.97(−12)	1.84(−14)
7	1.69(−4)	1.51(−11)	6.33(−14)	1.46(−15)

In all four cases the number of square root operations needed to obtain satisfactory convergence of the series was $j = 2$.

Table I shows the superior behaviour of the accelerated series (2.5) and (2.6) comparing to the Taylor series (1.1).

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