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ORTHOGONALITY ON THE RADIAL RAYS IN THE COMPLEX PLANE AND SOME APPLICATIONS*

G.V. Milovanović
Faculty of Electronic Engineering, Department of Mathematics, P.O. Box 73
18000 Niš, Serbia – Yugoslavia

Abstract

We give an account on polynomials orthogonal on the radial rays in the complex plane, which were introduced in [6], and consider several special cases of these polynomials which are appeared in some problems in physics. As an application, we mention a physical problem connected to a non-linear diffusion equation. Assuming a logarithmic potential, we give also an electrostatic interpretation of the zeros of our polynomials. In these applications we use the second order linear differential equations for these orthogonal polynomials.

1 Introduction

Orthogonal polynomials (see [1, 16]) are very attractive in many applications in mathematics, physics, and some other computational and applied sciences. In particular, classical orthogonal polynomials play very important role in many problems in approximation theory and numerical analysis, as well as in many problems in physics and technics. There are also problems connected with other (nonstandard) classes of orthogonal polynomials. This paper is devoted to one such class of orthogonal polynomials and corresponding applications.

It is interesting to mention a physical problem connected to a non-linear diffusion equation. The equations for the dispersion of a buoyant contaminant can be approximated by the Erdogan-Chatwin equation

$$\partial_t c = \partial_y \Big\{ \Big[D_0 + (\partial_y c)^2 D_2 \Big] \partial_y c \Big\}, \tag{1.1}$$

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where D_0 is the dispersion coefficient appropriate for neutrally-buoyant contaminants, and the D_2 term represents the increased rate of dispersion associated with the buoyancy-driven currents (cf. [11]). The same equation was derived also by other authors for other physical contexts.

Smith [11] obtained analytic expressions for the similarity solutions of the equation (1.1) in the limit of strong non-linearity ($D_0 = 0$), i.e.,

$$\partial_t c = D_2 \partial_y \left[(\partial_y c)^3 \right],$$

both for a concentration jump and for a finite discharge. He also investigated the asymptotic stability of these solutions. It is interesting that the stability analysis for the finite discharge involves a family of orthogonal polynomials $Y_N(z)$, such that

$$(1 - z^4) Y_N'' - 6z^3 Y_N' + N(N+5) z^2 Y_N = 0. \tag{1.2}$$

The degree N is restricted to the values $0, 1, 4, 5, 8, 9, \ldots$, so that the first few (monic) polynomials are:

$$1,\ z,\ z^4 - \frac{1}{3},\ z^5 - \frac{5}{11} z,\ z^8 - \frac{14}{17} z^4 + \frac{21}{221},\ z^9 - \frac{18}{19} z^5 + \frac{3}{19} z,\ \ldots\ . \tag{1.3}$$

We will show that these polynomials are a special case of polynomials orthogonal on the radial rays in the complex plane.

This paper is organized as follows. In Section 2 we give a short account on general properties of polynomials orthogonal on the radial rays. Section 3 is devoted to some special cases of these polynomials. Finally, an application of these polynomials in electrostatic is discussed in Section 4.

2 Orthogonal polynomials on the radial rays

Suppose that we have M points in the complex plane, $z_s = a_s e^{i\varphi_s} \in \mathbf{C}$, $s = 0, 1, \ldots, M-1$, with different arguments φ_s. Some of a_s (or all) can be ∞. The case $M = 6$ is shown in Figure 2.1.

We define an inner product on these radial rays ℓ_s in the complex plane which connect the origin $z = 0$ and the points z_s, $s = 0, 1, \ldots, M-1$. Namely,

$$(f, g) = \sum_{s=0}^{M-1} e^{-i\varphi_s} \int_{\ell_s} f(z) \overline{g(z)}\, |w(z)|\, dz,$$

where $z \mapsto w(z)$ is a suitable complex weight function. This product can be expressed in the form

$$(f, g) = \sum_{s=0}^{M-1} \int_0^{a_s} f(xe^{i\varphi_s}) \overline{g(xe^{i\varphi_s})}\, |w(xe^{i\varphi_s})|\, dx. \tag{2.1}$$

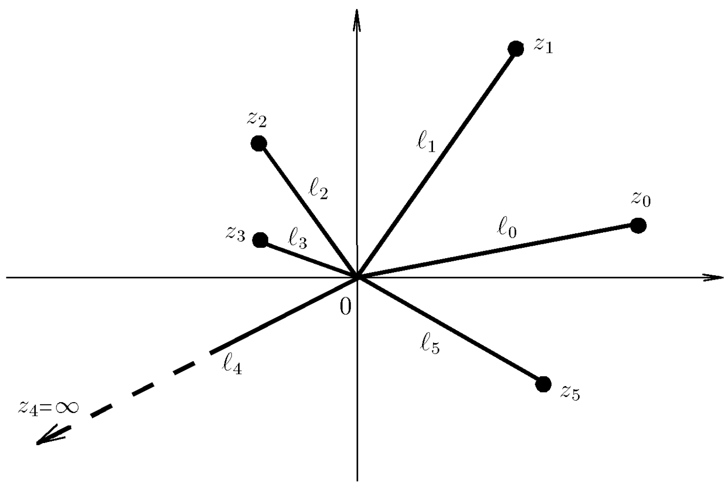

Figure 2.1: The rays in the complex plane ($M = 6$)

Notice that $\|f\|^2 = (f, f) > 0$, except when $f(z) = 0$.

The first consideration of such orthogonality was given in [6], with an even number of rays ($M = 2m$), $a_s = 1$, $\varphi_s = \pi s/m$, $s = 0, 1, \ldots, 2m-1$. The weight $z \mapsto w(z)$ was a holomorphic function such that $|w(x\varepsilon_s)| = w(x)$, $s = 0, 1, \ldots, 2m-1$, where $\varepsilon_s = \exp(i\varphi_s) = \exp(i\pi s/m)$ and $x \mapsto w(x)$ is a weight function on $(0, 1)$ ($w(x) \geq 0$ on $(0, 1)$ and $\int_0^1 w(x)\, dx > 0$). The inner product (2.1), in this symmetric case, reduces to

$$(f, g) = \int_0^1 \Big(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \Big) w(x)\, dx. \tag{2.2}$$

In [6] we investigated the existence and uniqueness of the (monic) polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product (2.2). Basing on the property of the inner product $(z^m f, g) = (f, z^m g)$, we can prove the following result:

Theorem 2.1 *Let the inner product* $(\cdot, \cdot)$ *be given by* (2.2) *and let the corresponding monic orthogonal polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *exist. Then they satisfy the recurrence relation*

$$\begin{aligned} \pi_{N+m}(z) &= z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \\ \pi_N(z) &= z^N, \quad N = 0, 1, \ldots, 2m-1, \end{aligned} \tag{2.3}$$

where

$$b_N = \frac{(\pi_N, z^m \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = \frac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}\,. \tag{2.4}$$

In a trivial case when $m = 1$, the inner product (2.2) becomes

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx,$$

and we have the standard case of polynomials orthogonal on $(-1, 1)$ with respect to the even weight function $x \mapsto w(x)$. Then (2.3) reduces to the well-known three-term recurrence relation for such kind of polynomials.

We consider now a simple case when $m = 2$ and $w(x) = (1 - x^4)^\alpha x^{4\gamma}$, where $\alpha > -1$ and $\gamma > -1/4$. The inner product (2.2) then becomes

$$(f, g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)}\,\Big]w(x)\,dx.$$

Using the moments determinants we can calculate directly the coefficient b_N in the recurrence relation (2.3):

$$b_{4n+\nu} = \begin{cases} \dfrac{n(n+\alpha)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } \ \nu = 0, 1 \\[2ex] \dfrac{(n+\beta_\nu)(n+\alpha+\beta_\nu)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } \ \nu = 2, 3. \end{cases} \tag{2.5}$$

Similar calculations for $w(x) = 1$ were given in [6].

Notice that $b_N \to 1/4$ as $N \to +\infty$, just like in Szegő's theory for orthogonal polynomials on the interval $(-1, 1)$.

Using the recurrence relation (2.3), i.e.,

$$\begin{aligned} \pi_{N+2}(z) &= z^2\pi_N(z) - b_N\pi_{N-2}(z), \quad N = 2, 3, \ldots, \\ \pi_0(z) &= 1,\ \pi_1(z) = z,\ \pi_2(z) = z^2,\ \pi_3(z) = z^3, \end{aligned}$$

and taking the coefficients b_N, given by (2.5), we obtain the corresponding sequence of orthogonal polynomials. For example, if we put $\alpha = \gamma = 0$, i.e., $w(x) = 1$, the first few polynomials are:

$$1,\ z,\ z^2,\ z^3,\ z^4 - \frac{1}{5},\ z^5 - \frac{3}{7}z,\ z^6 - \frac{5}{9}z^2,\ z^7 - \frac{7}{11}z^3,\ z^8 - \frac{10}{13}z^4 + \frac{5}{117},\ \ldots\ .$$

Similarly, for $\alpha = \gamma = 1/2$ we get

$$1,\ z,\ z^2,\ z^3,\ z^4 - \frac{1}{3},\ z^5 - \frac{5}{11}z,\ z^6 - \frac{7}{13}z^2,\ z^7 - \frac{3}{5}z^3,\ z^8 - \frac{14}{17}z^4 + \frac{21}{221},\ \ldots\ .$$

As we can see, the last polynomial sequence contains the sequence (1.3) given in Section 1. Namely, the polynomials $Y_N(z)$ are just our polynomials $\pi_N(z)$ for the particular values $\alpha = \gamma = 1/2$. Notice that our sequence of polynomials is complete.

Another cases when M is an arbitrary number of rays were investigated in [9, 10].

In the sequel we consider the case with the inner product (2.2), where the weight function w is given by

$$w(x) = (1 - x^{2m})^{\alpha} x^{2m\gamma}, \qquad \alpha > -1,\, \gamma > -\frac{1}{2m}\,. \tag{2.6}$$

Then we can prove ([6]:

Theorem 2.2 *The monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product* (2.2)*, with the weight function* (2.6)*, can be expressed in the form*

$$\pi_N(z) = 2^{-n} z^{\nu} \hat{P}_n^{(\alpha,\beta_\nu)}(2z^{2m} - 1), \quad N = 2mn + \nu,\ n = [N/2m], \tag{2.7}$$

where $\nu \in \{0, 1, \ldots, 2m-1\}$, $\beta_\nu = \gamma + (2\nu + 1 - 2m)/(2m)$, and $\hat{P}_n^{(\alpha,\beta)}(x)$ denotes the monic Jacobi polynomial orthogonal with respect to the weight $x \mapsto (1-x)^\alpha (1+x)^\beta$ on $(-1, 1)$. The polynomials $\pi_N(z)$ satisfy the recurrence relation (2.3)*, where*

$$b_{2mn+\nu} = \begin{cases} \dfrac{n(n+\alpha)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } 0 \le \nu \le m-1, \\[2ex] \dfrac{(n+\beta_\nu)(n+\alpha+\beta_\nu)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } m \le \nu \le 2m-1. \end{cases} \tag{2.8}$$

We can see that for $m = 2$ and $\alpha = \gamma = 1/2$, formulae (2.8) reduce to (2.5).

According to the previous theorem we see that for $N = 2mn + \nu$, where $\nu \in \{0, 1, \ldots, 2m-1\}$, we have an equality (up to a multiplicative constant) of the form

$$\pi_N(z) = \pi_N^{(\alpha,\gamma)}(z) \asymp z^{\nu} P_n^{(\alpha,\beta_\nu)}(2z^{2m} - 1),$$

where $\beta_\nu = \beta_0 + \nu/m$, $\beta_0 = \gamma - 1 + 1/(2m)$. For $N = 2mn$, i.e., $\nu = 0$,

$$\pi_N(z) = \pi_{2mn}^{(\alpha,\gamma)}(z) \asymp P_n^{(\alpha,\beta_0)}(2z^{2m} - 1), \tag{2.9}$$

and for an arbitrary ν, we have $\pi_{2mn+\nu}^{(\alpha,\gamma)}(z) \asymp z^{\nu} P_n^{(\alpha,\beta_0+\nu/m)}(2z^{2m} - 1)$, i.e.,

$$\pi_{2mn+\nu}^{(\alpha,\gamma)}(z) = z^{\nu} \pi_{2mn}^{(\alpha,\gamma+\nu/m)}(z). \tag{2.10}$$

In order to get a linear differential equation for $\pi_N(z)$ we take $N = 2mn$ and start from the corresponding differential equation for the Jacobi polynomials $y = P_n^{(\alpha,\beta)}(x)$,

$$(1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0, \tag{2.11}$$

where $\beta = \beta_0 = \gamma - 1 + 1/(2m)$. Putting $x = 2z^{2m} - 1$ and using (2.9), we find

$$\mathrm{D}P_n^{(\alpha,\beta)}(x) \asymp \pi_N'(z)\,\frac{dz}{dx}\,, \qquad \mathrm{D}^2 P_n^{(\alpha,\beta)}(x) \asymp \pi_N''(z)\Big(\frac{dz}{dx}\Big)^2 + \pi_N'(z)\,\frac{d^2 z}{dx^2}\,.$$

Then, according to (2.11), we obtain the following result:

Theorem 2.3 *The polynomials $\pi_N(z)$ orthogonal with respect to the inner product* (2.2)*, with the weight function* (2.6)*, satisfy a second order linear homogeneous differential equation of the form*

$$(1 - z^{2m})z^2 Y'' + C(z)zY' + Az^{2m}Y = 0, \tag{2.12}$$

where $N = 2mn$ and

$$A = N[N + 2m(\alpha + \gamma) + 1], \quad C(z) = 2[m(\gamma - 1) + 1 - (m(\alpha + \gamma) + 1)z^{2m}].$$

Now, it is easy to get the corresponding differential equation for each N ($= 2mn + \nu$). According to (2.10) and (2.12), with $\gamma := \gamma + \nu/m$, we obtain (see [7]):

Theorem 2.4 *The polynomials $\pi_N(z)$ orthogonal with respect to the inner product* (2.2)*, with the weight function* (2.6)*, satisfy a second order linear homogeneous differential equation of the form*

$$(1 - z^{2m})z^2 Y'' + C(z)zY' + (Az^{2m} - B)Y = 0, \tag{2.13}$$

where $N = 2mn + \nu$, $\nu \in \{0, 1, \ldots, 2m - 1\}$, and

$$\begin{aligned} C(z) &= 2[m(\gamma - 1) + 1 - (m(\alpha + \gamma) + 1)z^{2m}], \\ A &= N[N + 2m(\alpha + \gamma) + 1], \quad B = \nu[\nu + 2m(\gamma - 1) + 1]. \end{aligned}$$

Notice that for $m = 2$, $\alpha = \gamma = 1/2$, equation (2.13) becomes

$$(1 - z^4)Y'' - 6z^3 Y' + [N(N + 5)z^2 - \nu(\nu - 1)z^{-2}]Y = 0,$$

where $N = 4n + \nu$, $\nu \in \{0, 1, 2, 3\}$. Evidently, for $N = 4n$ and $N = 4n + 1$ ($n \in \mathbf{N}_0$), this equation reduces to equation (1.2 derived by Smith [11]. Some similar differential equations with polynomial solutions were also obtained by Smith [12].

A study of zero distribution for polynomials $\pi_N(z)$ was given in [10]. We located the regions in which these zeros are contained, and also we analyzed cases when the zeros are on the rays.

The case of infinite rays was considered in [5] and [7].

3 Electrostatic interpretation of zeros

For a symmetric case given by Theorem 2.1 we proved the following result ([6]):

Theorem 3.1 *Let $N = 2mn + \nu$, $n = [N/2m]$, $\nu \in \{0, 1, \ldots, 2m - 1\}$. All zeros of the polynomial $\pi_N(z)$ are simple and located symmetrically on the radial rays ℓ_s, $s = 0, 1, \ldots, 2m - 1$, with the possible exception of a multiple zero of order ν at the origin $z = 0$.*

An electrostatic interpretation of the zeros of Jacobi polynomials was given by Stieltjes in 1885 (see [13, 14, 15] for details). Stieltjes considered an electrostatic problem with particles of charge p and q ($p, q > 0$) fixed at $x = 1$ and $x = -1$, respectively, and n unit charges confined to the interval $[-1, 1]$ at points $x_1, x_2, \dots, x_n$. Assuming a logarithmic potential, he proved that the electrostatic equilibrium arises when x_k are zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(x)$. In that case, the Hamiltonian

$$H(x_1, x_2, \dots, x_n) = -\sum_{k=1}^{n}\Big(\log(1-x_k)^p + \log(1+x_k)^q\Big) - \sum_{1\le k<j\le n} \log|x_k - x_j|$$

becomes a minimum. Obviously, $H(x_1, x_2, \dots, x_n)$ can be interpreted as the energy of the previous electrostatic system. Stieltjes' approach is closely connected with the calculation of the discriminant of the classical orthogonal polynomials (cf. [8, pp. 65–69]).

In [7] we considered a symmetric electrostatic problem with $2m$ positive point charges all of strength q which are placed at the fixed points

$$\xi_k = \exp\Big(\frac{k\pi i}{m}\Big) \qquad (k = 0, 1, \dots, 2m-1) \tag{3.1}$$

and a charge of strength p ($> -m + 1/2$) at the origin $z = 0$. Also we have N positive free unit charges, positioned at z_1, z_2, …, z_N. Assuming a logarithmic potential, we found these points in electrostatic equilibrium.

Theorem 3.2 *An electrostatic system of* $2m$ *positive point charges all of strength* q*, which are placed at the fixed points* ξ_k *given by* (3.1)*, and a charge of strength* p ($> -m + 1/2$) *at the origin* $z = 0$*, as well as* N *positive free unit charges, positioned at* z_1*,* z_2*, …,* z_N*, is in electrostatic equilibrium if these points* z_k *are zeros of the polynomial* $\pi_N(z)$ *orthogonal with respect to the inner product* (2.2)*, with the weight function* $w(x) = (1 - x^{2m})^{2q-1} x^{2m+2(p-1)}$.

Some other references in this direction are [2, 3, 4].

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