Extremal Problems for Restricted Polynomial Classes in L^r Norm

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Abstract

We consider extremal problems of Markov-Bernstein type in integral norms, especially for restricted polynomial classes, which were introduced and studied by the late Professor Arun K. Varma. Beside some results on extremal problems of Markov and Bernstein type for the non-restricted polynomial class $\mathcal{P}_n$ in L^r norms, with a special attention to the case $r = 2$, we give a short account of L^2 inequalities of Markov type for curved majorants and on Bernstein inequalities in mixed norms. Also, we consider extremal problems for some classes of nonnegative polynomials on $[0, +\infty)$ and $[-1, 1]$ with respect to the generalized Laguerre and Jacobi measure, respectively.

1 Introduction

There are many results on extremal problems and inequalities of Markov-Bernstein type with algebraic polynomials. The first result of Markov type for polynomials of the second degree was connected with some investigations of the well-known Russian chemist Mendeleev [22]. A general case in the class $\mathcal{P}_n$ of all algebraic polynomials of degree at most n was considered by A. A. Markov [20]. Taking the uniform norm $\|f\|_\infty = \max\limits_{-1\le t\le 1} |f(t)|$ he solved the extremal problem

$$A_n = \sup_{P\in\mathcal{P}_n} \frac{\|P'\|_\infty}{\|P\|_\infty}.$$

The best constant is $A_n = n^2$ and the extremal polynomial $P^*(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant. The best constant can be expressed also as

$A_n = T_n(1)$. Thus, the classical Markov's inequality can be expressed in the form

$$\|P'\|_\infty \le n^2 \|P\|_\infty \qquad (P \in \mathcal{P}_n).$$

In 1892, younger brother V. A. Markov [21] found the best possible inequality for k-th derivative,

$$\|P^{(k)}\|_\infty \le T_n^{(k)}(1) \|P\|_\infty \qquad (P \in \mathcal{P}_n),$$

where the extremal polynomial is also T_n. The best constant can be expressed in the form

$$T_n^{(k)}(1) = \|T_n^{(k)}\|_\infty = \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (n^2 - i^2).$$

A version of this remarkable paper in German was published in 1916.

In 1912 Bernstein [2] considered another type of these inequalities taking $\|f\| = \max_{|z| \le 1} |f(z)|$. He proved the inequality

$$\|P'\| \le n \|P\| \qquad (P \in \mathcal{P}_n),$$

with equality case when $P(z) = cz^n$ (c is an arbitrary constant).

There are several different forms of this Bernstein's inequality. A standard form of that can be done as

$$|P'(t)| \le \frac{n}{\sqrt{1-t^2}}\,, \qquad -1 < t < 1. \tag{1}$$

The equality is attained at the points $t = t_\nu = \cos \frac{(2\nu-1)\pi}{2n}$, $\nu = 1, \ldots, n$, if and only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

Combining the inequalities of Markov and Bernstein we can state the following result:

Theorem 1 *If $P \in \mathcal{P}_n$ then*

$$|P'(t)| \le \min\Big\{ n^2, \frac{n}{\sqrt{1-t^2}} \Big\} \|P\|_\infty, \qquad -1 \le t \le 1.$$

Several monographs and papers have been published in this area (cf. Durand [8], Govil [11], [12], Milovanović [24], [25], Milovanović, Mitrinović and Rassias [28], [29], Mohapatra, O'Hara and Rodriguez [31], Rahman and Schmeisser [33], [34]).

In this survey we consider extremal problems of Markov-Bernstein type in integral norms, especially for restricted polynomial classes, which were introduced and studied by the late Professor Arun K. Varma. The paper is organized as follows. In Section 2 we give some definitions and preliminary results on extremal problems of Markov and Bernstein type for the non-restricted polynomial class $\mathcal{P}_n$ in L^r norms, with a special attention to the case $r = 2$. Sections 3 and 4 are devoted to L^2 inequalities of Markov type for curved majorants and Bernstein inequalities in mixed norms, respectively. In Section 5 we consider extremal problems for the class of nonnegative polynomials on $[0, +\infty)$ with respect to the generalized Laguerre measure. Finally, Section 6 is devoted to the corresponding extremal problems for Lorentz classes of nonnegative polynomials on the interval $(-1, 1)$ with respect to the Jacobi measure.

2 Extremal Problems in L^r Norm on $\mathcal{P}_n$

The classical Markov and Bernstein inequalities and corresponding extremal problems were generalized for various domains, various norms and for various subclasses for polynomials, both algebraic and trigonometric (for details see Chapter 6 in [29]).

Let

$$\|f\|_r = \Big(\int_{\mathbb{R}} |f(t)|^r \, d\lambda(t)\Big)^{1/r}, \quad r \geq 1, \tag{2}$$

where $d\lambda(t)$ is a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\lambda(t)$, $k = 0, 1, \ldots$, exist and are finite and $\mu_0 > 0$. In a special case $r = 2$, (2) reduces to

$$\|f\|_2 = \Big(\int_{\mathbb{R}} |f(t)|^2 \, d\lambda(t)\Big)^{1/2}. \tag{3}$$

In that case we have an inner product defined by

$$(f, g) = \int_{\mathbb{R}} f(t)\overline{g(t)} \, d\lambda(t)$$

such that $\|f\|_2 = \sqrt{(f, f)}$. Then also, there exists a unique set of (monic) orthogonal polynomials $\pi_k(\cdot) = \pi_k(\,\cdot\,; d\lambda)$, $k \geq 0$, with respect to $(\cdot, \cdot)$, such that

$$\pi_k(t) = t^k + \text{lower degree terms}, \qquad (\pi_k, \pi_m) = \|\pi_k\|_2^2 \, \delta_{km},$$

where δ_{km} is Kronecker's delta. In this paper we deal with the measures of the classical orthogonal polynomials $d\lambda(t) = w(t)\,dt$, where the weight function $t \mapsto w(t)$ satisfy the differential equation

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where

$$A(t) = \begin{cases} 1-t^2, & \text{if } (a,b)=(-1,1), \\ t, & \text{if } (a,b)=(0,+\infty), \\ 1, & \text{if } (a,b)=(-\infty,+\infty), \end{cases} \tag{4}$$

and $B(t)$ is a polynomial of the first degree. For such classical weights we will write $w \in CW$.

Based on this definition, the classical orthogonal polynomials $\{Q_k\}$ on (a,b) can be specificated as the *Jacobi polynomials* $P_k^{(\alpha,\beta)}(t)$ $(\alpha,\beta > -1)$ on $(-1,1)$, the *generalized Laguerre polynomials* $L_k^s(t)$ $(s > -1)$ on $(0,+\infty)$, and finally as the *Hermite polynomials* $H_k(t)$ on $(-\infty,+\infty)$. The classical orthogonal polynomial $Q_k(t)$ satisfies the second order linear differential equation of hyphergeometric type $A(t)y'' + B(t)y' + \lambda_k y = 0$, where λ_k is a constant. The weight functions, the constants λ_k and the corresponding polynomials $B(t)$ are given in Table 1.

TABLE 1

(a,b)	$w(t)$	$B(t)$	λ_k
$(-1,1)$	$(1-t)^\alpha(1+t)^\beta$	$\beta-\alpha-(\alpha+\beta+2)t$	$k(k+\alpha+\beta+1)$
$(0,+\infty)$	$t^s e^{-t}$	$s+1-t$	k
$(-\infty,+\infty)$	e^{-t^2}	$-2t$	$2k$

The first results on extremal problems in the L^2-norm and corresponding Markov's inequalities

$$\|P'\|_2 \le A_n \|P\|_2 \qquad (P \in \mathcal{P}_n), \tag{5}$$

were given by E. Schmidt [35] and Turán [37]:

Theorem 2 *Let* $\|\,.\,\|_2$ *be defined by* (3). (a) *If* $(a,b) = (-\infty,+\infty)$ *and* $d\lambda(t) = e^{-t^2}\,dt$ *the best constant in* (5) *is given by* $A_n = \sqrt{2n}$. *An extremal polynomial is Hermite's polynomial* H_n.

(b) *Let* $(a,b) = (0,+\infty)$ *and* $d\lambda(t) = e^{-t}\,dt$. *Then* (5) *holds with*

$$A_n = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

The extremal polynomial is

$$P(t) = \sum_{\nu=1}^{n} \sin\frac{\nu\pi}{2n+1} L_\nu(t),$$

where L_ν *is Laguerre polynomial.*

Theorem 2 (b), in this form, was formulated by Turán [37].

An important generalization of A. A. Markov's inequality for algebraic polynomials in an integral norm was given by Hille, Szegő, and Tamarkin [16], who proved the following result:

Theorem 3 *Let $r \geq 1$, $(a, b) = (-1, 1)$, $P \in \mathcal{P}_n$, and let $\|\,.\,\|_r$ be given by* (2)*. Then*

$$\|P'\|_r \leq An^2 \|P\|_r, \tag{6}$$

where the constant $A = A(n, r)$ is given by

$$A(n, r) = 2(r-1)^{1/r-1}\Big(r + \frac{1}{n}\Big)\Big(1 + \frac{r}{nr - r + 1}\Big)^{n-1+1/r},$$

for $r > 1$, and

$$A(n, 1) = 2\Big(1 + \frac{1}{n}\Big)^{n+1}.$$

The factor n^2 in (6) cannot be replaced by any function tending to infinity more slowly. Namely, for each n, there exist polynomials $P(t)$ of degree n such that the left side of (6) is $\leq Bn^2$, where B is a constant of the same nature as $A = A(n, r)$.

The constant $A(n, r)$ in Theorem 3 is not the best possible. We can see that $A(n, r) \leq 6 \exp(1 + 1/e)$, for every n and $r \geq 1$. Also,

$$A(n, r) \;\to\; \begin{cases} 2(1 + 1/(n-1))^{n-1} < 2e & (n \;\text{ fixed},\; r \to +\infty), \\ 2e & (r = 1,\; n \to +\infty), \\ 2er(r-1)^{(1/r)-1} & (r > 1 \;\text{ fixed},\; n \to +\infty). \end{cases}$$

Some improvements of the constant $A(n, r)$ have recently been obtained by Goetgheluck [10]. He found that

$$A(n, 1) = \sqrt{\frac{8}{\pi}}\Big(1 + \frac{3}{4n}\Big)^2,$$

as well as a very complicated expression for $r > 1$.

Recently Guessab and Milovanović [14] have considered a weighted L^2-analogues of the Bernstein's inequality (see Theorem 1), which can be stated in the following form:

$$\|\sqrt{1-t^2}\, P'(t)\|_\infty \leq n\|P\|_\infty. \tag{7}$$

Using the norm $\|f\|^2 = (f, f)$, with $w \in CW$, they determined the best constant $C_{n,m}(w)$ $(1 \leq m \leq n)$ in inequality

$$\|A^{m/2} P^{(m)}\| \leq C_{n,m}(w)\|P\|, \tag{8}$$

where A is defined by (4).

Theorem 4 *For all polynomials $P \in \mathcal{P}_n$ the inequality* (8) *holds, with the best constant*

$$C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}},$$

where $\lambda_{n,k} = -(n-k)\big(\frac{1}{2}(n+k-1)A''(0) + B'(0)\big)$.

The equality is attained in (8) *if and only if P is a constant multiple of the classical polynomial $Q_n(t)$ orthogonal with respect to the weight function $w \in CW$.*

We note that $\lambda_{n,0} = \lambda_n$, where λ_n is given in Table 2.1. In some special cases we have:

(1) Let $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$ on $(-1,1)$ (Jacobi case). Then

$$\|(1-t^2)^{m/2}P^{(m)}\| \le \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\ \|P\|,$$

with equality if and only if $P(t) = cP_n^{(\alpha,\beta)}(t)$.

(2) Let $w(t) = t^s e^{-t}$ $(s > -1)$ on $(0, +\infty)$ (generalized Laguerre case). Then

$$\|t^{m/2}P^{(m)}\| \le \sqrt{n!/(n-m)!}\ \|P\|,$$

with equality if and only if $P(t) = cL_n^s(t)$.

(3) The Hermite case with the weight $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$ is the simplest. Then the best constant is $C_{n,m}(w) = 2^{m/2}\sqrt{n!/(n-m)!}$.

In connection with the previous results is also the following characterization of the classical orthogonal polynomials given by Agarwal and Milovanović [1].

Theorem 5 *For all $P \in \mathcal{P}_n$ the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}\,P'\|^2 \le \lambda_n^2\|P\|^2 + \|AP''\|^2 \tag{9}$$

holds, with equality if only if $P(t) = cQ_n(t)$, where Q_n is the classical orthogonal polynomial with respect to the weight function $w \in CW$ and c is an arbitrary real constant.

The Hermite case was considered by Varma [45]. Then, the inequality (9) reduces to

$$\|P'\|^2 \le \frac{1}{2(2n-1)}\ \|P''\|^2 + \frac{2n^2}{2n-1}\ \|P\|^2.$$

In the generalized Laguerre case, the inequality (9) becomes

$$\|\sqrt{t}\,P'\|^2 \le \frac{n^2}{2n-1}\;\|P\|^2 + \frac{1}{2n-1}\;\|tP''\|^2,$$

where $w(t) = t^s e^{-t}$ on $(0, +\infty)$.

In the Jacobi case the inequality (9) reduces to the inequality

$$\begin{aligned}\big((2n-1)(\alpha+\beta) \quad &+ 2(n^2+n-1)\big)\|\sqrt{1-t^2}\,P'\|^2 \\ &\le n^2(n+\alpha+\beta+1)^2\|P\|^2 + \|(1-t^2)P''\|^2.\end{aligned}$$

In the simplest case, when $\alpha = \beta = 0$ (Legendre case), we obtain

$$\|\sqrt{1-t^2}\,P'\|^2 \le \frac{n^2(n+1)^2}{2(n^2+n-1)}\;\|P\|^2 + \frac{1}{2(n^2+n-1)}\;\|(1-t^2)P''\|^2.$$

In the Chebyshev case ($\alpha = \beta = -1/2$), we get

$$\|\sqrt{1-t^2}\,P'\|^2 \le \frac{n^4}{2n^2-1}\;\|P\|^2 + \frac{1}{2n^2-1}\;\|(1-t^2)P''\|^2,$$

where $\|f\|^2 = \int_{-1}^{1}(1-t^2)^{-1/2}f(t)^2\,dt$.

The corresponding result for trigonometric polynomials was obtained by Varma [47].

3 L^2 Inequalities of Markov Type for Curved Majorants

Answering to a question of P. Turán[1], Rahman, Pirrre and Rahman, Rahman and Schmeisser, Videnskiĭ and others (see Chapter 6 in [29]) gave several inequalities in the uniform norm on $[-1, 1]$. The first who started with the corresponding inequalities in L^2 norm was Professor Varma. In order to present his results, at first, for polynomials $P \in \mathcal{P}_n$ we define

$$\|P\|_* = \sup_{1<t<1} \frac{|P(t)|}{\sqrt{1-t^2}}\,, \tag{10}$$

or generally,

$$\|P\|_\varphi = \sup_{1<t<1} \frac{|P(t)|}{\varphi(t)}\,,$$

[1]Professor Paul Turán asked this question in 1970 at a conference on *Constructive Function Theory* held in Varna, Bulgaria.

where the majorant $t \mapsto \varphi(t)$ is a nonnegative function on $[-1,1]$. Taking a norm $\|\,.\,\|$ for polynomials on $[-1,1]$, Turán's problem can be stated in the form: *If* $\|P\|_* \le 1$, *or* $\|P\|_\varphi \le 1$, *how large can* $\|P^{(m)}\|$ *be?*

In the case of uniform norm, a general majorant

$$\varphi(t) = (1-t)^{\lambda/2}(1+t)^{\mu/2},$$

where λ, μ are nonnegative integers, was used by Pierre and Rahman [32]. Varma [46], [49] and Varma, Mills and Smith [50] considered L^2 inequalities using the majorants

$$\varphi(t) = \sqrt{1-t^2} \qquad \text{(circular majorant)}$$

and

$$\varphi(t) = 1-t^2 \qquad \text{(parabolic majorant)}.$$

For a circular majorant Varma [46] proved:

Theorem 6 *Let* $P \in \mathcal{P}_{n+1}$ $(n \ge 2)$ *and let* $\|\,.\,\|_*$ *be defined by* (10). *If* $\|P\|_* \le 1$ *then*

$$\int_{-1}^{1} (P'(t))^2 \sqrt{1-t^2}\, dt \le \frac{\pi}{4}(n^2+1),$$

with equality if $P(t) = p_0(t) = (1-t^2)U_{n-1}(t)$, $U_k(t) = \sin(k+1)\theta/\sin\theta$, $t = \cos\theta$.

Under same conditions, Varma [46] also proved the following inequality

$$\int_{-1}^{1} (P'(t))^2\, dt \le \frac{2n^2(2n^2-1)}{4n^2-1} + 2 + 4\sum_{k=1}^{n} \frac{1}{2k-1}\,,$$

which is at least asymptotically best possible.

Recently, Varma [49] proved:

Theorem 7 *Under same conditions as in the previous theorem, we have*

$$\int_{-1}^{1} (P^{(j)}(t))^2(1-t^2)^{1/2}\, dt \le \int_{-1}^{1} (p_0^{(j)}(t))^2(1-t^2)^{1/2}\, dt \qquad (j=2,3)$$

and

$$\int_{-1}^{1} (P'(t))^2(1-t^2)^{-1/2}\, dt \le \int_{-1}^{1} (p_0'(t))^2(1-t^2)^{-1/2}\, dt,$$

with equality if $P(t) = (1-t^2)U_{n-1}(t)$.

In the L^2 norm for real algebraic polynomials of degree $n+2$ that have the parabolic majorant

$$|P(t)| \le 1 - t^2, \qquad -1 \le t \le 1, \tag{11}$$

Varma, Mills and Smith [50] proved the following results:

Theorem 8 *If $P \in \mathcal{P}_{n+2}$ $(n \ge 1)$ and (11) is satisfied then*

$$\int_{-1}^{1} (P'(t))^2 \, dt \le \int_{-1}^{1} (q_0'(t))^2 \, dt, \tag{12}$$

where $q_0(t) = \pm(1-t^2)T_n(t)$, $T_n(t) = \cos n\theta$ and $t = \cos\theta$. Further, equality in (12) occurs if and only if $P(t) = q_0(t)$.

Theorem 9 *If $P \in \mathcal{P}_{n+2}$ $(n \ge 1)$ and (11) is satisfied then*

$$\int_{-1}^{1} (P''(t))^2 \, dt \le \int_{-1}^{1} (q_0''(t))^2 \, dt,$$

with equality if and only if $P(t) = q_0(t)$.

Similarly, Varma [49] proved:

Theorem 10 *Let P be any member of the set of those algebraic polynomials of degree $n+2$ which have only real zeros, all of them in the interval $[-1,1]$, and for which (11) is satisfied. Then*

$$\int_{-1}^{1} (P'(t))^2 (1-t^2)^{-1/2} \, dt \le \int_{-1}^{1} (q_0'(t))^2 (1-t^2)^{-1/2} \, dt$$

and

$$\int_{-1}^{1} (P''(t))^2 (1-t^2)^{1/2} \, dt \le \int_{-1}^{1} (q_0''(t))^2 (1-t^2)^{1/2} \, dt,$$

with equalities if and only if $P(t) = q_0(t) = \pm(1-t^2)T_n(t)$.

Theorem 11 *If $P \in \mathcal{P}_{n+2}$ $(n \ge 1)$ and (11) is satisfied then*

$$\int_{-1}^{1} (P'''(t))^2 (1-t^2)^{1/2} \, dt \le \int_{-1}^{1} (q_0'''(t))^2 (1-t^2)^{1/2} \, dt,$$

with equality if and only if $P(t) = q_0(t) = \pm(1-t^2)T_n(t)$.

It is interesting to remark that the last theorem does not require that the zeros of $P(t)$ are real and lie inside $[-1,1]$.

Also, we mention here an interesting auxiliary result proved by Varma [49]:

Theorem 12 *Let $Q \in \mathcal{P}_{n-1}$ and let*

$$|Q(t)| \le (1-t^2)^{-1/2}, \qquad -1 < t < 1.$$

Then

$$\int_{-1}^{1} (Q'(t))^2 (1-t^2)^{3/2}\, dt \le \frac{\pi}{2}(n^2-1),$$

with equality if and only if $Q(t) = \pm \sin n\theta / \sin\theta$, $t = \cos\theta$.

4 Bernstein Inequality in Mixed Norms

In order to find L^2 generalizations of Bernstein inequality (1), i.e., (7), Varma [48] considered the class H_n of all real polynomials of degree n bounded by 1 on the interval $[-1,1]$ and proved:

Theorem 13 *If $P \in H_n$ then we have*

$$\int_{-1}^{1} (1-t^2)(P'(t))^2\, dt \le n^2\Big(1 + \frac{1}{4n^2-1}\Big) = \int_{-1}^{1} (1-t^2)(T_n'(t))^2\, dt,$$

with equality only for $P(t) = \pm T_n(t)$.

This result can be interpreted in the form (7) using mixed norms $\|\,.\,\|_\infty$ and $\|\,.\,\|_2$, defined by (3) with $d\lambda(t) = dt$ on the interval $(-1,1)$. Thus, for for all $P \in \mathcal{P}_n$ we have

$$\|A^{1/2}P'\|_2 \le C_n \|P\|_\infty, \tag{13}$$

where A is defined by (4) and

$$C_n = \left(\int_{-1}^{1} (1-t^2)(T_n'(t))^2\, dt\right)^{1/2} = \frac{2n^2}{\sqrt{4n^2-1}}\,.$$

Defining

$$\|f\|_\alpha = \left(\int_{-1}^{1} f(t)^2 (1-t^2)^\alpha\, dt\right)^{1/2} \qquad (\alpha > -1),$$

we can consider inequalities of the form

$$\|A^{1/2}P'\|_{\alpha-1} = \|P'\|_\alpha \le C_n(\alpha)\|P\|_\infty, \tag{14}$$

Using the Varma's method, Shen [36] proved that (14) holds for $\alpha = 3/2$ and $\alpha = 1/2$, with best constants

$$C_n(3/2) = \left(\int_{-1}^{1}(1-t^2)^{3/2}(T_n'(t))^2\,dt\right)^{1/2} = \frac{n\sqrt{\pi}}{2}$$

and

$$C_n(1/2) = \left(\int_{-1}^{1}(1-t^2)^{1/2}(T_n'(t))^2\,dt\right)^{1/2} = n\sqrt{\frac{\pi}{2}},$$

respectively. Furthermore, it was proved that this inequality holds also for $\alpha = -1/2$ and $\alpha = 0$. Namely, applying the n-point Gauss-Chebyshev quadrature formula to the integral $\|P'\|^2_{-1/2}$, we have

$$\int_{-1}^{1}(1-t^2)^{-1/2}(P'(t))^2\,dt = \frac{\pi}{n}\sum_{\nu=1}^{n}(P'(\tau_\nu))^2, \tag{15}$$

where $\tau_\nu = \cos((2\nu-1)\pi/(2n))$, $\nu = 1,\ldots,n$ are zeros the Chebyshev polynomial T_n. Then by Bernstein's inequality (1), it follows that

$$|P'(\tau_\nu)| \le n(1-\tau_\nu^2)^{-1/2}, \qquad \nu = 1,\ldots,n.$$

Since $T_n'(\tau_\nu) = n(1-\tau_\nu^2)^{-1/2}(-1)^{\nu-1}$, $\nu = 1,\ldots,n$, we have that $|P'(\tau_\nu)| \le |T'(\tau_\nu)|$. Using (15) we conclude that (14) is valid for $\alpha = -1/2$. Here, $C_n(-1/2) = n\sqrt{n\pi}$.

The case $\alpha = 0$ follows from the following theorem (for $r = 2$) proved by Bojanov [3]:

Theorem 14 *Let $P \in \mathcal{P}_n$ and $r \in [1,+\infty)$. Then*

$$\|P'\|_r \le \|T_n'\|_r\|P\|_\infty.$$

Equality is attained only for $P(t) = cT_n(t)$, where c is an arbitrary constant.

Since

$$\int_{-1}^{1}(T_n'(t))^2\,dt = 2n^2\sum_{k=1}^{n}\frac{1}{2k-1}$$

we find that

$$C_n(0) = n\sqrt{\sum_{k=1}^{n}\frac{2}{2k-1}}\,.$$

Recently, Bojanov [4] proved (14) for $|\alpha| \le 1/2$ with the best constant $C_n(\alpha) = \|T_n'\|_\alpha$. Also, he considered the corresponding problem for higher derivatives. The inequalities with the second derivative were investigated by Varma [48] and [49].

Theorem 15 *If $P \in H_n$ then we have*

$$\int_{-1}^{1} (P''(t))^2 (1-t^2)^{-1/2}\, dt \le \int_{-1}^{1} (T_n''(t))^2 (1-t^2)^{-1/2}\, dt.$$

Theorem 16 *If $P \in H_n$ then we have*

$$\int_{-1}^{1} (P''(t))^2 (1-t^2)^{3/2}\, dt \le \int_{-1}^{1} (T_n''(t))^2 (1-t^2)^{3/2}\, dt = \frac{\pi}{2}\, n^2(n^2-1).$$

In order to prove the last theorem, Varma [49] used the Bernstein inequality (1) and Theorem 12 with $Q(t) = P'(t)/n$.

Theorem 13 can be also interpreted in terms of trigonometric polynomials t_n of degree n with real coefficients such that

$$\|t_n\|_\infty = \max_{0 \le \theta \le \pi} |t_n(\theta)| \le 1.$$

Theorem 17 *Let t_n be a trigonometric polynomial of degree n with real coefficients such that $\|t_n\|_\infty \le 1$. Then*

$$\int_0^\pi (t_n'(\theta))^2 \sin\theta\, d\theta \le n^2\Big(1 + \frac{1}{4n^2-1}\Big),$$

with equality only for $t_n(t) = \pm\cos n\theta$.

Putting $t_n(\theta) = P(\cos\theta)$, this result reduces to Theorem 13.

Recently, Chen [5] investigated the following quantity:

$$\sup_{\|t_n\|_\infty \le 1} \int_0^\pi (t_n^{(k)}(\theta))^2 w(\theta)\, d\theta, \tag{16}$$

where $w(\theta) = \sin^j\theta$ and j is a positive integer. The case $k = 1$ and $j = 2$ was investigated by Shen [36]. The solution of (16) gives us the best constant in (14) for $\alpha = (j+1)/2$. For example, we have

$$C_n(2) = n\sqrt{\frac{2}{3}\Big(1 - \frac{9}{(4n^2-1)(4n^2-9)}\Big)}$$

and

$$C_n(5/2) = \frac{n}{4}\sqrt{3\pi} \qquad (n \ge 3).$$

At the end of this section we mention that there are several opposite inequalities of the previous (see Labelle [18], Lupaş [19], Daugavet and Rafal'son [7], Konjagin [17], etc.).

5 L^2 Inequalities With Generalized Laguerre Measure for Nonnegative Polynomials

Several results on inequalities of Markov and Turán type in L^2 norm on the restricted polynomial classes were obtained by Professor A. K. Varma [39]–[44] and [51].

In 1981 Varma [43] investigated the problem of determining the best constant $C_n(\alpha)$ in the L^2 inequality

$$\|P'\|^2 \le C_n(\alpha)\|P\|^2, \tag{17}$$

for polynomials with nonnegative coefficients, with respect to the generalized Laguerre weight function $t \mapsto w(t) = t^\alpha e^{-t}$ $(\alpha > -1)$ on $[0, +\infty)$.

Theorem 18 *Let P_n be an algebraic polynomial of degree exactly equal to n with nonnegative coefficients. Then for $\alpha \ge (\sqrt{5} - 1)/2$,*

$$\int_0^\infty (P_n'(t))^2 t^\alpha e^{-t}\, dt \le \frac{n^2}{(2n+\alpha)(2n+\alpha-1)} \int_0^\infty P_n(t)^2 t^\alpha e^{-t}\, dt. \tag{18}$$

The equality holds for $P_n(t) = t^n$. For $0 \le \alpha \le 1/2$,

$$\int_0^\infty (P_n'(t))^2 t^\alpha e^{-t}\, dt \le \frac{1}{(2+\alpha)(1+\alpha)} \int_0^\infty P_n(t)^2 t^\alpha e^{-t}\, dt. \tag{19}$$

Moreover, (19) *is also the best possible in the sense that for $P_n(t) = t^n + \lambda t$ the value of the expression on the left can be made arbitrarily close to the one on the right by choosing λ positive and sufficiently large.*

Using some shortness we will renew the key points in Varma's proof. At first, we write

$$P_n(t) = a_n t^n + P_{n-1}(t), \qquad P_{n-1}(t) = \sum_{k=0}^{n-1} a_k t^k, \quad a_k \ge 0.$$

Introducing the following inner product and norm by

$$(f, g) = \int_0^{+\infty} f(t)g(t)t^\alpha e^{-t}\, dt \quad \text{and} \quad \|f\| = \sqrt{(f, f)}, \tag{20}$$

respectively, we have

$$\|P_n'\|^2 = \|P_{n-1}'\|^2 + a_n^2 n^2 \Gamma(2n+\alpha-1) + 2na_n(P_{n-1}', t^{n-1})$$

and

$$\|P_n\|^2 = a_n^2 \Gamma(2n+\alpha+1) + \|P_{n-1}\|^2 + 2a_n(P_{n-1}, t^n).$$

Putting

$$b_n = \frac{n^2}{(2n+\alpha)(2n+\alpha-1)} \tag{21}$$

and

$$\lambda_n = 2n(P'_{n-1}, t^{n-1}) - 2b_n(P_{n-1}, t^n),$$

Varma obtained

$$\|P'_n\|^2 - b_n\|P_n\|^2 = \lambda_n a_n + \|P'_{n-1}\|^2 - b_n\|P_{n-1}\|^2. \tag{22}$$

Also, he derived that

$$\lambda_n = \frac{2}{n}\, b_n \sum_{k=0}^{n-1} a_k \mu_{kn} \Gamma(k+n+\alpha-1), \tag{23}$$

where $\mu_{kn} = (k-n)[n(n-k) + (2\alpha-1)n + \alpha(\alpha-1)]$, $0 \le k \le n-1$.

Clearly, for $\alpha \ge 0$ we have

$$\mu_{kn} \le -\alpha(2n+\alpha-1) \le 0, \qquad k = 0, 1, \ldots, n-1. \tag{24}$$

Using (23) and (24), Varma claimed that

$$\lambda_n \le 0, \qquad n = 1, 2, \ldots\ . \tag{25}$$

Also, he noted that for every $n = 2, 3, \ldots$,

$$b_n \ge b_{n-1} \qquad \text{for} \quad \alpha \ge \frac{\sqrt{5}-1}{2} \tag{26}$$

and

$$b_n < b_{n-1} \qquad \text{for} \quad 0 \le \alpha \le \frac{1}{2}\,. \tag{27}$$

Using these ideas Varma completed his proof of Theorem 18. Namely, for $\alpha \ge (\sqrt{5}-1)/2$ he obtained from (22), (25) and (26) that

$$\Phi_k \le \Phi_{k-1} \qquad (k = 2, \ldots, n),$$

where we put $\Phi_k = \|P'_k\|^2 - b_n\|P_k\|^2$. Adding all these inequalities Varma concluded that $\Phi_n \le \Phi_1$.

A simple computation shows that for every $P_1(t) = a_1 t + a_0$, $a_1 > 0$, $a_0 \ge 0$, $\|P'_1\|^2 \le b_1\|P_1\|^2$, i.e., $\Phi_1 \le 0$, with equality if $P_1(t) = a_1 t$, $a_1 > 0$. So, Varma proved that $\Phi_n \le 0$, i.e., (18).

Using (27) instead of (26), Varma got the following inequalities

$$\Phi_k \le \Phi_{k-1} + (b_{k-1} - b_k)\|P_n\|^2 \quad (k = 2, \ldots, n).$$

In a similar way, he concluded that

$$\Phi_n \le \Phi_1 + (b_1 - b_n)\|P_n\|^2 \le (b_1 - b_n)\|P_n\|^2,$$

i.e., $\|P_n'\|^2 \le b_1\|P_n\|^2$.

The case $\alpha = 1$ was considered by Varma [42]. The cases $\alpha \in (-1, 0)$ and $\alpha \in (1/2, (\sqrt{5}-1)/2)$ were not solved in the paper of Varma [43]. Xie [52] tried to solve this problem for $\alpha \in (1/2, (\sqrt{5}-1)/2)$. In fact, he proved the following complicated and crude result:

Theorem 19 *Let $b_n = b_n(\alpha)$ be given by (21) and*

$$\alpha_n = \frac{1 - 2n - 4n^2 + \sqrt{16n^4 + 32n^3 + 20n^2 + 4n + 1}}{2(2n+1)} \quad (n \ge 1).$$

Then for each polynomial P of degree n with nonnegative coefficients,

$$\|P'\|^2 \le b_n(\alpha)\|P\|^2 \qquad (\alpha \ge \alpha_1)$$

and

$$\|P'\|^2 \le \begin{cases} b_1(\alpha)\|P\|^2 & (\alpha_\nu \le \alpha < \alpha_{\nu-1},\ n \le \nu), \\ [b_1(\alpha) + b_n(\alpha) - b_\nu(\alpha)]\|P\|^2 & (\alpha_\nu \le \alpha < \alpha_{\nu-1},\ n > \nu), \end{cases}$$

where $\nu = 2, 3, \ldots$.

In the paper [23], we gave a complete solution to Varma's problem (17) determining

$$C_n(\alpha) = \sup_{P \in W_n} \frac{\|P'\|^2}{\|P\|^2}, \tag{28}$$

for all $\alpha \in (-1, +\infty)$, where W_n is defined in the following way:

$$W_n = \Big\{ P \ \Big| \ P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu, \quad a_\nu \ge 0\ (\nu = 0, 1, \ldots, n-1),\ a_n > 0 \Big\}.$$

We denote by W_n^0 the subset of W_n for which $a_0 = 0$ (i.e., $P(0) = 0$). Note that the supremum in (28) is attained for some $P \in W_n^0$. Indeed,

$$\sup_{P \in W_n} \frac{\|P'\|}{\|P\|} = \sup_{\substack{P \in W_n^0 \\ a_0 \ge 0}} \frac{\|P'\|}{\|P + a_0\|} = \sup_{P \in W_n^0} \frac{\|P'\|}{\|P\|}.$$

Let $(.,.)$ and $\|\,.\,\|$ be defined by (20). The following theorem (see Milovanović [23]) gives the solution of the extremal problem (17), i.e., (28).

Theorem 20 *The best constant $C_n(\alpha)$ defined in (28) is*

$$C_n(\alpha) = \begin{cases} \dfrac{1}{(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n), \\ \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & (\alpha_n \le \alpha < +\infty), \end{cases} \tag{29}$$

where

$$\alpha_n = \frac{1}{2}(n+1)^{-1}\big((17n^2+2n+1)^{1/2} - 3n + 1\big). \tag{30}$$

In our proof we take that $P \in W_n^0$, i.e., $P(t) = \sum\limits_{\nu=1}^{n} a_\nu t^\nu$, $a_\nu \ge 0$, and put $I_n(\alpha) = \|\,.\,\|$. Then

$$P(t)^2 = \sum_{\nu=2}^{2n} b_\nu t^\nu \qquad (b_\nu \ge 0)$$

and

$$\|P\|^2 = I_n(\alpha) = \sum_{\nu=2}^{2n} b_\nu \Gamma(\nu+\alpha+1),$$

where Γ is the gamma function.

The inequality

$$t\big(P'(t)^2 - P(t)P''(t)\big) \le P'(t)P(t) \quad (P \in W_n;\ t \ge 0)$$

(see [23]) or [29, Subsection 2.1.5]) and a simple application of integration by parts give us

$$\|P'\| \le \frac{1}{4}\big\{I_n(\alpha) + (1-2\alpha)I_n(\alpha-1) + (\alpha-1)^2 I_n(\alpha-2)\big\},$$

i.e.,

$$\|P'\| \le \sum_{\nu=2}^{2n} H_\nu(\alpha) b_\nu \Gamma(\nu+\alpha+1),$$

where

$$H_\nu(\alpha) = \frac{\nu^2}{4(\nu+\alpha)(\nu+\alpha-1)}.$$

Therefore, $\|P'\|^2 \le \big(\max\limits_{2\le\nu\le 2n} H_\nu(\alpha)\big)\|P\|^2$, so

$$C_n(\alpha) \le \max_{2\le\nu\le 2n} H_\nu(\alpha).$$

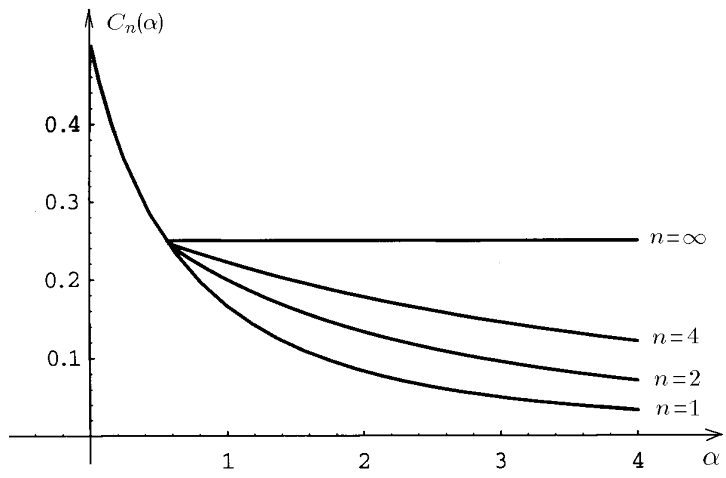

Figure 1: The constant $C_n(\alpha)$ for $n = 1, 2, 4$ and $n = \infty$

Determining the maximum of $f(x) = x^2/\big((x+\alpha)(x+\alpha-1)\big)$ on the interval $[2, 2n]$, we find that

$$\max_{2\le\nu\le 2n} H_\nu(\alpha) = \begin{cases} H_2(\alpha) & \text{if } -1 < \alpha \le \alpha_n, \\ H_{2n}(\alpha) & \text{if } \alpha_n \le \alpha < +\infty, \end{cases}$$

where α_n is given by (30).

We can also show that $C_n(\alpha)$, as it is defined in (29), is the best possible, i.e. that $C_n(\alpha) = \max\limits_{2\le\nu\le 2n} H_\nu(\alpha)$ (see [23]).

The best constant $C_n(\alpha)$ for $n = 1, 2, 3$ and $n = \infty$ as a funcion of α is displayed in Figure 1. An enlarged nontrivial part of that is given in Figure 2. We can see that:

(a) $C_n(\alpha_n - 0) = C_n(\alpha_n + 0)$;

(b) $C_{n+1}(\alpha) \ge C_n(\alpha)$;

(c) The sequence $\{\alpha_n\}$ is decreasing, i.e.,

$$\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_\infty,$$

where

$$\alpha_1 = (\sqrt{5}-1)/2,\ \alpha_2 = (\sqrt{73}-5)/6,\ \alpha_3 = (\sqrt{10}-2)/2,\ \text{etc.},$$

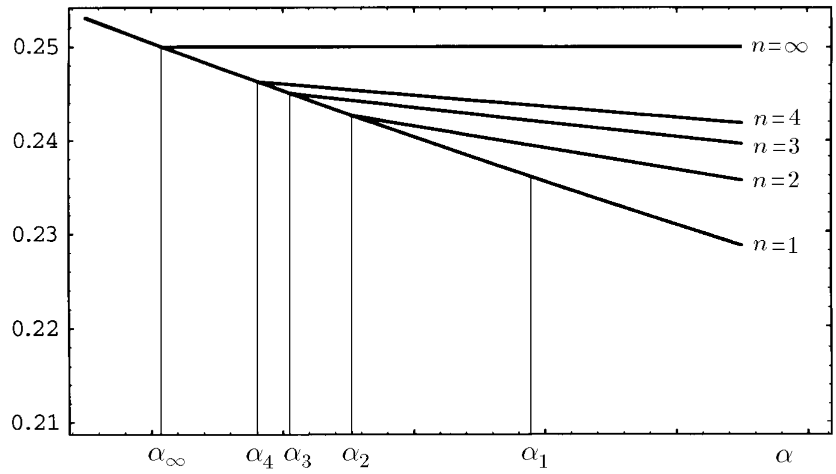

Figure 2: Enlarged nontrivial part in Figure 1

and

$$\alpha_\infty = \lim_{n\to\infty} \alpha_n = (\sqrt{17}-2)/2 = 0.561552812\ldots\ .$$

G. V. Milovanović and I. Ž. Milovanović [27] solved the following extremal problem for higher derivatives

$$C_{n,k}(\alpha) = \sup_{P\in W_n} \frac{\|P^{(k)}\|^2}{\|P\|^2} \qquad (1 \le k \le n). \tag{31}$$

Theorem 21 *The best constant $C_{n,k}(\alpha)$ is given by*

$$C_{n,k}(\alpha) = \begin{cases} \dfrac{(k!)^2}{(\alpha+1)_{2k}}, & -1 < \alpha \le \alpha_{n,k}, \\[2ex] \dfrac{n^2(n-1)^2\cdots(n-k+1)^2}{(2n+\alpha)^{(2k)}}, & \alpha \ge \alpha_{n,k}, \end{cases}$$

where $\alpha_{n,k}$ is the unique positive root of the equation

$$\frac{(2n+\alpha)^{(2k)}}{(2k+\alpha)^{(2k)}} = \binom{n}{k}^2.$$

Here $(p)_\nu = p(p+1)\cdots(p+\nu-1)$ and $p^{(\nu)} = p(p-1)\cdots(p-\nu+1)$.

In the special case, when $n \to +\infty$, the best constant $C_{n,k}$, defined in Theorem 21, reduces to

$$C_k^*(\alpha) = \lim_{n\to\infty} C_{n,k}(\alpha) = \begin{cases} \dfrac{(k!)^2}{(\alpha+1)_{2k}}, & -1 < \alpha \le \alpha_k^*, \\ \\ \dfrac{1}{4^k}, & \alpha_k^* \le \alpha < +\infty, \end{cases}$$

where α_k^* is the unique positive root of the equation $(\alpha+1)_{2k} = 4^k(k!)^2$. We note that $\alpha_1^* = \alpha_\infty = (\sqrt{17}-3)/2$.

The corresponding extremal problem for polynomials with nonnegative coefficients, with respect to the Freud's weight $t \mapsto w(t) = t^\alpha \exp(-t^s)$, $\alpha > -1,\ \ s > 0$, on the interval $(0, +\infty)$, was investigated by Milovanović and Djordjević [26]. In this case, using the same method, it was proved that for $P \in W_n^0$

$$\|P\|^2 = (P,P) = \frac{1}{s}\sum_{\nu=2}^{2n} b_\nu \Gamma\Big(\frac{\alpha+\nu+1}{s}\Big)$$

and

$$\|P'\|^2 = (P',P') \le \frac{1}{s}\sum_{\nu=2}^{2n} H_\nu(\alpha;s) b_\nu \Gamma\Big(\frac{\alpha+\nu+1}{s}\Big),$$

where $(f,g) = \int_0^\infty w(t)f(t)g(t)\,dt$ and

$$H_\nu(\alpha;s) = \frac{\nu^2}{2}\cdot\frac{\Gamma\Big(\dfrac{\alpha+\nu-1}{s}\Big)}{\Gamma\Big(\dfrac{\alpha+\nu+1}{s}\Big)}.$$

The corresponding best constant we will denote by $C_n(\alpha;s)$. If $s = 2$ it gets a simple result (see Milovanović and Djordjević [26]):

Theorem 22 *The best constant* $C_n(\alpha;2)$ *is given by*

$$C_n(\alpha;2) = \begin{cases} \dfrac{2}{\alpha+1}, & -1 < \alpha \le -\dfrac{n-1}{n+1}, \\ \\ \dfrac{2n^2}{2n+\alpha-1}, & -\dfrac{n-1}{n+1} \le \alpha < +\infty. \end{cases}$$

Putting $\alpha = 0$ we obtain the following inequality

$$\int_0^\infty e^{-t^2} P'(t)^2\,dt \le \frac{2n^2}{2n-1}\int_0^\infty e^{-t^2} P(t)^2\,dt$$

for each $P \in W_n$.

The case when s is an arbitrary positive number is more complicated. The following conjecture was stated by Milovanović and Djordjević [26]:

Conjecture 23 *Let $s \geq 1$ and let $\alpha_n\,(> -1)$ be the unique root of the equation*

$$\frac{\Gamma\Big(\dfrac{\alpha+1}{s}\Big)}{\Gamma\Big(\dfrac{\alpha+3}{s}\Big)} = n^2 \frac{\Gamma\Big(\dfrac{\alpha+2n-1}{s}\Big)}{\Gamma\Big(\dfrac{\alpha+2n+1}{s}\Big)}.$$

The best constant $C_n(\alpha; s)$ is given by

$$C_n(\alpha; s) = \begin{cases} H_2(\alpha; s), & -1 < \alpha \leq \alpha_n, \\ H_{2n}(\alpha; s), & \alpha_n \leq \alpha < +\infty. \end{cases}$$

Recently Guessab, Milovanović and Arino [15] considered the extremal problem (31) in L^r-norm,

$$\|P\|_r = \Big(\int_0^\infty |P(t)|^r t^\alpha e^{-t}\, dt\Big)^{1/r}, \qquad r \geq 1.$$

For every $r \in \mathbb{N}$, using the previous method they determined the best constant in the inequality

$$\|P^{(m)}\|_r^r \leq C_{n,r}^{(m)}(\alpha) \|P\|_r^r \qquad (P \in W_n). \tag{32}$$

Theorem 24 *Let $r \in \mathbb{N}$ and let $\alpha_{n,r,m}\,(> -1)$ be the unique root of the equation*

$$\frac{\Gamma(\alpha+1)}{\Gamma(mr+\alpha+1)} = \binom{n}{m}^r \frac{\Gamma((n-m)r+\alpha+1)}{\Gamma(nr+\alpha+1)}.$$

Then the best constant $C_{n,r}^{(m)}(\alpha)$ in (32) is given by

$$C_{n,r}^{(m)}(\alpha) = \begin{cases} (m!)^r \dfrac{\Gamma(\alpha+1)}{\Gamma(mr+\alpha+1)}, & -1 < \alpha \leq \alpha_{n,r,m}, \\ \\ \big(n^{(m)}\big)^r \dfrac{\Gamma((n-m)r+\alpha+1)}{\Gamma(nr+\alpha+1)}, & \alpha_{n,r,m} \leq \alpha < +\infty, \end{cases}$$

where $n^{(m)} = n(n-1)\cdots(n-m+1)$.

Our method of proving this theorem works only when r is an integer. We also use the fact that

$$\sup_{P \in W_n} \frac{\|P^{(m)}\|_r}{\|P\|_r} = \sup_{\substack{P \in W_n^0 \\ a_0,\ldots,a_{m-1} \geq 0}} \frac{\|P^{(m)}\|_r}{\|P + Q_{m-1}\|_r} = \sup_{P \in W_n^0} \frac{\|P^{(m)}\|_r}{\|P\|_r},$$

where $Q_{m-1}(t) = \sum\limits_{k=0}^{m-1} a_k t^k$ $(a_k \geq 0)$ and W_n^0 is a subset of W_n such that

$$P(0) = P'(0) = \cdots = P^{(m-1)}(0) = 0.$$

The case $r = 3$ and $m = 1$ was considered earlier by Guessab and Milovanović [13]. In that case we have that the best constant $C_{n,3}^{(1)}(\alpha)$ given by

$$C_{n,3}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{(3+\alpha)(2+\alpha)(1+\alpha)} & (-1 < \alpha \le \alpha_n), \\ \dfrac{n^3}{(3n+\alpha)(3n+\alpha-1)(3n+\alpha-2)} & (\alpha_n \le \alpha < +\infty), \end{cases}$$

where α_n is the unique positive root of the equation

$$(n^2+n+1)\alpha^3 + 3(2n^2+2n-1)\alpha^2 + (11n^2-16n+2)\alpha - 3n(7n-2) = 0.$$

In the simplest case ($r = 1$, $m = 1$), we have

$$C_{n,1}^{(1)}(\alpha) = \begin{cases} \dfrac{1}{\alpha+1}, & -1 < \alpha \le 0, \\ \dfrac{n}{\alpha+n}, & \alpha \ge 0. \end{cases}$$

Recently, this case was also considered by Chen [6].

For $P \in W_n$ and for positive integers r and p ($r \le p$), Varma [51] proved the L^r inequality

$$\int_0^{+\infty} |P_n'(t)|^r t^{p-1} e^{-t}\, dt \le \frac{n^r(nr+p-r-1)!}{(nr+p-1)!} \int_0^{+\infty} |P_n(t)|^r t^{p-1} e^{-t}\, dt,$$

with equality if and only if $P_n(t) = ct^n$.

In the case $p = 1$, he obtained the best constant in the form $1/r!$, with extremal polynomial $P_n(t) = ct$.

Evidently, he did not know our more general result given in Theorem 24 (see [15]). We believe that this theorem holds for every real $r \ge 1$.

6 Extremal Problems for Lorentz Classes of Polynomials

In this section we consider the extremal problems of Markov's type for non-negative algebraic polynomials on $[-1, 1]$ in L^2 metric with Jacobi weight $w(t) = (1-t)^\alpha(1+t)^\beta$ ($\alpha, \beta > -1$). These problems were investigated by Varma [42], Erdős and Varma [9], Milovanović and Petković [30], Chen [6], and Underhill and Varma [38].

Let L_n be the Lorentz class of algebraic polynomials of the form

$$P(t) = \sum_{\nu=0}^{n} b_\nu (1-t)^\nu (1+t)^{n-\nu}, \quad b_\nu \ge 0 \ (\nu = 0, 1, \ldots, n).$$

A subset of the Lorentz class L_n for which $P^{(i-1)}(-1) = P^{(i-1)}(1) = 0$ $(i = 1, \ldots, k)$ will be denoted by $L_n^{(k)}$. Notice that $L_n^{(0)} \supset L_n^{(1)} \supset \cdots$, where $L_n^{(0)} \equiv L_n$. The corresponding representation of a polynomial P from $L_n^{(k)}$ is

$$P(t) = \sum_{\nu=k}^{n-k} b_\nu (1-t)^\nu (1+t)^{n-\nu},$$

where $b_\nu \geq 0$ $(\nu = k, \ldots, n-k)$.

Let $w(t) = (1-t)^\alpha (1+t)^\beta$, $\alpha, \beta > -1$, and $\|f\|^2 = (f, f)$, where

$$(f, g) = \int_{-1}^{1} w(t) f(t) g(t)\, dt \qquad (f, g \in L^2(-1, 1)).$$

For the determination of the best constant

$$C_n^{(k)}(\alpha, \beta) = \sup_{P \in L_n^{(k)} \setminus \{0\}} \frac{\|P'\|^2}{\|P\|^2}, \tag{33}$$

where $k = 0, 1, \ldots, [n/2]$, Milovanović and Petković [30] used the following inequality

$$(1-t^2)(P'(t)^2 - P''(t)P(t)) \leq nP(t)^2 - 2tP(t)P'(t),$$

which holds for every $t \in [-1, 1]$ and $P \in L_n$ (see also [29, Subsection 2.1.5]). They proved:

Theorem 25 *Let $P \in L_n$ and $\alpha, \beta \geq 1$, then the best constant $C_n^{(0)}(\alpha, \beta)$, defined in* (33)*, is given by*

$$C_n^{(0)}(\alpha, \beta) = \frac{n^2(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}{4(2n + \lambda)(2n + \lambda - 1)},$$

where $\lambda = \min(\alpha, \beta)$.

In a special case we obtain:

Corollary 26 *Let $P \in L_n$, then*

$$C_n^{(0)}(1, 1) = \frac{n(n+1)(2n+3)}{4(2n+1)}.$$

This result was proved earlier by Erdős and Varma [9] (see, also, Varma [42]).

In the same paper [30], Milovanović and Petković proved the following assertion for the class of polynomials $L_n^{(k)}$ $(1 \leq k \leq [n/2])$.

Theorem 27 *Let $P \in L_n^{(k)}$ $(1 \le k \le [n/2])$, $\alpha, \beta > -1$, then*

$$C_n^{(k)}(\alpha,\beta) = \frac{1}{16}(2n+\alpha+\beta)(2n+\alpha+\beta+1)\max\bigl(H_{2k}(\alpha,\beta), H_{2n-2k}(\alpha,\beta)\bigr),$$

where $H_\nu(\alpha,\beta) \equiv f(\nu)$ and f is given by

$$\begin{aligned} f(x) &= \frac{(\alpha-1)^2}{(x+\alpha-1)(x+\alpha)} + \frac{(\beta-1)^2}{(2n-x+\beta-1)(2n-x+\beta)} \\ &\quad + \frac{2n+\alpha+\beta-2\alpha\beta}{(x+\alpha)(2n-x+\beta)}. \end{aligned}$$

Especially interesting cases appear when $\alpha = \beta$.

Theorem 28 *Let $P \in L_n^{(k)}$, $k \ge 1$, $\alpha = \beta > -1$, then*

$$C_n^{(k)}(\alpha,\beta) = \frac{(n+\alpha)(2n+2\alpha+1)q(n,k,\alpha)}{2(2k+\alpha-1)(2k+\alpha)(2n-2k+\alpha-1)(2n-2k+\alpha)},$$

where

$$q(n,k,\alpha) = \alpha(\alpha-1)n^2 + 2k(n-k)(n-1+3\alpha-2\alpha^2).$$

In the special cases when $\alpha = 0$ (Legendre case), $\alpha = -1/2$ (Chebyshev case), and $\alpha = 1$, we have:

Corollary 29 *Let $P \in L_n^{(k)}$, $k \ge 1$, then*

$$C_n^{(k)}(0,0) = \frac{n(n-1)(2n+1)}{4(2k-1)(2n-2k-1)},$$

$$C_n^{(k)}(-1/2,-1/2) = \frac{2n(2n-1)[3n^2+8k(n-k)(n-3)]}{(4k-3)(4k-1)(4n-4k-3)(4n-4k-1)},$$

$$C_n^{(k)}(1,1) = \frac{n(n+1)(2n+3)}{4(2k+1)(2n-2k+1)}. \tag{34}$$

From Corollary 26 we see that (34) holds and for $k = 0$ too.

For $k = 1$, the best constants in Corollary 29 reduce to

$$C_n^{(1)}(0,0) = \frac{n(n-1)(2n+1)}{4(2n-3)}, \tag{35}$$

$$C_n^{(1)}(-1/2,-1/2) = \frac{2n(2n-1)(11n^2-32n+24)}{3(4n-5)(4n-7)},$$

and

$$C_n^{(1)}(1,1) = \frac{n(n+1)(2n+3)}{12(2n-1)}\,.$$

It is of interest to note that Erdős and Varma [9] proved that the best constant in the Lorentz class L_n ($n \ge 2$) for $\alpha = \beta = 0$ is the same one as that in (35), i.e. $C_n^{(0)}(0,0) = C_n^{(1)}(0,0)$.

Recently, Underhill and Varma [38] provided a new proof of the ultraspherical case, without the requirement that $P(\pm 1) = 0$ for $-a < \alpha < 1$. Namely, they proved:

Theorem 30 *Let $P \in L_n$, $n \ge 2$, $\alpha > -1$, and let α_n be the unique positive solution of the equation*

$$2\alpha^4 + (8n-5)\alpha^3 + (12n^2 - 17n + 4)\alpha^2$$
$$+(8n^3 - 20n^2 + 11n - 1)\alpha - 2n(2n^2 - 5n + 4) = 0.$$

Then for $\alpha \ge \alpha_n$ we have

$$C_n^{(0)}(\alpha,\alpha) = \frac{n^2(2n+2\alpha+1)(n+\alpha)}{2(2n+\alpha)(2n+\alpha-1)}\,,$$

and for $-1 < \alpha \le \alpha_n$,

$$C_n^{(0)}(\alpha,\alpha) = \frac{(2n+2\alpha+1)(n+\alpha)A(n,\alpha)}{2(\alpha+1)(\alpha+2)(2n+\alpha-2)(2n+\alpha-3)}\,,$$

where $A(n,\alpha) = \alpha(\alpha-1)n^2 + 2(n-1)\big(n - (\alpha-1)(2\alpha-1)\big)$.

Also, they considered the corresponding problem in L^4 norm with the ultraspherical weight $t \mapsto (1-t^2)^3$ on $(-1,1)$.

At the end we mention a result for polynomials with non-negative coefficients

$$S_n = \Big\{\, P \;\Big|\; P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu,\ a_\nu \ge 0 \ (\nu = 0,1,\ldots,n)\Big\},$$

given by Chen [6]:

Theorem 31 *Let $P \in S_n$ and $\alpha > -1$. Then*

$$\int_{-1}^{1} (P'(t))^2 (1-t^2)^\alpha\,dt \le \frac{2n+2\alpha+1}{2n-1}\, n^2 \int_{-1}^{1} (P(t))^2 (1-t^2)^\alpha\,dt,$$

with equality when $P(t) = t^n$.

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