GENERALIZED HERMITE POLYNOMIALS ON THE RADIAL RAYS IN THE COMPLEX PLANE

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Abstract. In this paper we consider a class of polynomials $\{\pi_N(z)\}$ orthogonal on some radial rays in the complex plane with respect to the inner product $(f,g) = \int_0^{+\infty}\Bigl(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\Bigr)w(x)\,dx$, where the weight function is given by $w(x) = x^{2m\gamma}\exp\bigl(-x^{2m}\bigr)$ and $\gamma > -1/(2m)$ is a parameter. Such polynomials generalize the well-known Hermite polynomials ($m = 1$ and $\gamma = 0$). We give the basic properties of new class of polynomials. A recurrence relation, a representation and the connection with generalized Laguerre polynomials orthogonal on $(0,+\infty)$, as well as a zero distribution, are given. A linear second-order differential equation for $\pi_N(z)$ is also derived.

1. Introduction

The Hermite polynomials $\{H_n(x)\}_{n=0}^{+\infty}$ are orthogonal on the real line with respect to the inner product

$$(f,g) = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}e^{-x^2}\,dx. \tag{1.1}$$

It is well known that these polynomial can be expressed in terms of the generalized Laguerre polynomials $\{L_n^{(s)}(x)\}_{n=0}^{+\infty}$, which are orthogonal on the half line with respect to the inner product

$$(f,g) = \int_0^{+\infty} f(t)\overline{g(t)}t^s e^{-t}\,dt. \tag{1.2}$$

Namely, we have (cf. [10, p. 120 and p. 147])

$$H_{2k}(x) = c_k L_k^{(-1/2)}(x^2) \quad \text{and} \quad H_{2k+1}(x) = d_k x L_k^{(1/2)}(x^2),$$

where c_k and d_k are constants.

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In this paper we give a generalization of the previous fact, i.e., we introduce a new class of orthogonal polynomials on some selected radial rays in the complex plane. The paper is organized as follows. In Section 2 we introduce the orthogonality on the radial rays and give some basic properties of such polynomials $\{\pi_N(z)\}$. A linear second-order differential equation for $\pi_N(z)$ is obtained in Section 3.

An orthogonality on the rays of finite length with an arbitrary weight function, as well as an analogue of the Jacobi polynomials were considered in [15] and [16]. An orthogonality on the unit circle was considered by Geronimus [6–9] (see also Szegő [20] and Nevai [19]. Also, an orthogonality on the semicircle with respect to a complex valued (non-Hermitian) inner product was introduced ([4–5]) and further studied in [2–3], [11–14]. A generalization to a circular arc was given in [1] and further investigations in [18].

2. Orthogonality on the Radial Rays

Let $m \in \mathbb{N}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{2m-1}$ be $(2m)$th roots of unity, i.e., $\varepsilon_s = \exp(i\pi s/m)$, $s = 0, 1, \dots, 2m-1$. We study orthogonal polynomials relative to the inner product

$$(f, g) = \int_0^{+\infty} \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\, dx, \tag{2.1}$$

where

$$w(x) = x^{2m\gamma} \exp\bigl(-x^{2m}\bigr), \quad \gamma > -\frac{1}{2m}\,. \tag{2.2}$$

Here we have $2m$ radial rays in the complex plane ℓ_s, $s = 0, 1, \dots, 2m-1$, which connect the origin $z = 0$ and $z = \infty$ with $2m$ different angles $\varphi_s = \pi s/m$, $s = 0, 1, \dots, 2m-1$.

In the case $m = 1$, (2.1) becomes

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)} w(x)\, dx, \tag{2.3}$$

so we have the standard case of polynomials orthogonal on the real line with respect to the weight function $x \mapsto w(x) = |x|^{2\gamma} \exp(-x^2)$. For $\gamma = 0$, (2.3) reduces to (1.1).

The inner product (2.1) has the following property:

Lemma 2.1. $(z^m f, g) = (f, z^m g)$.

Proof. Since $\varepsilon_s^m = \varepsilon_s^{-m} = (-1)^s$ we have

$$\begin{aligned} (z^m f, g) &= \int_0^{+\infty} \left(\sum_{s=0}^{2m-1} x^m \varepsilon_s^m f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\, dx \\ &= \int_0^{+\infty} \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{x^m \varepsilon_s^m g(x\varepsilon_s)} \right) w(x)\, dx \\ &= (f, z^m g). \quad \square \end{aligned}$$

The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \left(\sum_{s=0}^{2m-1} \varepsilon_s^{p-q} \right) \int_0^{+\infty} x^{p+q} w(x)\, dx, \quad p, q \geq 0,$$

i.e.,

$$\mu_{p,q} = (z^p, z^q) = \frac{1}{2} \left(\sum_{s=0}^{2m-1} \varepsilon_s^{p-q} \right) \Gamma\left(\gamma + \frac{p+q+1}{2} \right), \qquad p, q \geq 0, \tag{2.4}$$

where Γ is the gamma function.

If $p = 2mn + \nu$, $n = [p/(2m)]$, and $0 \leq \nu \leq 2m-1$, it is easy to verify that

$$\sum_{s=0}^{2m-1} \varepsilon_s^p = \sum_{s=0}^{2m-1} \varepsilon_s^\nu = \begin{cases} 2m & \text{if } \nu = 0, \\ 0 & \text{if } 1 \leq \nu \leq 2m-1. \end{cases}$$

Thus, $\mu_{p,q}$ in (2.4) is different from zero only if $p \equiv q \pmod{2m}$; otherwise $\mu_{p,q} = 0$. Using the moment determinants

$$\Delta_0 = 1, \quad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \quad N \geq 1,$$

we can prove the following existence result for the (monic) orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (2.1). This result holds in a general case and its proof can be found in [15].

Theorem 2.2. *If $\Delta_N > 0$ for all $N \geq 1$ the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$, orthogonal with respect to the inner product* (2.1)*, exist uniquely.*

It is well known that an orthogonal sequence of polynomials satisfies a three-term recurrence relation if the inner product has the property $(zf, g) = (f, zg)$. In our case the corresponding property is given by $(z^m f, g) = (f, z^m g)$ (see Lemma 2.1) and the following result holds:

Theorem 2.3. *Let the inner product $(\cdot, \cdot)$ be given by* (2.1) *and let the corresponding system of monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ exist. They satisfy the recurrence relation*

$$\begin{aligned} \pi_{N+m}(z) &= z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \\ \pi_N(z) &= z^N, \quad N = 0, 1, \ldots, 2m-1, \end{aligned} \tag{2.5}$$

where
$$b_{2mn+\nu} = \begin{cases} n+1+\alpha_\nu & \text{if } 0 \le \nu \le m-1, \\ n & \text{if } m \le \nu \le 2m-1. \end{cases}$$

Let $\hat{L}_n^{(s)}(t)$ be the monic generalized Laguerre polynomials orthogonal with respect to the weight $t \mapsto t^s e^{-t}$ on $(0, +\infty)$. They satisfy the three-term recurrence relation (cf. [17, p. 46])

$$\hat{L}_{n+1}^{(s)}(t) = \big(t - (2n+s+1)\big)\hat{L}_n^{(s)}(t) - n(n+s)\hat{L}_{n-1}^{(s)}(t),$$

as well as the following relations (see [20])

$$t\hat{L}_{n-1}^{(s+1)}(t) = \hat{L}_n^{(s)}(t) + (n+s)\hat{L}_{n-1}^{(s)}(t), \quad \hat{L}_n^{(s)}(t) = \hat{L}_n^{(s-1)}(t) - n\hat{L}_{n-1}^{(s)}(t).$$

We can conclude and easily prove that $\pi_N(z)$ are incomplete polynomials with the following representation (see Milovanović [15]):

Theorem 2.4. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (2.1)*, where the weight function is given by* (2.2)*, can be expressed in the form*
$$\pi_N(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^{2m}), \quad N = 2mn + \nu,\ n = [N/2m],$$
where $\nu \in \{0, 1, \dots, 2m-1\}$*,* $\alpha_\nu = \gamma + (2\nu + 1 - 2m)/(2m)$*, and* $\hat{L}_n^{(s)}(t)$ *denotes the monic generalized Laguerre polynomial orthogonal with respect to the weight* $t \mapsto t^s e^{-t}$ *on* $(0, +\infty)$*.*

The next result gives the zero distribution of the polynomials $\pi_N(z)$ (see [15]):

Theorem 2.5. *Let* $N = 2mn + \nu$*,* $n = [N/2m]$*,* $\nu \in \{0, 1, \dots, 2m-1\}$*. All zeros of the polynomial* $\pi_N(z)$ *are simple and located symmetrically on the radial rays* l_s*,* $s = 0, 1, \dots, 2m-1$*, with the possible exception of a multiple zero of order* ν *at the origin* $z = 0$*.*

3. Differential Equation

Like the generalized Laguerre polynomial $L_n^{(s)}(t)$, the polynomial $\pi_N(z)$ satisfies a second order linear homogeneous differential equation.

Theorem 3.1. *The polynomial* $\pi_N(z)$ *orthogonal with respect to the inner product* (2.1) *satisfies the differential equation*

$$z^2 y'' + B(z)y' + C(z)y = 0, \tag{3.1}$$

where

$$\begin{aligned} B(z) &= 2z\big[1 + m(\gamma - 1 - z^{2m})\big], \\ C(z) &= 2mNz^{2m} - \nu(\nu + 2m(\gamma - 1) + 1), \end{aligned} \tag{3.2}$$

and $N = 2mn + \nu$, $n = [N/(2m)]$, $\nu \in \{0, 1, \dots, 2m-1\}$.

Proof. Let $N = 2mn + \nu$, $n = [N/(2m)]$, $\nu \in \{0, 1, \dots, 2m-1\}$. Starting from the representation of the orthogonal polynomial $\pi_N(z)$ given by

$$\pi_N(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^{2m}),$$

where $\alpha_\nu = \gamma + (2\nu + 1 - 2m)/(2m)$, $\gamma > -1/(2m)$, we find

$$\begin{aligned} 2mt\mathrm{D}\hat{L}_n^{(\alpha_\nu)}(t) &= z^{-\nu}\left[z\pi_N'(z) - \nu\pi_N(z)\right], \\ 4m^2t^2\mathrm{D}^2\hat{L}_n^{(\alpha_\nu)}(t) &= z^{-\nu}\left[z^2\pi_N''(z) - (2\nu + 2m - 1)z\pi_N'(z) + \nu(\nu + 2m)\pi_N(z)\right], \end{aligned}$$

where $t = z^{2m}$ and D is the standard differentiation operator $\mathrm{D} = \dfrac{d}{dt}$.

Now, using the generalized Laguerre differential equation

$$t\mathrm{D}^2\hat{L}_n^{(\alpha_\nu)}(t) + (\alpha_\nu + 1 - t)\hat{L}_n^{(\alpha_\nu)}(t) + n\hat{L}_n^{(\alpha_\nu)}(t) = 0,$$

we obtain

$$z^2y'' + 2z\big[1 + m(\gamma - 1 - z^{2m})\big]y' + \big[2mNz^{2m} - \nu(\nu + 2m(\gamma - 1) + 1)\big]y = 0,$$

i.e., (3.1), where $y = \pi_N(z)$. $\square$

Remark 3.1. For $m = 1$, the equation (3.1) reduces to the Hermite equation

$$y'' - 2zy' + 2Ny = 0.$$

Remark 3.2. A simple case could be if we choose the parameter γ in the weight function (2.2) such that the coefficient $B(z)$ in (3.2) reduces to a monomial of degree $2m + 1$. Namely, if $\gamma = (m-1)/m$, the equation (3.1) reduces to

$$y'' - 2mz^{2m-1}y' + \left[2mNz^{2m-2} - \frac{\nu(\nu - 1)}{z^2}\right]y = 0.$$

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