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ERROR BOUNDS OF SOME GAUSS–TURÁN–KRONROD QUADRATURES WITH GORI–MICCHELLI WEIGHTS FOR ANALYTIC FUNCTIONS

Gradimir V. Milovanović[1], Miodrag M. Spalević[2], Miroslav S. Pranić[3]

Dedicated to Professor Boško Jovanović on the occasion of his 60th birthday

[1]*Faculty of Electronic Engineering, Department of Mathematics, University of Niš, P. O. Box 73, 18000 Niš, Serbia*
[2]*Faculty of Science, Department of Mathematics and Informatics, P. O. Box 60, 34000 Kragujevac, Serbia*
[3]*Faculty of Science, Department of Mathematics and Informatics, M. Stojanovića 2, 51000 Banja Luka, Bosnia and Herzegovina*

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Abstract. We study the kernels $K_{n,s}(z)$ of the remainder term $R_{n,s}(f)$ of some Gauss–Turán–Kronrod quadrature rules for analytic functions when the weight function is the chosen subclass of Gori–Micchelli weight functions. We investigate the location on the elliptic contours where the modulus of the kernel attains its maximum value, which leads to effective error bounds of Gauss–Turán–Kronrod quadratures.

INTRODUCTION

Let w be an integrable (nonnegative) weight function on the interval $(-1, 1)$, $n \in \mathbf{N}$ and $s \in \mathbf{N}_0$. It is well known that the Gauss–Turán quadrature formula with multiple nodes,

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f)\,, \tag{1}$$

is exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The nodes τ_ν in (9) must be zeros of the s-orthogonal polynomials with respect to the weight function $w(t)$. The s-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $w(t)$ are polynomials which satisfy the following orthogonality conditions

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k w(t)dt = 0\,, \qquad k = 0, 1, \ldots, n-1.$$

Numerically stable methods for constructing nodes τ_ν and coefficients $A_{i,\nu}$ can be found in [4], [10], [14]. For more details on quadrature formulae with multiple nodes see [5] and [9].

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (9) admits the contour integral representation (see [17], [11])

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_{\Gamma} K_{n,s}(z) f(z)\,dz. \tag{2}$$

The *kernel* is given by

$$K_{n,s}(z; w) = \frac{\rho_{n,s}(z; w)}{[\pi_{n,s}(z)]^{2s+1}}\,, \qquad z \notin [-1, 1],$$

where

$$\rho_{n,s}(z; w) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\, w(t)dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|$ (see [11, Lemma 2.1]).

The integral representation (2) leads to a general error estimate, by using Hölder inequality,

$$\begin{aligned} |R_{n,s}(f)| &= \frac{1}{2\pi}\left|\oint_{\Gamma} K_{n,s}(z) f(z)\,dz\right| \\ &\le \frac{1}{2\pi}\left(\oint_{\Gamma} |K_{n,s}(z)|^{r} |dz|\right)^{1/r}\left(\oint_{\Gamma} |f(z)|^{r'} |dz|\right)^{1/r'}, \end{aligned} \tag{3}$$

i.e.,

$$|R_{n,s}(f)| \le \frac{1}{2\pi} \, \|K_{n,s}\|_r \, \|f\|_{r'}, \tag{4}$$

where $1 \le r \le +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} \left(\oint_\Gamma |f(z)|^r |dz| \right)^{1/r}, & 1 \le r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ ($r' = 1$) gives

$$|R_{n,s}(f)| \le \frac{1}{2\pi} \Bigl(\max_{z\in\Gamma} |K_{n,s}(z)| \Bigr) \|f\|_1. \tag{5}$$

L^∞-error bounds of type (5) for Gaussian quadratures ($s = 0$) of analytic functions were studied in [2], [3], [18], [19], [7]. The general case ($s \in \mathbf{N}$) was studied in [11] and [16].

On the other side, for $r = 1$ ($r' = +\infty$), the estimate (4) is reduced to

$$|R_{n,s}(f)| \le \frac{1}{2\pi} \, \left(\oint_\Gamma |K_{n,s}(z)||dz| \right) \|f\|_\infty \,. \tag{6}$$

L^1-error bounds of type (6) for Gaussian quadratures ($s = 0$) of analytic functions were studied in [6]. The general case ($s \in \mathbf{N}$) is studied in [12].

Following the well-known idea of Kronrod, the formula (9) can be extended to the interpolatory quadrature formula (see [15])

$$\int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1} \sum_{j=0}^{2s_\mu} C_{j,\mu} f^{(i)}(\tau^*_\mu) + E_{n,s}(f)\,, \tag{7}$$

where τ_ν are the same nodes as in (9), $s_\mu \in \mathbf{N}_0$ ($\mu = 1, \ldots, n+1$) and the new nodes τ^*_μ and new weights $B_{i,\nu}$,$C_{j,\mu}$ are chosen to maximize the degree of exactness of (7), which is greater than or equal to $2ns + 3n + 1 + 2\sum_{\mu=1}^{n+1} s_\mu$. The nodes τ^*_μ must satisfy the orthogonality conditions

$$\int_{-1}^{1} w(t) t^m \Omega_n(t)\, dt = 0, \quad m = 0, 1, \ldots, n, \tag{8}$$

where

$$\Omega_n(z) = [\pi_{n,s}(z)]^{2s+1} \prod_{\mu=1}^{n+1} (z - \tau^*_\mu)^{2s_\mu+1}.$$

The case $s_\mu = 0$ $(\mu = 1, \dots, n+1)$ was considered in [8], [20], [13].

If the integrand f is analytic on D and continuous on $\overline{D}$, then $E_{n,s}(f)$ for quadratures (7) can be expressed again in the form (2), with $K_{n,s}(z;w) = \rho_{n,s}(z;w)/\Omega_n(z)$ and

$$\rho_{n,s}(z;w) = \int_{-1}^{1} \frac{\Omega_n(z)}{z-t}\, w(t)dt.$$

Usually for these quadratures we introduce the so-called *generalized Stieltjes polynomial* as $\hat{\pi}_{n+1}(t) = \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)$.

In this paper we study the kernels of some Gauss–Turán–Kronrod quadrature rules for analytic functions when the weight function is the chosen subclass of Gori–Micchelli weight functions. In fact, we investigate the location on the elliptic contours where the modulus of the kernel attains its maximum value, which leads to effective error bounds of such quadratures. It is given in the next section. In the rest of this section we give a remark on the existence problem according to a mistake in the proof of Proposition 2.1 in [15].

Namely, the existence of the real rule (7) depends on the existence of the corresponding Stieltjes polynomial, i.e., on the fact whether all its zeros are real, simple and different from τ_ν. This can be derived as a consequence of the conditions (2.5) in [15, Proposition 2.1]. The existence of the requested Sieltjes polynomial in general case depends on the weight function w. This is a very difficult question which is settled only partially in the theory of ordinary Gauss-Kronrod quadrature formulae (see, for instance, [1], [21] and reference therein). A proof of existence and uniqueness of general real Kronrod extensions of Gaussian quadrature formulae with multiple nodes (contrary to an observation in Abstract and the last paragraph of Introduction in [15]) cannot be given. However, the existence of the real rule (7) can be proved in some special cases (see [15, Theorems 3.1 – 3.4]). It can be shown that when the rule (7) with all nodes belonging to $[-1,1]$ exists, it is unique. The proof is given as follows.

The quadrature rule of type (7) can be written in the form

$$\int_{-1}^{1} w(t)\, f(t)\, dt = \sum_{\nu=1}^{2\ell+1} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f)\,, \tag{9}$$

where the nodes τ_ν, $\nu = 1, 2, \ldots, \ell$ are fixed. It will have a degree of exactness equal to $N = 3\ell + 2\sum_{\nu=1}^{2\ell+1} s_\nu + 1$, if and only if $K = 2\ell + 1 + 2\sum_{\nu=1}^{2\ell+1} s_\nu$ coefficients $A_{i,\nu}$ satisfy the following system of $N+1$ linear equations

$$\sum_{\nu=1}^{2\ell+1}\sum_{i=0}^{2s_\nu} A_{i,\nu} u_j^{(i)}(\tau_\nu) = \int_{-1}^{1} w(t) u_j(t)dt, \quad j = 0, 1, \ldots, N, \tag{10}$$

where $u_j(t)$, $j = 0, 1, \ldots, N$, are linearly independent functions from the space $\mathcal{P}_N$. Let us take that the matrix $[u_j^{(i)}(\tau_\nu)]$ with $N+1$ rows and K columns has rank $N+1-q$ ($q > 1$ because of $N > K$). If all nodes are mutually different, this is equivalent to the following statement: the boundary differential problem

$$\frac{d^{N+1}u}{dt^{N+1}} = 0, \;\; u_j^{(i)}(\tau_\nu) = 0, \quad (i = 0, 1, \ldots, 2s_\nu; \; \nu = 1, 2, \ldots, 2\ell + 1), \tag{11}$$

has q linearly independent solutions U_k, $k = 0, 1, \ldots, q - 1$ (see [5, p. 41-43]).

The system (10) is consistent if and only if the q conditions

$$\int_{-1}^{1} w(t) U_k(t) dt = 0, \quad k = 0, 1, \ldots, q - 1, \tag{12}$$

are satisfied. In that case (10) has $\infty^{K-(N+1-q)}$ solutions.

It is easy to see that the problem (11) has the following $\ell + 1$ linearly independent nontrivial solutions

$$t^k \prod_{\nu=1}^{2\ell+1} (t - \tau_\nu)^{2s_\nu+1}, \quad k = 0, 1, \ldots, \ell.$$

Denoting them as $U_k(t)$, the conditions (12) become the conditions (8). The system (10) has one and only one solution since

$$K - (N + 1 - q) = (2\ell + 1 + 2\sum_{\nu=1}^{2\ell+1} s_\nu) + (\ell + 1) - (3\ell + 2\sum_{\nu=1}^{2\ell+1} s_\nu + 2) = 0.$$

MAIN RESULTS

We take the contour Γ as an ellipse with foci at the points ± 1 and a sum of semi-axes $\varrho > 1$,

$$\mathcal{E}_\varrho = \left\{ z \in \mathbf{C} : z = \frac{1}{2}\left(\varrho e^{i\theta} + \varrho^{-1} e^{-i\theta}\right), \; 0 \le \theta \le 2\pi \right\}.$$

In this paper we consider the case when the weight function w is the Gori–Micchelli weight function

$$w(t) = w_{n,\ell}(t) = \frac{U_{n-1}^{2\ell}(t)}{n^{2\ell}}\,(1-t^2)^{\ell-1/2},\ \ \ell \in \{0,1,\dots,s\}, \tag{13}$$

where $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$ is the Chebyshev polynomial of the second kind. If we take $n \geq 2$, $s_1 = s_{n+1} = (s-\ell)/2$ and $s_\mu = s-\ell$ $(\mu = 2,\dots,n)$, there holds $\hat{\pi}_{n+1}(t) = (1-t^2)U_{n-1}(t)$ (see [15, Theorem 3.1]).

In [15, p. 301] it was shown that

$$\rho_{n,s,\ell}(z;w) = \frac{\pi}{n^{2\ell}\,2^{4s+1}\,u^{2n}}\,Z_{n,s,\ell}(u)\,,$$

where

$$Z_{n,s,\ell}(u) = \sum_{j=0}^{s}(-1)^j\binom{2s+1}{s-j}\frac{1}{u^{4nj}}$$

and

$$\Omega_{n,s,\ell}(z;w) = \frac{1}{2^{2s+1}}\,\frac{(1-z^2)^{s+1-\ell}\,[U_{2n-1}(z)]^{2s+1}}{[U_{n-1}(z)]^{2\ell}}\,.$$

Now we can derive the explicit representation of the kernel

$$K_{n,s,\ell}(z;w) = \frac{\pi}{n^{2\ell}\,2^{2s}\,u^{2n}}\,\frac{[U_{n-1}(z)]^{2\ell}\,Z_{n,s,\ell}(u)}{(1-z^2)^{s+1-\ell}\,[U_{2n-1}(z)]^{2s+1}}\,. \tag{14}$$

Using equalities

$$|1-z^2| = 2^{-1}\,(a_2-\cos 2\theta), \quad |U_{n-1}(z)| = \left(\frac{a_{2n}-\cos 2n\theta}{a_2-\cos 2\theta}\right)^{1/2},$$

where $a_j = a_j(\varrho) = (\varrho^j + \varrho^{-j})/2\,,\ j \in \mathbf{N}\,,\ \rho > 1$, we get

$$|K_{n,s,\ell}(z;w)| = \frac{\pi}{n^{2\ell}\,2^{s+\ell-1}\,\varrho^{2n}}\,\frac{(a_{2n}-\cos 2n\theta)^{\ell}\,|Z_{n,s,\ell}(u)|}{(a_{4n}-\cos 4n\theta)^{s+1/2}(a_2-\cos 2\theta)^{1/2}}\,. \tag{15}$$

The graphs $\theta \mapsto |K_{n,s,\ell}(z)|$ $(z \in \mathcal{E}_\varrho)$ for some values of n, s, ℓ and ϱ are displayed in Figure 1.

Theorem 1. *For each fixed* $s \in \mathbf{N}_0$, $\varrho > 1$ *and* $\ell \in \mathbf{N}$ $(\ell \leq s)$ *there exists* $n_0 = n_0(\varrho,\ell) \in \mathbf{N}$ $(n_0 \geq 2)$ *such that*

$$\max_{z\in\mathcal{E}_\varrho}|K_{n,s,\ell}(z)| = \left|K_{n,s,\ell}\left(\frac{1}{2}(\varrho+\varrho^{-1})\right)\right|\,,$$

for each $n \geq n_0$. When $\ell = 0$, $|K_{n,s,\ell}(z)|$ attains its maximum on the real axis for all $n \in \mathbf{N}$ $(n \geq 2)$.

Proof. The weight function $w(t)$ is even, so we can take $\theta \in [0, \pi/2]$.

First, for each $n \in \mathbf{N}$ and $\varrho > 1$ we prove

$$\frac{|Z_{n,s,\ell}(\rho e^{i\theta})|}{(a_{4n} - \cos 4n\theta)^{1/2}} \leq \frac{Z_{n,s,\ell}(\rho)}{(a_{4n} - 1)^{1/2}}, \quad 0 \leq \theta \leq \pi/2.$$

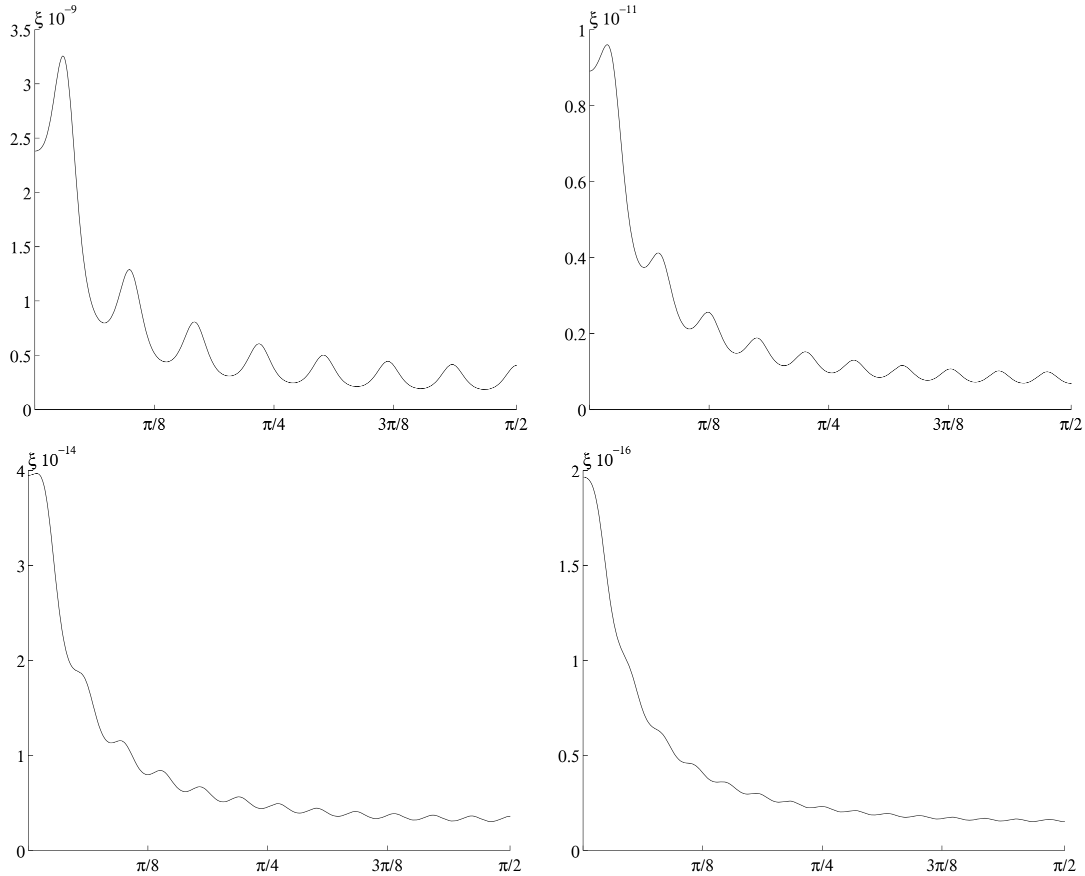

Figure 1: The function $\theta \mapsto |K_{n,3,2}(z)|$ $(z \in \mathcal{E}_{1.07})$ for $n = 15, 20, 25, 30$

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We note that

$$Z_{n,s,\ell}(u) = \sum_{\nu=0}^{[(s-1)/2]} \left(\sum_{k=2\nu}^{2\nu+1} (-1)^k \binom{2s+1}{s-k} u^{-4nk} \right) + \zeta_{n,s}(u)\,, \tag{16}$$

where

$$\zeta_{n,s}(u) := \begin{cases} 0, & \text{when s is even,} \\ u^{-4ns}, & \text{when s is odd.} \end{cases}$$

It is easy to prove that $|\zeta_{n,s}(\rho e^{i\theta})| = \zeta_{n,s}(\rho)$. Using the next notation

$$\begin{aligned} S_\nu(u) \; &:= \; \sum_{k=2\nu}^{2\nu+1} (-1)^k \binom{2s+1}{s-k} u^{-4nk} \\ &= \; \binom{2s+1}{s-2\nu} u^{-8\nu n} - \binom{2s+1}{s-2\nu-1} u^{-4(2\nu+1)n} \\ &= \; \binom{2s+1}{s-2\nu} u^{-8\nu n} (1 - \alpha u^{-4n}) \, , \end{aligned}$$

where

$$\alpha = \frac{s-2\nu}{s+2\nu+2} \quad \text{and} \quad 0 < \alpha < 1 \, ,$$

we get

$$|S_\nu(\rho e^{i\theta})| = \binom{2s+1}{s-2\nu} \rho^{-8\nu n} \sqrt{1 - 2q\cos 4n\theta + q^2} \, ,$$

where $q = \alpha \varrho^{-4n}$, $0 < q < 1$.

The next step is to prove

$$\frac{|S_\nu(\rho e^{i\theta})|}{(a_{4n} - \cos 4n\theta)^{1/2}} \le \frac{S_\nu(\rho)}{(a_{4n} - 1)^{1/2}} \, . \tag{17}$$

Using the previous facts, (17) is reduced to

$$(a_{4n} - 1)(1 - 2q\cos 4n\theta + q^2) \le (a_{4n} - \cos 4n\theta)(1-q)^2 \, ,$$

i.e.,

$$1 - 2q a_{4n} + q^2 \ge 0 \, .$$

It is easy to prove that

$$1 - 2q a_{4n} + q^2 = (1-\alpha)(1 - \alpha\rho^{-8n}) \, ,$$

which proves (17).

Now, from (16) and (17) it follows

$$\begin{aligned}\frac{|Z_{n,s,\ell}(\rho e^{i\theta})|}{(a_{4n}-\cos 4n\theta)^{1/2}} &\leq \sum_{\nu=0}^{[(s-1)/2]} \frac{|S_\nu(\rho e^{i\theta})|}{(a_{4n}-\cos 4n\theta)^{1/2}} + \frac{|\zeta_{n,s}(\rho e^{i\theta})|}{(a_{4n}-\cos 4n\theta)^{1/2}} \\ &\leq \sum_{\nu=0}^{[(s-1)/2]} \frac{S_\nu(\rho)}{(a_{4n}-1)^{1/2}} + \frac{\zeta_{n,s}(\rho)}{(a_{4n}-1)^{1/2}} \\ &= \frac{Z_{n,s,\ell}(\rho)}{(a_{4n}-1)^{1/2}}\,.\end{aligned}$$

To complete the proof it remains to prove

$$\frac{(a_{2n}-\cos 2n\theta)^{\ell}}{(a_{4n}-\cos 4n\theta)^{s}(a_2-\cos 2\theta)^{1/2}} \leq \frac{(a_{2n}-1)^{\ell}}{(a_{4n}-1)^{s}(a_2-1)^{1/2}}\,, \tag{18}$$

for sufficiently large n ($n \geq n_0(\varrho,\ell)$) and $\theta \in (0,\pi/2]$. Since

$$a_{4n}-\cos 4n\theta = 2\,(a_{2n}-\cos 2n\theta)\,(a_{2n}+\cos 2n\theta) \geq a_{4n}-1,$$

it suffices to prove

$$(a_{2n}+\cos 2n\theta)^{2\ell}(a_2-\cos 2\theta) \geq (a_{2n}+1)^{2\ell}(a_2-1), \tag{19}$$

for sufficiently large n ($n \geq n_0(\varrho,\ell)$) and $\theta \in (0,\pi/2]$. When $\ell = 0$, (19) holds obviously.

Using the following transformations

$$a_2-\cos 2\theta = (a_2-1)+2\sin^2\theta$$

and

$$(a_{2n}+\cos 2n\theta)^{2\ell} = (a_{2n}+1)^{2\ell} - 2\sin^2 n\theta\cdot F_{\rho,\ell}(n,\theta),$$

where

$$F_{\rho,\ell}(n,\theta) = \sum_{k=1}^{2\ell}(-2)^{k-1}\binom{2\ell}{k}(a_{2n}+1)^{2\ell-k}\sin^{2k-2} n\theta \quad (\geq 0),$$

(19) is reduced to

$$2\sin^2\theta\,(a_{2n}+1)^{2\ell} - 2\sin^2 n\theta\,[(a_2-1)+2\sin^2\theta]\,F_{\rho,\ell}(n,\theta) \;\geq 0. \tag{20}$$

After dividing the inequality (20) by $2\sin^2\theta$, it is reduced to

$$(a_{2n}+1)^{2\ell} - \frac{\sin^2 n\theta}{\sin^2\theta}\,[(a_2-1)+2\sin^2\theta]\,F_{\rho,\ell}(n,\theta) \;\geq 0. \tag{21}$$

Now, from the well-known fact

$$\left|\frac{\sin n\theta}{\sin\theta}\right| \le n\,,$$

it follows

$$\frac{\sin^2 n\theta}{\sin^2\theta}\left[(a_2-1)+2\sin^2\theta\right] = (a_2-1)\frac{\sin^2 n\theta}{\sin^2\theta}+2\sin^2 n\theta \le (a_2-1)n^2+2. \tag{22}$$

In the same way as in [16, Theorem 2.2] one can show that

$$F_{n,\ell}(\varrho,\theta) \le \sum_{k=0}^{\ell-1} 4^k \binom{2\ell}{2k+1}(a_{2n}+1)^{2\ell-2k-1} - \frac{1}{2}\sum_{k=1}^{\ell-1} 4^k\binom{2\ell}{2k}(a_{2n}+1)^{2\ell-2k}\ . \tag{23}$$

Using (22) and (23), we conclude that the left-hand side of (21) is greater than or equal to $G(n) \equiv G_{\varrho,\ell}(n)$, where

$$\begin{aligned} G_{\varrho,\ell}(n) &:= (a_{2n}+1)^{2\ell} - \left[(a_2-1)n^2+2\right]\times \\ &\times\left[\sum_{k=0}^{\ell-1} 4^k\binom{2\ell}{2k+1}(a_{2n}+1)^{2\ell-2k-1} - \frac{1}{2}\sum_{k=1}^{\ell-1} 4^k\binom{2\ell}{2k}(a_{2n}+1)^{2\ell-2k}\right]. \end{aligned} \tag{24}$$

Since $G_{\varrho,\ell}(n)$ (ϱ, ℓ – are fixed) is continuous on $\mathbf{R}$ and $\lim_{n\to+\infty} G_{\varrho,\ell}(n) = +\infty$, it follows that $G_{\varrho,\ell}(n) > 0$ for each $n > t$, where t is the largest zero of $G_{\varrho,\ell}(n)$. For n_0 we can take $[t]+1$. $\square$

The proof of Theorem 1 is not only of a theoretical, but also of a practical importance. We can use the function $G_{\varrho,\ell}(n)$ (G does not depend on s) from the proof to estimate n_0. Numerical values of $[t]+1$ (t is the largest zero of $G_{\varrho,\ell}$) for some values of ϱ, s and ℓ are presented in Table 1. The smallest possible (s.p.) values of n_0 are also presented. We can see that the smallest possible n_0 is estimated by $[t]+1$ very well.

A typical graph illustrating the relationship between n and $G_{\varrho,\ell}(n)$ is displayed in Figure 2 (left).

Theorem 2. *For each fixed $s \in \mathbf{N}_0$, $n \in \mathbf{N}$ ($n \ge 2$) and $\ell \in \mathbf{N}$ ($\ell \le s$) there exists $\varrho_0 = \varrho_0(n,\ell) > 1$ such that*

$$\max_{z\in\mathcal{E}_\varrho} |K_{n,s,\ell}(z)| = \left|K_{n,s,\ell}\left(\frac{1}{2}(\varrho+\varrho^{-1})\right)\right|$$

for each $\varrho > \varrho_0$.

Table 1:

ℓ	ϱ	the s.p. n_0				
ℓ	ϱ	$s=2$	$s=3$	$s=4$	$s=5$	$[t]+1$
2	1.05	44	43	42	41	46
	1.08	28	28	27	26	30
	1.1	23	22	22	21	24
	1.2	12	12	12	12	13
	1.3	9	9	8	8	9
	1.5	6	6	6	6	6
ℓ	ϱ	$s=4$	$s=5$	$s=6$	$s=7$	$[t]+1$
4	1.05	57	56	56	56	58
	1.08	36	36	36	36	37
	1.1	29	29	29	29	30
	1.2	16	16	16	15	16
	1.3	11	11	11	11	11
	1.5	7	7	7	7	7

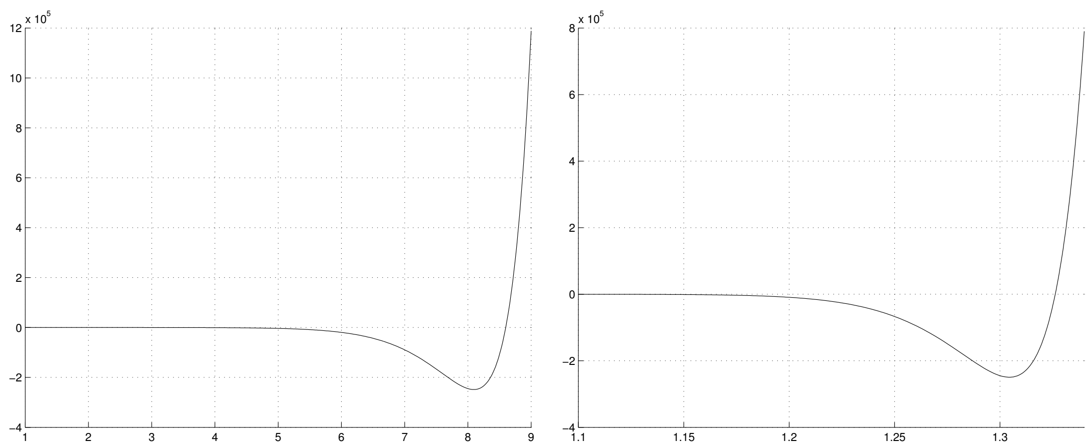

Figure 2: The function $G_{1.3,2}(n)$ (left) and $G_{8,2}(\varrho)$(right).

Proof. We can repeat the same computation which led to (24), where we can fix n and let ϱ be a variable. Since $G_{n,\ell}(\varrho)$ (n, ℓ – are fixed) is continuous when $\varrho > 1$ and $\lim_{\varrho\to+\infty} G_{n,\ell}(n) = +\infty$, it follows that $G_{n,\ell}(\varrho) > 0$ for all $\varrho > r$, where r is the largest zero of $G_{n,\ell}(\varrho)$. For ϱ_0 we can take r. $\square$

We can use the function $G_{n,\ell}(\varrho)$ from the proof to estimate ϱ_0. Numerical values of r (r is the largest zero of $G_{n,\ell}$) for some values of n, s and ℓ are presented in

Table 2. The smallest possible (s.p.) values of ϱ_0 are also presented. We can see that the smallest possible ϱ_0 is estimated by r very well.

Table 2:

ℓ	n	the s.p. ϱ_0			r
		$s=1$	$s=2$	$s=3$	
1	3	1.4459	1.4080	1.3935	1.7969
	5	1.2277	1.2111	1.2047	1.4052
	10	1.1047	1.0977	1.0949	1.1828
	20	1.0507	1.0474	1.0461	1.0873
ℓ	n	$s=3$	$s=4$	$s=5$	r
3	3	2.4733	2.4687	2.4641	2.5175
	5	1.6720	1.6697	1.6673	1.6938
	10	1.2859	1.2849	1.2839	1.2949
	20	1.1332	1.1328	1.1323	1.1372

A typical graph illustrating the relationship between ϱ and $G_{n,\ell}(\varrho)$ is displayed in Figure 2 (right).

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