

SOME FINITE SUMMATION FORMULAS INVOLVING MULTIVARIABLE HYPERGEOMETRIC POLYNOMIALS

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Abstract

The main purpose of this paper is to present a family of finite summation formulas and to apply it in order to derive several functional relationships involving various multivariable hypergeometric polynomials and the Gauss hypergeometric function. A number of special and limit cases of these functional relationships are also considered.

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1. Introduction and Preliminaries

In terms of a bounded multiple sequence $\{\Omega(k_1, \dots, k_r)\}$ of essentially arbitrary (real or complex) parameters, let

$$\Theta_{n_1, \dots, n_r}^{m_1, \dots, m_r}(x_1, \dots, x_r) := \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1 k_1} \cdots (-n_r)_{m_r k_r} \cdot \Omega(k_1, \dots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \quad (1.1)$$

$$(n_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; m_j \in \mathbb{N}; j = 1, \dots, r),$$

where $[\kappa]$ denotes the greatest integer in $\kappa \in \mathbb{R}$ and $(\lambda)_k$ is the Pochhammer symbol (or, more precisely, the *shifted* factorial, since $(1)_k = k!$ ($k \in \mathbb{N}_0$)) defined, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0; \lambda \neq 0) \\ \lambda(\lambda + 1) \cdots (\lambda + k - 1) & (k \in \mathbb{N}), \end{cases} \quad (1.2)$$

\mathbb{N} being the set of *positive* integers.

For different choices of the multiple sequence $\{\Omega(k_1, \dots, k_r)\}$ and with

$$m_j = 1 \quad (j = 1, \dots, r),$$

the multivariable polynomials [*cf.* Equation (1.1)]

$$\Theta_{n_1, \dots, n_r}^{m_1, \dots, m_r}(x_1, \dots, x_r)$$

would readily yield, as special cases, various classes of orthogonal and biorthogonal polynomials associated with hypergeometric functions of two and more variables (see, for details, [1], [6], [7], and [12] to [18]). Motivated essentially by these and sundry other occurrences of special multivariable hypergeometric polynomials in the mathematical and physical sciences literature, we first propose to derive here a family of finite summation formulas involving the polynomials defined by (1.1) and then show how this general result can be applied in order to deduce several functional relationships between various multivariable hypergeometric polynomials and the Gauss hypergeometric function which corresponds to the familiar special case

$$p - 1 = q = 1$$

of the generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by

$$\begin{aligned} {}_pF_q [(\alpha_p); (\beta_q); z] &= {}_pF_q \left[\begin{matrix} (\alpha_p); \\ (\beta_q); \end{matrix} z \right] \\ &:= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \end{aligned} \quad (1.3)$$

$$(p, q \in \mathbb{N}_0; p \leq q + 1; p \leq q \text{ and } |z| < \infty;$$

$$p = q + 1 \text{ and } |z| < 1; p = q + 1, |z| = 1, \text{ and } \Re(\Xi) > 0),$$

where (*and throughout this paper*) we find it to be convenient to abbreviate the p -parameter array:

$$\alpha_1, \dots, \alpha_p \quad (p \in \mathbb{N})$$

by (α_p) , the array being empty when $p = 0$, with similar interpretations for (β_q) , *et cetera*, and

$$\Xi := \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j \quad (\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}).$$

2. Finite Summation Formulas

We begin by recalling the multinomial theorem in the form (*cf.*, *e.g.*, [3, p. 13, Eq. 2.3 (9)]):

$$\sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} x_1^{n_1} \cdots x_r^{n_r} = (x_1 + \dots + x_r)^n \quad (2.1)$$

$$(n, n_j \in \mathbb{N}_0; j = 1, \dots, r; r \in \mathbb{N} \setminus \{1\}),$$

where, *and in what follows*,

$$\binom{n}{n_1, \dots, n_r} := \frac{n!}{n_1! \cdots n_r!}. \quad (2.2)$$

Since

$$(-n_j)_{m_j k_j} = (-1)^{m_j k_j} \frac{n_j!}{(n_j - m_j k_j)!} \quad (2.3)$$

$$(0 \leq k_j \leq [n_j/m_j]; j = 1, \dots, r),$$

by virtue of the definition (1.2), we can make use of the multinomial theorem (2.1) in conjunction with the definition (1.1) to show that

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} \Theta_{n_1, \dots, n_r}^{m_1, \dots, m_r} (x_1, \dots, x_r) t_1^{n_1} \dots t_r^{n_r} \\
&= T^n \sum_{k_1, \dots, k_r=0}^{m_1 k_1 + \dots + m_r k_r \leq n} (-n)_{m_1 k_1 + \dots + m_r k_r} \Omega(k_1, \dots, k_r) \\
&\quad \cdot \frac{\{x_1 (t_1/T)^{m_1}\}^{k_1}}{k_1!} \dots \frac{\{x_r (t_r/T)^{m_r}\}^{k_r}}{k_r!}
\end{aligned} \tag{2.4}$$

$$(T := t_1 + \dots + t_r; \ n, n_j \in \mathbb{N}_0; \ m_j \in \mathbb{N}; \ j = 1, \dots, r).$$

With a view to applying the general finite summation formula (2.4) to the following familiar *special* case of the (Srivastava-Daoust) generalized Lauricella functions, defined by (*cf.*, *e.g.*, [9, p. 38, Eq. 1.4 (24)])

$$\begin{aligned}
& F_{q; q_1; \dots; q_r}^{p; p_1; \dots; p_r} \left[\begin{array}{c} (\alpha_p) : (\gamma'_{p_1}); \dots; (\gamma^{(r)}_{p_r}); \\ (\beta_q) : (\delta'_{q_1}); \dots; (\delta^{(r)}_{q_r}); \end{array} \middle| z_1, \dots, z_r \right] \\
&:= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_{k_1+\dots+k_r} \prod_{j=1}^{p_1} (\gamma'_j)_{k_1} \dots \prod_{j=1}^{p_r} (\gamma_j^{(r)})_{k_r}}{\prod_{j=1}^q (\beta_j)_{k_1+\dots+k_r} \prod_{j=1}^{q_1} (\delta'_j)_{k_1} \dots \prod_{j=1}^{q_r} (\delta_j^{(r)})_{k_r}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_r^{k_r}}{k_r!}, \tag{2.5}
\end{aligned}$$

whenever the multiple hypergeometric series in (2.5) converges or terminates, we conveniently set

$$m_1 = \dots = m_r = 1,$$

and we find from (2.4) that

$$\begin{aligned}
 & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} F_{q:q_1+1; \dots; q_r+1}^{p:p_1+1; \dots; p_r+1} \left[\begin{array}{l} (\alpha_p) : -n_1, (\gamma'_{p_1}); \dots; \\ (\beta_q) : (\delta'_{q_1}); \dots; \\ -n_r, (\gamma^{(r)}_{p_r}) \\ x_1, \dots, x_r \\ (\delta^{(r)}_{q_r}); \end{array} \right] t_1^{n_1} \dots t_r^{n_r} \\
 &= T^n F_{q:q_1; \dots; q_r}^{p+1:p_1; \dots; p_r} \left[\begin{array}{l} -n, (\alpha_p) : (\gamma'_{p_1}); \dots; (\gamma^{(r)}_{p_r}); \\ (\beta_q) : (\delta'_{q_1}); \dots; (\delta^{(r)}_{q_r}); \\ \frac{x_1 t_1}{T}, \dots, \frac{x_r t_r}{T} \end{array} \right] \quad (2.6) \\
 & \quad (T := t_1 + \dots + t_r; n, n_j \in \mathbb{N}; j = 1, \dots, r).
 \end{aligned}$$

Next, by appealing appropriately to the multiple series identity (*cf.* [8]; see also [9, p. 39]):

$$\begin{aligned}
 & \sum_{n_1, \dots, n_r=0}^{\infty} \omega(n_1 + \dots + n_r) (\lambda_1)_{n_1} \dots (\lambda_r)_{n_r} \frac{z^{n_1}}{n_1!} \dots \frac{z^{n_r}}{n_r!} \\
 &= \sum_{n=0}^{\infty} \omega(n) (\lambda_1 + \dots + \lambda_r)_n \frac{z^n}{n!} \quad (2.7)
 \end{aligned}$$

and its multivariable hypergeometric form:

$$\begin{aligned}
 & F_{q:0; \dots; 0}^{p:1; \dots; 1} \left[\begin{array}{l} (\alpha_p) : \lambda_1; \dots; \lambda_r; \\ (\beta_q) : -; \dots; -; \end{array} \right] z, \dots, z \\
 &= {}_{p+1}F_q \left[\begin{array}{l} (\alpha_p), \lambda_1 + \dots + \lambda_r; \\ (\beta_q); \end{array} \right] z, \quad (2.8)
 \end{aligned}$$

the second members of (2.4) and (2.6) can be simplified considerably in the special cases:

$$\begin{aligned}
 \Omega(k_1, \dots, k_r) &= (\lambda_1)_{k_1} \dots (\lambda_r)_{k_r} \omega(k_1 + \dots + k_r), \quad m_j = m, \quad \text{and} \quad x_j = \left(\frac{\tau}{t_j} \right)^m \quad (2.9) \\
 & \quad (m \in \mathbb{N}; k_j \in \mathbb{N}_0; \lambda_j \in \mathbb{C}; j = 1, \dots, r)
 \end{aligned}$$

and

$$p_j - 1 = q_j = 0 \quad \left(\gamma_{p_j}^{(j)} = \lambda_j \right) \quad \text{and} \quad x_j t_j = \tau \quad (j = 1, \dots, r), \quad (2.10)$$

respectively. We thus find from (2.4) that

$$\begin{aligned} & \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} \Phi_{n_1, \dots, n_r}^{\lambda_1, \dots, \lambda_r} \left(m; \frac{\tau}{t_1}, \dots, \frac{\tau}{t_r} \right) t_1^{n_1} \dots t_r^{n_r} \\ &= T^n \sum_{k=0}^{[n/m]} (-n)_{mk} (\lambda_1 + \dots + \lambda_r)_k \omega(k) \frac{(\tau/T)^{mk}}{k!} \end{aligned} \quad (2.11)$$

$$(T := t_1 + \dots + t_r; m \in \mathbb{N}; n, n_j \in \mathbb{N}_0; \lambda_j \in \mathbb{C}; j = 1, \dots, r),$$

where [cf. Equation (1.1)]

$$\begin{aligned} \Phi_{n_1, \dots, n_r}^{\lambda_1, \dots, \lambda_r} (m; x_1, \dots, x_r) &:= \sum_{k_1=0}^{[n_1/m]} \dots \sum_{k_r=0}^{[n_r/m]} (-n_1)_{mk_1} (\lambda_1)_{k_1} \dots (-n_r)_{mk_r} (\lambda_r)_{k_r} \\ &\quad \cdot \omega(k_1 + \dots + k_r) \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!} \end{aligned} \quad (2.12)$$

$$(n_j \in \mathbb{N}_0; \lambda_j \in \mathbb{C}; j = 1, \dots, r; m \in \mathbb{N})$$

in terms of a bounded sequence $\{\omega(n)\}_{n=0}^{\infty}$ of essentially arbitrary (real or complex) parameters. Furthermore, under the constraints given by (2.10), the finite summation formula (2.6) similarly yields

$$\begin{aligned} & \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} F_{q:0; \dots; 0}^{p:2; \dots; 2} \left[\begin{array}{l} (\alpha_p) : -n_1, \lambda_1; \dots; -n_r, \lambda_r; \\ (\beta_q) : \text{---}; \dots; \text{---}; \end{array} \frac{\tau}{t_1}, \dots, \frac{\tau}{t_r} \right] t_1^{n_1} \dots t_r^{n_r} \\ &= T^n {}_{p+2}F_q \left[\begin{array}{l} -n, \lambda_1 + \dots + \lambda_r, (\alpha_p); \\ (\beta_q); \end{array} \frac{\tau}{T} \right] \end{aligned} \quad (2.13)$$

$$(T := t_1 + \dots + t_r; n, n_j \in \mathbb{N}_0; \lambda_j \in \mathbb{C}; j = 1, \dots, r).$$

3. Applications to Lauricella Polynomials in Several Variables

The various finite summation formulas, which we presented in the preceding section, can indeed be applied in order to derive the corresponding results for many of the orthogonal and biorthogonal multivariable hypergeometric polynomials which were referred to in Section 1. We leave the details involved in the derivations of these consequences of our results as an exercise for the interested reader. We turn instead to some multivariable hypergeometric

polynomials which are associated, in particular, with the Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ of r variables, where (cf. [1, p. 114]; see also [9, p. 33])

$$\begin{aligned}
 & F_A^{(r)} [a, b_1, \dots, b_r; c_1, \dots, c_r; z_1, \dots, z_r] \\
 &= F_{0:1;\dots;1}^{1:1;\dots;1} \left[\begin{array}{c} a : b_1; \dots ; b_r; \\ z_1, \dots, z_r \\ - : c_1; \dots ; c_r; \end{array} \right] \\
 &:= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(a)_{k_1+\dots+k_r} (b_1)_{k_1} \cdots (b_r)_{k_r}}{(c_1)_{k_1} \cdots (c_r)_{k_r}} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \quad (3.1)
 \end{aligned}$$

$$(|z_1| + \dots + |z_r| < 1; c_j \notin \mathbb{Z}_0^-; j = 1, \dots, r),$$

$$\begin{aligned}
 & F_B^{(r)} [a_1, \dots, a_r, b_1, \dots, b_r; c; z_1, \dots, z_r] \\
 &= F_{1:0;\dots;0}^{0:2;\dots;2} \left[\begin{array}{c} - : a_1, b_1; \dots ; a_r, b_r; \\ z_1, \dots, z_r \\ c : \text{---}; \dots ; \text{---}; \end{array} \right] \\
 &:= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(a_1)_{k_1} \cdots (a_r)_{k_r} (b_1)_{k_1} \cdots (b_r)_{k_r}}{(c)_{k_1+\dots+k_r}} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \quad (3.2)
 \end{aligned}$$

$$(\max \{|z_1|, \dots, |z_r|\} < 1; c \notin \mathbb{Z}_0^-),$$

$$\begin{aligned}
 & F_C^{(r)} [a, b; c_1, \dots, c_r; z_1, \dots, z_r] \\
 &= F_{0:1;\dots;1}^{2:0;\dots;0} \left[\begin{array}{c} a, b : \text{---}; \dots ; \text{---}; \\ z_1, \dots, z_r \\ \text{---} : c_1; \dots ; c_r; \end{array} \right] \\
 &:= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(a)_{k_1+\dots+k_r} (b)_{k_1+\dots+k_r}}{(c_1)_{k_1} \cdots (c_r)_{k_r}} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \quad (3.3)
 \end{aligned}$$

$$\left(|z_1|^{\frac{1}{2}} + \dots + |z_r|^{\frac{1}{2}} < 1; c_j \notin \mathbb{Z}_0^-; j = 1, \dots, r \right),$$

and

$$\begin{aligned}
& F_D^{(r)} [a, b_1, \dots, b_r; c; z_1, \dots, z_r] \\
&= F_{1:0; \dots; 0}^{1:1; \dots; 1} \left[\begin{array}{c} a : b_1; \dots; b_r; \\ z_1, \dots, z_r \\ c : -; \dots; -; \end{array} \right] \\
&:= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(a)_{k_1+\dots+k_r} (b_1)_{k_1} \dots (b_r)_{k_r}}{(c)_{k_1+\dots+k_r}} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_r^{k_r}}{k_r!} \\
& \quad (\max \{|z_1|, \dots, |z_r|\} < 1; c \notin \mathbb{Z}_0^-).
\end{aligned} \tag{3.4}$$

First of all, in its special case when

$$p - 1 = q = 0 \quad (\alpha_1 = a) \quad \text{and} \quad p_j = q_j - 1 = 0 \quad (\delta_j^{(j)} = c_j; j = 1, \dots, r),$$

our summation formula (2.6) yields

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} F_A^{(r)} [a, -n_1, \dots, -n_r; c_1, \dots, c_r; x_1, \dots, x_r] t_1^{n_1} \dots t_r^{n_r} \\
&= T^n F_C^{(r)} \left[-n, a; c_1, \dots, c_r; \frac{x_1 t_1}{T}, \dots, \frac{x_r t_r}{T} \right] \\
& \quad (T := t_1 + \dots + t_r; n, n_j \in \mathbb{N}_0; c_j \notin \mathbb{Z}_0^-; j = 1, \dots, r).
\end{aligned} \tag{3.5}$$

In the case of the Lauricella hypergeometric polynomials associated with $F_B^{(r)}$, (2.13) with

$$p = q - 1 = 0 \quad (\beta_1 = c) \quad \text{and} \quad \lambda_j \mapsto b_j \quad (j = 1, \dots, r)$$

leads us immediately to the functional relationship:

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} F_B^{(r)} \left[-n_1, \dots, -n_r, b_1, \dots, b_r; c; \frac{\tau}{t_1}, \dots, \frac{\tau}{t_r} \right] t_1^{n_1} \dots t_r^{n_r} \\
&= T^n {}_2F_1 \left(-n, b_1 + \dots + b_r; c; \frac{\tau}{T} \right) \\
& \quad (T := t_1 + \dots + t_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r; c \notin \mathbb{Z}_0^-).
\end{aligned} \tag{3.6}$$

For the Lauricella hypergeometric polynomials associated with $F_D^{(r)}$, we similarly find from (2.6) with

$$p = q = 1 \quad (\alpha_1 = a; \beta_1 = c) \quad \text{and} \quad p_j = q_j = 0 \quad (j = 1, \dots, r)$$

that

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} F_D^{(r)} [a, -n_1, \dots, -n_r; c; x_1, \dots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\ &= T^n {}_2F_1 \left(-n, a; c; \frac{x_1 t_1 + \cdots + x_r t_r}{T} \right) \end{aligned} \quad (3.7)$$

$$(T := t_1 + \cdots + t_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r; c \notin \mathbb{Z}_0^-).$$

Since

$$F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad \text{and} \quad F_D^{(2)} = F_1, \quad (3.8)$$

each of the functional relationships (3.5), (3.6), and (3.7) (*with* $r = 2$) can immediately be rewritten in terms of one or the other of Appell's hypergeometric functions F_1 , F_2 , F_3 , and F_4 of two variables (*cf.* [1, p. 14]; see also [9, pp. 22-23]).

Next, for the second set of the Carlitz-Srivastava polynomials defined by [2, Part II, p. 143, Eq. (27)]

$$\begin{aligned} & \mathcal{F}_D^{(r)} [(\alpha : \vartheta_j) : (-n_j, m_j); (\gamma : \varphi_j); x_1, \dots, x_r] \\ &:= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1 k_1} \cdots (-n_r)_{m_r k_r} \frac{(\alpha)_{k_1 \vartheta_1 + \cdots + k_r \vartheta_r}}{(\gamma)_{k_1 \varphi_1 + \cdots + k_r \varphi_r}} \\ & \quad \cdot \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \end{aligned} \quad (3.9)$$

$$(n_j \in \mathbb{N}_0; m_j \in \mathbb{N}; \vartheta_j, \varphi_j \in \mathbb{R}^+; j = 1, \dots, r; \gamma \notin \mathbb{Z}_0^-),$$

the general result (2.4) readily yields

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} \mathcal{F}_D^{(r)} [(\alpha : \vartheta_j) : (-n_j, m_j); (\gamma : \varphi_j); x_1, \dots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\ &= T^n \sum_{k_1, \dots, k_r=0}^{m_1 k_1 + \cdots + m_r k_r \leq n} (-n)_{m_1 k_1 + \cdots + m_r k_r} \frac{(\alpha)_{k_1 \vartheta_1 + \cdots + k_r \vartheta_r}}{(\gamma)_{k_1 \varphi_1 + \cdots + k_r \varphi_r}} \\ & \quad \cdot \frac{\{x_1 (t_1/T)^{m_1}\}^{k_1}}{k_1!} \cdots \frac{\{x_r (t_r/T)^{m_r}\}^{k_r}}{k_r!} \end{aligned} \quad (3.10)$$

$$(T := t_1 + \cdots + t_r; n, n_j \in \mathbb{N}_0; m_j \in \mathbb{N}; j = 1, \dots, r; \gamma \notin \mathbb{Z}_0^-).$$

In the particular case when

$$m_j = m, \quad \vartheta_j = \rho, \quad \text{and} \quad \varphi_j = \sigma \quad (m, \rho, \sigma \in \mathbb{N}), \quad (3.11)$$

(3.10) would reduce to the form:

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} \mathcal{F}_D^{(r)} [(\alpha : \rho) : (-n_j, m); (\gamma : \sigma); x_1, \dots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\ &= T^n {}_{m+\rho}F_\sigma \left[\begin{array}{c} \Delta(m; -n), \Delta(\rho; \alpha); \\ \Delta(\sigma; \gamma); \end{array} \frac{m^m \rho^\rho}{\sigma^\sigma} \left\{ x_1 \left(\frac{t_1}{T} \right)^m + \cdots + x_r \left(\frac{t_r}{T} \right)^m \right\} \right] \end{aligned} \quad (3.12)$$

$$(T := t_1 + \cdots + t_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r; m, \rho, \sigma \in \mathbb{N}; \gamma \notin \mathbb{Z}_0^-),$$

where $\Delta(m; \lambda)$ abbreviates the array of m parameters:

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \dots, \frac{\lambda+m-1}{m} \quad (m \in \mathbb{N}).$$

For $m = \rho = \sigma = 1$, (3.12) would obviously correspond to the functional relationship (3.7).

Lastly, for Erdélyi's *multivariable* extension of the classical Laguerre polynomials defined by (cf., e.g., [2, Part II, p. 144, Eq. (29)])

$$\begin{aligned} L_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &:= \frac{(\alpha+1)_{n_1+\dots+n_r}}{n_1! \cdots n_r!} \\ &\cdot F_{1:0; \dots; 0}^{0:1; \dots; 1} \left[\begin{array}{c} \text{---} : -n_1; \dots; -n_r; \\ \alpha+1 : \text{---}; \dots; \text{---}; \end{array} x_1, \dots, x_r \right], \end{aligned} \quad (3.13)$$

it is easily seen from (2.6) with

$$p = q - 1 = 0 \quad (\beta_1 = \alpha + 1) \quad \text{and} \quad p_j = q_j = 0 \quad (j = 1, \dots, r)$$

that

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} L_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) t_1^{n_1} \cdots t_r^{n_r} \\ &= \binom{\alpha+n}{n} T^n {}_1F_1 \left(-n; \alpha+1; \frac{x_1 t_1 + \cdots + x_r t_r}{T} \right) \end{aligned} \quad (3.14)$$

$$(T := t_1 + \cdots + t_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r; \alpha \in \mathbb{Z}^- := \mathbb{Z}_0^- \setminus \{0\}).$$

Clearly, since [cf. Equations (3.4) and (3.13)]

$$\begin{aligned} & L_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\ &= \lim_{|\lambda| \rightarrow \infty} \left\{ F_D^{(r)} \left[\lambda, -n_1, \dots, -n_r; \alpha+1; \frac{x_1}{\lambda}, \dots, \frac{x_r}{\lambda} \right] \right\}, \end{aligned} \quad (3.15)$$

our last functional relationship (3.14) can be deduced as a limit case of (3.7) when

$$c = \alpha + 1, \quad x_j \mapsto \frac{x_j}{a} \quad (j = 1, \dots, r), \quad \text{and} \quad |a| \rightarrow \infty.$$

4. Further Extensions and Consequences

The multinomial theorem (2.1), which provided one of the tools used in our present investigation, is itself a limit case of the following well-known multiple sum [3, p. 13, Eq. 2.3(5)]:

$$\sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} = (\lambda_1 + \cdots + \lambda_r)_n \quad (4.1)$$

$$(n, n_j \in \mathbb{N}_0; \lambda_j \in \mathbb{C}; j = 1, \dots, r),$$

which, for $r = 2$, is equivalent to the Chu-Vandermonde summation theorem:

$$\sum_{k=0}^n \binom{\lambda}{n-k} \binom{\mu}{k} = \binom{\lambda + \mu}{n} \quad (n \in \mathbb{N}_0; \lambda, \mu \in \mathbb{C}). \quad (4.2)$$

In fact, since

$$\lim_{|\lambda| \rightarrow \infty} \left\{ (\lambda)_n \left(\frac{z}{\lambda} \right)^n \right\} = z^n = \lim_{|\mu| \rightarrow \infty} \left\{ \frac{(\mu z)^n}{(\mu)_n} \right\} \quad (4.3)$$

$$(n \in \mathbb{N}_0; |z| < \infty),$$

upon setting $\lambda_j = \lambda x_j$ ($j = 1, \dots, r$), dividing both sides by λ^n , and letting $|\lambda| \rightarrow \infty$, (4.1) yields the multinomial theorem (2.1).

If we apply the general result (4.1) in place of the multinomial theorem (2.1), we find from the definition (1.1) that

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \Theta_{n_1, \dots, n_r}^{m_1, \dots, m_r}(x_1, \dots, x_r) \\ &= (\lambda_1 + \cdots + \lambda_r)_n \sum_{k_1, \dots, k_r=0}^{m_1 k_1 + \dots + m_r k_r \leq n} \frac{(-n)_{m_1 k_1 + \dots + m_r k_r} (\lambda_1)_{m_1 k_1} \cdots (\lambda_r)_{m_r k_r}}{(\lambda_1 + \cdots + \lambda_r)_{m_1 k_1 + \dots + m_r k_r}} \\ & \quad \cdot \Omega(k_1, \dots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \end{aligned} \quad (4.4)$$

$$(n, n_j \in \mathbb{N}_0; m_j \in \mathbb{N}; \lambda_j \in \mathbb{C}; j = 1, \dots, r),$$

which provides a seemingly interesting extension of (2.4).

Finally, in order to give a conveniently simple hypergeometric form of (2.4), we set

$$\Lambda := \lambda_1 + \cdots + \lambda_r \quad \text{and} \quad m_j = 1 \quad (j = 1, \dots, r),$$

and choose the multiple sequence $\{\Omega(k_1, \dots, k_r)\}$ as in the definition (2.5). We thus obtain

$$\begin{aligned} & \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \\ & \cdot F_{q: q_1; \dots; q_r}^{p: p_1+1; \dots; p_r+1} \left[\begin{array}{c} (\alpha_p) : -n_1, (\gamma'_{p_1}); \cdots; -n_r, (\gamma_{p_r}^{(r)}); \\ (\beta_q) : \quad (\delta'_{q_1}); \cdots; \quad (\delta_{q_r}^{(r)}); \end{array} \right. \left. x_1, \dots, x_r \right] \\ & = (\Lambda)_n F_{q+1: q_1; \dots; q_r}^{p+1: p_1+1; \dots; p_r+1} \left[\begin{array}{c} -n, (\alpha_p) : \lambda_1, (\gamma'_{p_1}); \cdots; \lambda_r, (\gamma_{p_r}^{(r)}); \\ \Lambda, (\beta_q) : \quad (\delta'_{q_1}); \cdots; \quad (\delta_{q_r}^{(r)}); \end{array} \right. \left. x_1, \dots, x_r \right] \quad (4.5) \end{aligned}$$

$$(\Lambda := \lambda_1 + \cdots + \lambda_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r),$$

provided that each side of (4.5) exists. In its special case when

$$p = q = 0 \quad \text{and} \quad p_j = q_j = 1 \quad \left(\gamma_{p_j}^{(j)} = \mu_j; \delta_{p_j}^{(j)} = \nu_j; j = 1, \dots, r \right),$$

the hypergeometric summation formula (4.5) reduces immediately to the elegant form:

$$\begin{aligned} & \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \\ & \cdot {}_2F_1(-n_1, \mu_1; \nu_1; x_1) \cdots {}_2F_1(-n_r, \mu_r; \nu_r; x_r) \\ & = (\Lambda)_n F_{1:1; \dots; 1}^{1:2; \dots; 2} \left[\begin{array}{c} -n : \lambda_1, \mu_1; \cdots; \lambda_r, \mu_r; \\ \Lambda : \quad \nu_1; \cdots; \quad \nu_r; \end{array} \right. \left. x_1, \dots, x_r \right] \quad (4.6) \end{aligned}$$

$$(\Lambda := \lambda_1 + \cdots + \lambda_r; n, n_j \in \mathbb{N}_0; \Lambda, \nu_j \notin \mathbb{Z}_0^-; j = 1, \dots, r).$$

A *further* special case of (4.6) when

$$\nu_j = \lambda_j \quad (j = 1, \dots, r)$$

was proven, in a markedly different way, by Toscano [11, p. 241, Eq. (11.4)]; indeed, in this special case, the right-hand side of (4.6) would involve the Lauricella function $F_D^{(r)}$ defined by (3.4).

For the Lauricella hypergeometric function $F_A^{(r)}$ defined by (3.1), our summation formula (4.5) *with*

$$p - 1 = q = 0 \quad (\alpha_1 = \alpha) \quad \text{and} \quad p_j = q_j - 1 = 0 \quad \left(\delta_{q_j}^{(j)} = \mu_j; j = 1, \dots, r \right)$$

would readily yield

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \\ & \quad \cdot F_A^{(r)} [\alpha, -n_1, \dots, -n_r; \mu_1, \dots, \mu_r; x_1, \dots, x_r] \\ & = (\Lambda)_n F_{1:1;\dots;1}^{2:1;\dots;1} \left[\begin{array}{c} -n, \alpha : \lambda_1; \dots; \lambda_r; \\ \Lambda : \mu_1; \dots; \mu_r; \end{array} \middle| x_1, \dots, x_r \right] \end{aligned} \quad (4.7)$$

$$(\Lambda := \lambda_1 + \dots + \lambda_r; n, n_j \in \mathbb{N}_0; \Lambda, \lambda_j \notin \mathbb{Z}_0^-; j = 1, \dots, r),$$

which, in the special case when

$$\mu_j = \lambda_j \quad (j = 1, \dots, r),$$

reduces at once to the form:

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \\ & \quad \cdot F_A^{(r)} [\alpha, -n_1, \dots, -n_r; \lambda_1, \dots, \lambda_r; x_1, \dots, x_r] \\ & = (\Lambda)_n {}_2F_1(-n, \alpha; \Lambda; x_1 + \dots + x_r) \end{aligned} \quad (4.8)$$

$$(\Lambda := \lambda_1 + \dots + \lambda_r; n, n_j \in \mathbb{N}_0; \Lambda, \lambda_j \notin \mathbb{Z}_0^-; j = 1, \dots, r).$$

Lastly, we consider the basic multivariable Appell polynomials

$$E_{n_1, \dots, n_r}(a, b_1, \dots, b_r; x_1, \dots, x_r)$$

defined by (*cf.* [4, p. 26]; see also [5])

$$\begin{aligned} E_{n_1, \dots, n_r}(a, b_1, \dots, b_r; x_1, \dots, x_r) & := (-1)^n \frac{(b_1)_{n_1} \cdots (b_r)_{n_r}}{(a + b_1 + \dots + b_r + n)_n} \\ & \quad \cdot F_A^{(r)} [a + b_1 + \dots + b_r + n, -n_1, \dots, -n_r; b_1, \dots, b_r; x_1, \dots, x_r] \\ & \quad (n := n_1 + \dots + n_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r) \end{aligned} \quad (4.9)$$

or, equivalently, by

$$\begin{aligned} E_{n_1, \dots, n_r}(a, b_1, \dots, b_r; x_1, \dots, x_r) & := x_1^{n_1} \cdots x_r^{n_r} \\ & \quad \cdot F_B^{(r)} \left[-n_1, \dots, -n_r, 1 - b_1 - n_1, \dots, 1 - b_r - n_r; \right. \\ & \quad \left. 1 - a - b_1 - \dots - b_r - 2n; \frac{1}{x_1}, \dots, \frac{1}{x_r} \right] \end{aligned} \quad (4.10)$$

$$(n := n_1 + \dots + n_r; n, n_j \in \mathbb{N}_0; j = 1, \dots, r).$$

Indeed these basic multivariable Appell polynomials are *orthogonal* over the simplex

$$\mathcal{T}_r := \{(x_1, \dots, x_r) : x_1 + \dots + x_r \leq 1 \quad (x_j \geq 0; j = 1, \dots, r)\} \quad (4.11)$$

with the weight function

$$w(x_1, \dots, x_r) := x_1^{b_1-1} \cdots x_r^{b_r-1} (1 - x_1 - \cdots - x_r)^a. \quad (4.12)$$

Making use of the definition (4.10), it is not difficult to deduce the following summation formula from the functional relationship (3.6):

$$\begin{aligned} & \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, \dots, n_r} E_{n_1, \dots, n_r} \left(a+1, b_1 - n_1 + 1, \dots, b_r - n_r + 1; \frac{t_1}{\tau}, \dots, \frac{t_r}{\tau} \right) \\ &= \binom{a+B+r+n}{n}^{-1} P_n^{(a+r, B-n)} \left(\frac{2T}{\tau} - 1 \right) \\ & \quad (n, n_j \in \mathbb{N}_0; j = 1, \dots, r; B := b_1 + \cdots + b_r; T := t_1 + \cdots + t_r), \end{aligned} \quad (4.13)$$

where $P_n^{(\alpha, \beta)}(x)$ denotes the classical Jacobi polynomials defined by (cf., e.g., [10, p. 68])

$$P_n^{(\alpha, \beta)}(x) := \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2} \right)^k \left(\frac{x+1}{2} \right)^{n-k} \quad (4.14)$$

or, in terms of the Gauss hypergeometric function, by

$$P_n^{(\alpha, \beta)}(x) := \binom{\alpha + \beta + 2n}{n} \left(\frac{x+1}{2} \right)^n {}_2F_1 \left(-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{x+1} \right). \quad (4.15)$$

Many other interesting corollaries and consequences of our main summation formulas can be deduced in a similar manner.

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