Journal of Computational and Applied Mathematics 140 (2002) 619–637

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Quadrature formulae connected to σ-orthogonal polynomials ☆

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Received 23 August 2000; received in revised form 25 March 2001

Abstract

Let $\mathrm{d}\lambda(t)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k=\int_{\mathbb{R}} t^k\,\mathrm{d}\lambda(t)$, $k=0,1,\ldots,$ exist and are finite, and $\mu_0>0$. Quadrature formulas of Chakalov–Popoviciu type with multiple nodes

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t)=\sum_{v=1}^{n}\sum_{i=0}^{2s_v} A_{i,v} f^{(i)}(\tau_v)+R(f),$$

where $\sigma=\sigma_n=(s_1,s_2,\ldots,s_n)$ is a given sequence of nonnegative integers, are considered. A such quadrature formula has maximum degree of exactness $d_{\max}=2\sum_{v=1}^{n}s_v+2n-1$ if and only if

$$\int_{\mathbb{R}}\prod_{v=1}^{n}(t-\tau_v)^{2s_v+1}t^k\,\mathrm{d}\lambda(t)=0,\quad k=0,1,\ldots,n-1.$$

The proof of the uniqueness of the extremal nodes $\tau_1,\tau_2,\ldots,\tau_n$ was given first by Ghizzetti and Ossicini (Rend. Mat. 6(8) (1975) 1–15). Here, an alternative simple proof of the existence and the uniqueness of such quadrature formulas is presented. In a study of the error term $R(f)$, an influence function is introduced, its relevant properties are investigated, and in certain classes of functions the error estimate is given. A numerically stable iterative procedure, with quadratic convergence, for determining the nodes τ_v, $v=1,2,\ldots,n$, which are the zeros of the corresponding σ-orthogonal polynomial, is presented. Finally, in order to show a numerical efficiency of the proposed procedure, a few numerical examples are included.

MSC: primary 65D30; 65D32; secondary 41A55

Keywords: Nonnegative measure; Moments; Weights; Quadratures with multiple nodes; Nodes; Degree of precision; Error term; Influence function; Convergence; Orthogonal polynomials; s- and σ-orthogonal polynomials

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☆ This work was supported in part by the Serbian Ministry of Science and Technology.

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0377-0427/02/$ - see front matter
PII: S0377-0427(01)00476-9

1. Introduction and preliminaries

A quadrature formula of the form

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t) = \sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) + R(f), \tag{1.1}$$

where $A_{i,v} = A_{i,v}^{(n,s)}, \tau_v = \tau_v^{(n,s)}$ $(i = 0,1,\ldots,2s; v = 1,\ldots,n)$, which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, was considered firstly by P. Turán (see [25]), in the case when $\mathrm{d}\lambda(t) = \mathrm{d}t$ on $[-1,1]$. The case with a weight function $\mathrm{d}\lambda(t) = w(t)\,\mathrm{d}t$ on $[a,b]$ has been investigated by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, etc. (see [17] for references).

The nodes τ_v in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the following integral

$$\Phi \equiv \Phi(a_0, a_1, \ldots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2}\,\mathrm{d}\lambda(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In order to minimize Φ we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k\,\mathrm{d}\lambda(t) = 0, \quad k = 0,1,\ldots,n-1. \tag{1.2}$$

Such polynomials $\pi_n(t)$, which satisfy this new type of orthogonality called "*power orthogonality*" are known as s-orthogonal (or s-self associated) polynomials with respect to the measure $\mathrm{d}\lambda(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

An iterative (unstable) process for computing the coefficients of s-orthogonal polynomials in a special case, when the interval $[a,b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [26].

Milovanović [15] (see also [7,16,17]) gave a stable procedure for numerical construction of s-orthogonal polynomials with respect to $\mathrm{d}\lambda(t)$ on $\mathbb{R}$, taking advantage of the following interpretation of the "orthogonality conditions" (1.2):

$$\int_{\mathbb{R}} \pi_n(t) t^k \pi_n(t)^{2s}\,\mathrm{d}\lambda(t) = 0, \quad k = 0,1,\ldots,n-1,$$

i.e.,

$$\int_{\mathbb{R}} \pi_k^{s,n}(t) t^v\,\mathrm{d}\mu(t) = 0, \quad v = 0,1,\ldots,k-1,$$

where $\{\pi_k^{s,n}\}_{k\in\mathbb{N}_0}$ is a sequence of monic orthogonal polynomials with respect to the new nonnegative measure

$$\mathrm{d}\mu(t) = \mathrm{d}\mu^{s,n}(t) = (\pi_n(t))^{2s}\,\mathrm{d}\lambda(t).$$

Thus, we can conclude that such a sequence of polynomials $\{\pi_k^{s,n}\}_{k\in\mathbb{N}_0}$ exists. Since the measure $\mathrm{d}\mu(t)$ involves the unknown $\pi_n(t)$, these polynomials are implicitly defined (see Engels [4, pp. 214–226]). Of course, we are interested only in $\pi_n(\cdot) = \pi_n^{s,n}(\cdot)$. The polynomials $\pi_k^{s,n}(\cdot) = \pi_k(\cdot),\ k = 0,1,\ldots,$

satisfy a three-term recurrence relation

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, \ldots,$$
$$\pi_{-1}(t) = 0, \quad \pi_0(t) = 1$$

with coefficients α_k, β_k given by the well-known formulas

$$\alpha_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)} \quad (k \in \mathbb{N}_0), \quad \beta_k = \frac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})} \quad (k \in \mathbb{N}), \tag{1.3}$$

where $(f, g) = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}\mu(t)$. By convention, $\beta_0 = \int_{\mathbb{R}} \mathrm{d}\mu(t)$.

Using (1.3) we can form a system of nonlinear equations in unknown recurrence coefficients, and then apply Newton–Kantorovič method to solve it (see [15,7]). The convergence is quadratic.

In this paper, we consider the σ-orthogonal polynomials as a generalization of the s-orthogonal polynomials. The paper is organized as follows. In Section 2, we give an interpretation of σ-polynomials and their connection with quadratures with multiple nodes. In Section 3, a proof of the existence and the uniqueness of Chakalov–Popoviciu quadrature formulas is given. In Section 4, an influence function is introduced, its relevant properties are investigated, and in certain classes of functions the error estimate is given. Section 5 is devoted to a simple and numerically stable iterative method with quadratic convergence for constructing s- and σ-orthogonal polynomials and their zeros. Finally, in order to illustrate the efficiency of our method we give a few numerical examples in Section 5.

2. σ-orthogonal polynomials

Take now a sequence of nonnegative integers $\sigma = (s_1, s_2, \ldots)$. For any $n \in \mathbb{N}$ we denote the corresponding finite sequence $(s_1, s_2, \ldots, s_n)$ by σ_n and consider a generalization of Gauss–Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{2.1}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$ $(i = 0, 1, \ldots, 2s_\nu;\ \nu = 1, \ldots, n)$. Such formulas were derived independently by Chakalov [2,3] and Popoviciu [22]. A deep theoretical progress in this subject was made by Stancu (see [24] and references in it).

In this case, it is important to assume that the nodes $\tau_\nu (= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$\tau_1 < \tau_2 < \cdots < \tau_n, \quad \tau_\nu \in \operatorname{supp}(\mathrm{d}\lambda), \tag{2.2}$$

with odd multiplicities $2s_1 + 1, 2s_2 + 1, \ldots, 2s_n + 1$, respectively, in order to have uniqueness of Chakalov–Popoviciu quadrature formula (2.1) (cf. Karlin and Pinkus [13]). Then this quadrature formula has the maximum degree of exactness $d_{\max} = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1$ if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k\,\mathrm{d}\lambda(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{2.3}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov [2], Popoviciu [22], Morelli and Verna [20], and existence and uniqueness subject to (2.2) by Ghizzetti and Ossicini [9].

The conditions (2.3) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbb{N}_0}$,

$$\pi_{n,\sigma}(t)=\prod_{v=1}^{n}(t-\tau_v^{(n,\sigma)}),\quad \tau_1^{(n,\sigma)}<\tau_2^{(n,\sigma)}<\cdots<\tau_n^{(n,\sigma)},\quad \tau_v^{(n,\sigma)}\in\mathrm{supp}(\mathrm{d}\lambda),$$

such that

$$\int_{\mathbb{R}}\pi_{k,\sigma}(t)\prod_{v=1}^{n}(t-\tau_v^{(n,\sigma)})^{2s_v+1}\,\mathrm{d}\lambda(t)=0,\quad k=0,1,\ldots,n-1.$$

These polynomials are called σ-orthogonal polynomials and they correspond to the sequence $\sigma=(s_1,s_2,\ldots)$. We will often write simple τ_v or $\tau_v^{(n)}$ instead of $\tau_v^{(n,\sigma)}$. If we have $\sigma=(s,s,\ldots)$, the above polynomials reduce to the s-orthogonal polynomials.

The approach given in the previous section can be extended to the σ-orthogonal polynomials (see [12]), providing an algorithm for constructing such polynomials. For a given sequence $\sigma_n=(s_1,s_2,\ldots,s_n)$, the "orthogonality conditions" (2.3) are interpreted as

$$\int_{\mathbb{R}}\pi_{k,\sigma}^{(n)}(t)t^i\,\mathrm{d}\mu(t)=0,\quad i=0,1,\ldots,k-1,$$

where

$$\pi_{n,\sigma}(t)=\pi_{n,\sigma}^{(n)}(t)=\prod_{v=1}^{n}(t-\tau_v^{(n,\sigma)})\quad\text{and}\quad \mathrm{d}\mu(t)=\prod_{v=1}^{n}(t-\tau_v^{(n,\sigma)})^{2s_v}\,\mathrm{d}\lambda(t).$$

Thus, $\{\pi_{k,\sigma}^{(n)}\}$ is a sequence of standard orthogonal polynomials with respect to the measure $\mathrm{d}\mu(t)$ and $\pi_{n,\sigma}^{(n)}$ is the desired σ-orthogonal polynomial $\pi_{n,\sigma}$. Unfortunately, the Newton–Kantorovič method for solving the corresponding system of nonlinear equations cannot be applied in this case as in [15]. Therefore in [12], it was replaced by a version of the secant method. The speed of convergence was superlinear. In Section 5, we give an alternative approach with quadratic convergence.

Remark 2.1. A stable numerical procedure for calculating the coefficients $A_{i,v}$ in (1.1) was recently given by Gautschi and Milovanović [7]. Some alternative methods were proposed by Stroud and Stancu [24], Golub and Kautsky [10], and Milovanović and Spalević [18] (see also [23]). A generalization of methods from [7,18] to the general case when $s_v\in\mathbb{N}_0$, $v=1,\ldots,n$, was derived recently by Milovanović and Spalević [19].

3. A proof of the existence and the uniqueness of (2.1)

Let $[a,b]$ be the support of the nonnegative measure $\mathrm{d}\lambda(t)=w(t)\,\mathrm{d}t$, where $w(t)$ is the weight function. Consider the Chakalov–Popoviciu quadrature formula (2.1) for this case:

$$\int_a^b w(t)f(t)\,\mathrm{d}t=\sum_{v=1}^{n}\sum_{i=0}^{2s_v}A_{i,v}f^{(i)}(\tau_v)+R(f),\tag{3.1}$$

$\tau_\nu \in (a,b)$, with $R(f; w(t)\,\mathrm{d}t) = 0$ for all $f \in \mathscr{P}_{2(\sum_{\nu=1}^n s_\nu + n)-1}$. With $\mathscr{P}_k$ we denoted the set of all polynomials of degree at most k, $k \in \mathbb{N}_0$.

By using some results, which have been given by Ghizzetti and Ossicini [8], we will give an alternative proof of the existence and the uniqueness of the formula (3.1). The first proof of such kind was done in [9].

Define the generalized Gauss problem (see [8, pp. 41–43]). For all notions and notations we refer to [8].

Let us consider the elementary quadrature formula

$$\int_a^b w(t)f(t)\,\mathrm{d}t = \sum_{i=0}^{N-1}\sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \quad [E(f)=0 \Rightarrow R(f)=0],$$

where E is the linear differential operator of order N (see [8]). The question is whether, having fixed nonnegative integers $p_\nu (p_\nu \leqslant N-1)$, $\nu = 1,\ldots,n$, with $(\exists \nu = 1,\ldots,n)\, p_\nu \geqslant 1$, it is possible to make use of the arbitrary nature of these parameters to drop from the formula the values $f^{(i)}(\tau_\nu)$ of the derivatives of order higher than $N - p_\nu - 1$, $\nu = 1,\ldots,n$, that is whether there can exist a formula of the type

$$\int_a^b w(t)f(t)\,\mathrm{d}t = \sum_{\nu=1}^{n}\sum_{i=0}^{N-p_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{3.2}$$

such that $E(f)=0 \Rightarrow R(f)=0$. The answer give the following theorem, whose the proof can be done in similar way as one of Theorem 2.5.I in [8] (see also Problem 2 in [8, p. 45]).

Theorem 3.1. *Given the nodes* $\tau_1,\ldots,\tau_n$, *which satisfies*

$$a \leqslant \tau_1 < \tau_2 < \cdots < \tau_n \leqslant b, \tag{3.3}$$

the linear differential operator E *of order* N *and nonnegative integers* $p_\nu (p_\nu \leqslant N-1)$, $\nu = 1,\ldots,n$, *with* $(\exists \nu = 1,\ldots,n)\, p_\nu \geqslant 1$, *consider the homogeneous boundary differential problem*

$$E(f)=0; \quad f^{(i)}(\tau_\nu)=0, \quad i=0,1,\ldots,N-p_\nu-1,\ \nu=1,\ldots,n. \tag{3.4}$$

If this problem has no nontrivial solutions (*whence* $N \leqslant nN - \sum_{\nu=1}^n p_\nu$) *it is possible to write a quadrature formula of the type* (3.2) *in* $\infty^{nN-\sum_{\nu=1}^n p_\nu - N}$ *different ways. If on the other hand the problem* (3.4) *has* $q(\geqslant 1)$ *linearly independent solutions* $U_j(t)$ $(j=1,\ldots,q)$, *with* $N - nN + \sum_{\nu=1}^n p_\nu \leqslant q \leqslant p_\nu$ *for all* $\nu = 1,\ldots,n$, *then the formula* (3.2) *may apply only if the* q *conditions*

$$\int_a^b w(t)U_j(t)\,\mathrm{d}t = 0, \quad j=1,\ldots,q,$$

are satisfied; *if so there are* $\infty^{nN-\sum_{\nu=1}^n p_\nu - N + q}$ *possible formulae of form* (3.2).

Consider (3.1), with the conditions (3.3) for $\tau_\nu, \nu=1,\ldots,n$, for which $R(f)=0$ for all $f \in \mathscr{P}_{N-1}$, where $N = 2(\sum_{\nu=1}^n s_\nu + n)$. By virtue of Theorem 3.1 we must consider the boundary problem

$$\frac{\mathrm{d}^N f}{\mathrm{d}t^N} = 0$$

with $f^{(i)}(\tau_\nu)=0,\ i=0,1,\ldots,2s_\nu,\ \nu=1,\ldots,n$, and its nontrivial solutions which are

$$t^k\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1},\quad k=0,1,\ldots,n-1.$$

Therefore, (3.1) is possible if and only if

$$\int_a^b w(t)t^k\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\,\mathrm{d}t=0,\quad k=0,1,\ldots,n-1$$

and this shows that the nodes τ_ν must coincide with the zeros of the polynomial $\pi_{n,\sigma}(t)$ of the σ-orthogonal system relative to the measure $w(t)\,\mathrm{d}t$.

With such a choice of the nodes the formula (3.1) is unique since, by the notation of Theorem 3.1, we have that $nN-\sum_{\nu=1}^{n}p_\nu-N+q$, which in our case is $nN-\sum_{\nu=1}^{n}(N-2s_\nu-1)-N+n=0$.

4. The bounding formulas of the remainder in Chakalov–Popoviciu quadratures

Concerning the assumptions on $w(t), f(t)$ for the validity of (3.1) we have the following theorem:

Theorem 4.1. *Formula* (3.1) *is valid under the following hypotheses*:
(a) $w(t)\in L[a,b],\ f(t)\in AC^{N-1}[a,b]$, *if* $a,\ b$ *are finite*;
(b) $t^Nw(t)\in L[a,\infty),\ f(t)\in AC_{\mathrm{loc}}^{N-1}[a,\infty),\ f^{(N)}(t)\int_t^\infty\xi^{N-1}w(\xi)\,\mathrm{d}\xi\in L[a,\infty)$, *if* a *is finite*, $b=\infty$;
(c) $t^Nw(t)\in L(-\infty,\infty),\ f(t)\in AC_{\mathrm{loc}}^{N-1}(-\infty,\infty),\ f^{(N)}(t)\int_{-\infty}^{t}\xi^{N-1}w(\xi)\,\mathrm{d}\xi\in L(-\infty,0],\quad f^{(N)}(t)\int_t^\infty\xi^{N-1}w(\xi)\,\mathrm{d}\xi\in L[0,\infty)$, *if* $a=-\infty,\ b=\infty$.

The proof is the same as one of Theorem 4.13.I in [8, pp. 132–133] and will be omitted.

Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}$ for the remainder in (3.1) we have (see [8]):

$$R(f)=\int_a^b\Phi(t)f^{(N)}(t)\,\mathrm{d}t, \tag{4.1}$$

where the influence-function $\Phi(t)$ is expressed by

$$\Phi(t)=\varphi_i(t)\ \text{for}\ \tau_\nu<t<\tau_{\nu+1},\quad \nu=0,1,\ldots,n;\ \tau_0=a,\tau_{n+1}=b \tag{4.2}$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t)=w(t)$ (since N is even), are given by the formulae

$$\varphi_\nu(t)=\int_a^t w(\xi)v_0(t;\xi)\,\mathrm{d}\xi-\sum_{j=1}^{\nu}\sum_{i=0}^{2s_j}(-1)^iA_{i,j}v_i(t;\tau_j),\quad \nu=0,1\ldots,n, \tag{4.3}$$

where we put $v_k(t;\xi)=(t-\xi)^{N-k-1}/(N-k-1)!$. Notice that $v_{k+1}(t;\xi)=\mathrm{d}v_k(t;\xi)/\mathrm{d}t$. For $\varphi_n(t)$ we have

$$\varphi_n(t)=-\int_t^b w(\xi)v_0(t;\xi)\,\mathrm{d}\xi. \tag{4.4}$$

Without loss of generality, let us consider the case: $s_1 < s_2 < \cdots < s_n$, i.e.,

$$N - 2s_1 - 2 > N - 2s_2 - 2 > \cdots > N - 2s_n - 2.$$

From (4.2) and (4.3) it follows, differentiating k times (with $0 \leqslant k \leqslant N-1$):

$$\Phi^{(k)}(t) = \varphi_v^{(k)}(t) \quad \text{for } t \in (\tau_v, \tau_{v+1}), \quad v = 0, 1, \ldots, n, \tag{4.5}$$

where

$$\varphi_v^{(k)}(t) = \int_a^t w(\xi) v_k(t;\xi)\, d\xi - \sum_{j=1}^{v} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} v_{i+k}(t;\tau_j) \tag{4.6(1)}$$

for $0 \leqslant k \leqslant N - 2s_v - 2$, and for $\ell = v, v-1, \ldots, 1$,

$$\varphi_v^{(k)}(t) = \int_a^t w(\xi) v_k(t;\xi)\, d\xi - \sum_{j=1}^{\ell-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} v_{i+k}(t;\tau_j)$$

$$-\sum_{j=\ell}^{v} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} v_{i+k}(t;\tau_j) \tag{4.6(v+2–ℓ)}$$

for $N - 2s_\ell - 1 \leqslant k \leqslant N - 2s_{\ell-1} - 2$, where we put $s_0 = -\frac{1}{2}$.

For $v = n$ we have

$$\varphi_n^{(k)}(t) = -\int_t^b w(\xi) v_k(t;\xi)\, d\xi, \quad t \in (\tau_n, b). \tag{4.7}$$

Now, we can conclude that

$$\Phi^{(k)}(a) = 0, \ \Phi^{(k)}(b) = 0, \quad k = 0, 1, \ldots, N-1 \tag{4.8}$$

and that the functions $\Phi(t)$, $\Phi'(t), \ldots, \Phi^{(N-2s_n-2)}(t)$ are continuous in $[a,b]$, while $\Phi^{(N-2s_n-1)}(t), \ldots, \Phi^{(N-1)}(t)$ have discontinuities of first kind at the points $\tau_1, \tau_2, \ldots, \tau_n$.

From (4.6(1))–(4.6($v+1$)) we conclude

$$\Phi^{(k)}(t) > 0 \quad \text{for} \quad t \in (a, \tau_1), \quad k = 0, 1, \ldots, N-1. \tag{4.9}$$

From (4.7) we conclude

$$(-1)^k \Phi^{(k)}(t) > 0 \quad \text{for} \quad t \in (\tau_n, b), \quad k = 0, 1, \ldots, N-1. \tag{4.10}$$

Therefore, $\Phi(t) > 0$ on (a, τ_1) and (τ_n, b).

The same conclusions can be derived for an arbitrary case $\sigma = (s_1, s_2, \ldots, s_n)$, $s_v \in \mathbb{N}_0 (v = 1, \ldots, n)$. So, we have just proved that the influence function $\Phi(t)$ defined by (4.2) (together with (4.3) and (4.4)) belongs to the class $C^{N-2s_k-2}[a,b]$, where $N - 2s_k - 2 = \min_{1 \leqslant v \leqslant n} (N - 2s_v - 2)$.

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a,b]$. Using only Rolle theorem, we will give a direct proof of the positivity of the influence function on (a,b). Otherwise, if we identify the function $\Phi(t)$ as a monospline, then the property $\Phi(t) > 0$ on (a,b) is just a corollary from the Micchelli estimate [14] of the number of zeros for monosplines with multiplicities (cf. Braess [1, p. 241]).

We show that $\Phi^{(N-2s_\ell-2)}(t)$ $(\ell=1,\dots,n)$ has at most $2s_\ell+2$ zeros in each interval $[\tau_v,\tau_{v+1}]$, $v=1,\dots,n-1$. In fact, should it have $2s_\ell+3$ of them, for the Rolle theorem, $\Phi^{(N-2s_\ell-1)}(t)$ would have at least $2s_\ell+2$ zeros inside $[\tau_v,\tau_{v+1}]$, $\Phi^{(N-2s_\ell)}(t)$ would have at least $2s_\ell+1$ zeros and so on, until we may conclude that $\Phi^{(N-1)}(t)$ would have at least two zeros inside $[\tau_v,\tau_{v+1}]$. But this is absurd since from (4.5), (4.6($v+1$)) there follows that, for $t\in(\tau_v,\tau_{v+1})$, we have

$$\Phi^{(N-1)}(t)=\varphi_v^{(N-1)}(t)=\int_a^t w(\xi)\,\mathrm{d}\xi-\sum_{j=1}^{v}A_{0j}$$

and this function is increasing (for the hypothesis on $w(t)$).

(a) Firstly, consider the case

$$(s_1\leqslant s_2\leqslant\cdots\leqslant s_k)\wedge(s_k\geqslant s_{k+1}\geqslant\cdots\geqslant s_n),\quad k=1,\dots,n, \tag{4.11}$$

i.e., $(N-2s_1-2\geqslant\cdots\geqslant N-2s_k-2)\wedge(N-2s_k-2\leqslant\cdots\leqslant N-2s_n-2)$, where $k=1,\dots,n$. Since $N-2s_k-2=\min_{1\leqslant v\leqslant n}(N-2s_v-2)$, consider the function $\Phi^{(N-2s_k-2)}(t)$, which is continuous in $[a,b]$.

The function $\Phi^{(N-2s_1-2)}(t)$ is continuous in $[a,\tau_2]$, does not vanish in $(a,\tau_1]$, and has at most $2s_1+2$ zeros in $[\tau_1,\tau_2]$. Further, $\Phi^{(N-2s_v-2)}(t)$ $(2\leqslant v\leqslant k)$ is continuous in $[a,\tau_2]$, does not vanish in $(a,\tau_1]$, and let it have α_1 zeros in $(a,\tau_2]$. Applying Rolle theorem (using (4.8) for the point a) on $[a,\tau_2]$, we conclude that $\Phi^{(N-2s_v-1)}(t)$ has at least α_1 zeros in (a,τ_2), etc., $\Phi^{(N-2s_1-2)}(t)$ has at least α_1 zeros in (a,τ_2). Since $\alpha_1\leqslant 2s_1+2$, we have that $\Phi^{(N-2s_v-2)}(t)$ $(2\leqslant v\leqslant k)$ has at most $2s_1+2$ zeros in $(a,\tau_2]$.

The function $\Phi^{(N-2s_2-2)}(t)$ is continuous in $[a,\tau_3]$, in $(a,\tau_1]$ does not vanish, in $(a,\tau_2]$ has at most $2s_1+2$ zeros, on the basis of the preceding deduction, and has at most $2s_2+2$ zeros in $[\tau_2,\tau_3]$, therefore, has at most $(2s_1+2)+(2s_2+2)$ zeros in $(a,\tau_3]$. Further, $\Phi^{(N-2s_v-2)}(t)$ $(3\leqslant v\leqslant k)$ is continuous in $[a,\tau_3]$, does not vanish in $(a,\tau_1]$, in $(a,\tau_2]$ has at most $2s_1+2$ zeros and let it have α_2 zeros in $(a,\tau_3]$. Applying Rolle theorem (using (4.8) for the point a) on $[a,\tau_3]$, we conclude that $\Phi^{(N-2s_v-1)}(t)$ has at least α_2 zeros in (a,τ_3), etc., $\Phi^{(N-2s_2-2)}(t)$ has at least α_2 zeros in (a,τ_3). Since $\alpha_2\leqslant(2s_1+2)+(2s_2+2)$, we have that $\Phi^{(N-2s_v-2)}(t)$ $(3\leqslant v\leqslant k)$ has at most $(2s_1+2)+(2s_2+2)$ zeros in $(a,\tau_3]$.

Proceed in an analogous way, we prove that the function $\Phi^{(N-2s_k-2)}(t)$ has at most $(2s_1+2)+(2s_2+2)+\cdots+(2s_{k-1}+2)$ zeros in $(a,\tau_k]$. In a similar manner, we prove that $\Phi^{(N-2s_k-2)}(t)$ has at most $(2s_{k+1}+2)+(2s_{k+2}+2)+\cdots+(2s_n+2)$ zeros in $[\tau_k,b)$. Therefore, $\Phi^{(N-2s_k-2)}(t)$ has at most

$$\begin{aligned}&(2s_1+2)+\cdots+(2s_{k-1}+2)+(2s_{k+1}+2)+\cdots+(2s_n+2)\\&\quad=\sum_{v=1}^{n}(2s_v+2)-2s_k-2=N-2s_k-2\end{aligned} \tag{4.12}$$

zeros in $(a,b)=(a,\tau_k]\cup[\tau_k,b)$.

We then show that the influence-function does not vanish in (a,b) and therefore is positive, because it is such in $(a,\tau_1),(\tau_n,b)$. In fact, if $\Phi(t)$ should vanish at one point in (a,b), using (4.8) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, etc., $\Phi^{(N-2s_k-2)}(t)$ would vanish at least $N-2s_k-1$ times, in contraposition with the deduction (4.12), because $N-2s_k-1\leqslant N-2s_k-2$ gives $1\leqslant 0$.

Notice that in σ which satisfy (4.11) are included and ones which satisfy $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, or $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$, include the case $\sigma = (s, s, \ldots, s)$ (of s-orthogonal polynomials and Gauss–Turán quadrature formulas), which was considered in [8, pp. 131–139] (see also [21]).

For $n = 1, 2$, (4.11) represents the general case. In (b) we will consider and the general case for $n \geqslant 3$.

(b) We will give a proof for a sufficient general case. Then, proceed in analogous way, a proof for any other case can be performed. The consideration will be given in detail.

Let $n = 10$ and $s_5 > s_9 > s_7 > s_1 > s_{10} > s_4 > s_3 > s_8 > s_6 > s_2$, i.e.,

$$N - 2s_5 - 2 < N - 2s_9 - 2 < \cdots < N - 2s_2 - 2.$$

The zero τ_ν $(2 \leqslant \nu \leqslant n-1)$ of the σ-orthogonal polynomial $\pi_{n,\sigma}(t)$, we will call the *point of partition* of $[a, b]$ if for it holds $N - 2s_{\nu-1} - 2 \geqslant N - 2s_\nu - 2 < N - 2s_{\nu+1} - 2$. Also, τ_1 is the *point of partition* of $[a, b]$ if it holds that $N - 2s_1 - 2 = \min_{1 \leqslant \nu \leqslant n} (N - 2s_\nu - 2)$. Similarly, it is τ_n when $N - 2s_n - 2 = \min_{1 \leqslant \nu \leqslant n} (N - 2s_\nu - 2)$. Denote by I the *index set* whose elements are the indices of the points of partition of $[a, b]$. It is clear that $I \subset \{1, 2, \ldots, n\}$. Therefore, in our case, the points of partition of $[a, b]$ are τ_5, τ_7, τ_9, and $I = \{5, 7, 9\}$. Then $[a, b]$ by the points of partition we divide to the *intervals of partition*, in our case $[a, \tau_5], [\tau_5, \tau_7], [\tau_7, \tau_9], [\tau_9, b]$, on which we consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, $\nu \in I$. It is clear that $[a, b]$ can be represented as the union of the intervals of partition.

For $\nu \in I$, order in the decreasing sequence the values $N - 2s_\nu - 2$, and consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_7-2)}(t)$, $\Phi^{(N-2s_9-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, respectively.

(b.1) Firstly, consider $\Phi^{(N-2s_7-2)}(t)$, which is continuous in $[\tau_5, \tau_9] = [\tau_5, \tau_7] \cup [\tau_7, \tau_9]$.

Consequently, consider $[\tau_{p-1}, \tau_{p+1}]$ for $p = 6$ and $p = 8$. For $\nu \in \{p-1, p, p+1\}$ (the indices of the zeros belong to $[\tau_{p-1}, \tau_{p+1}]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last is one which corresponds to the point of partition τ_7, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_p-2)}(t)$, $\Phi^{(N-2s_7-2)}(t)$, respectively.

The function $\Phi^{(N-2s_p-2)}(t)$ is continuous in $[\tau_{p-1}, \tau_{p+1}] = [\tau_{p-1}, \tau_p] \cup [\tau_p, \tau_{p+1}]$ and has at most $(2s_p + 2) + (2s_p + 2)$ zeros in it.

Let $\Phi^{(N-2s_7-2)}(t)$ have $\alpha_7^{(p)}$ zeros in $[\tau_{p-1}, \tau_{p+1}]$. Then, applying Rolle theorem, we conclude that $\Phi^{(N-2s_7-1)}(t)$ has at least $\alpha_7^{(p)} - 1$ zeros in (τ_{p-1}, τ_{p+1}), etc., $\Phi^{(N-2s_p-2)}(t)$ has at least $\alpha_7^{(p)} - (2s_7 - 2s_p)$ zeros in (τ_{p-1}, τ_{p+1}). Therefore, we have $\alpha_7^{(p)} - (2s_7 - 2s_p) \leqslant (2s_p + 2) + (2s_p + 2)$, i.e., $\alpha_7^{(p)} \leqslant (2s_7 + 2) + (2s_p + 2)$.

Thus, $\Phi^{(N-2s_7-2)}(t)$ has at most

$$\alpha_7^{(6)} + \alpha_7^{(8)} = (2s_7 + 2) + (2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2)$$

zeros in $[\tau_5, \tau_9] = [\tau_5, \tau_7] \cup [\tau_7, \tau_9]$.

(b.2) Further, consider the function $\Phi^{(N-2s_9-2)}(t)$, which is continuous in $[\tau_5, b] = [\tau_5, \tau_9] \cup [\tau_9, b]$.

Firstly, consider $[\tau_5, \tau_9]$. Let $\Phi^{(N-2s_9-2)}(t)$ have α_9 zeros in it. Then, applying Rolle theorem, we conclude that $\Phi^{(N-2s_9-1)}(t)$ has at least $\alpha_9 - 1$ zeros in (τ_5, τ_9), etc., $\Phi^{(N-2s_7-2)}(t)$ has at least $\alpha_9 - (2s_9 - 2s_7)$ zeros in (τ_5, τ_9). Therefore, on the basis of (b.1), we have $\alpha_9 - (2s_9 - 2s_7) \leqslant (2s_6 + 2) + (2s_7 + 2) + (2s_7 + 2) + (2s_8 + 2)$, i.e., $\alpha_9 \leqslant (2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2) + (2s_9 + 2)$.

Now, consider $[\tau_9, b)$. For $v \in \{9, 10\}$ (the indices of zeros belong to $[\tau_9, b)$), order in a decreasing sequence the values $N - 2s_v - 2$ so that the last is one which corresponds to the point of partition τ_9, and then consider the functions $\Phi^{(N-2s_v-2)}(t)$, respectively. So, in our case we consider $\Phi^{(N-2s_{10}-2)}(t)$, $\Phi^{(N-2s_9-2)}(t)$, respectively.

The function $\Phi^{(N-2s_{10}-2)}(t)$ is continuous in $[\tau_9, b) = [\tau_9, \tau_{10}] \cup [\tau_{10}, b)$ and has at most $2s_{10}+2$ zeros in it. Let $\Phi^{(N-2s_9-2)}(t)$ have β_9 zeros in $[\tau_9, b)$. Then, applying Rolle theorem (with the conditions (4.8) for the point $t = b$), we conclude that $\Phi^{(N-2s_9-1)}(t)$ has at least $\beta_9 - 1$ zeros in (τ_9, b), etc., $\Phi^{(N-2s_{10}-2)}(t)$ has at least β_9 zeros in (τ_9, b). Therefore, we have $\beta_9 \leqslant 2s_{10} + 2$.

Thus, $\Phi^{(N-2s_9-2)}(t)$ has at most $(2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2) + (2s_9 + 2) + (2s_{10} + 2)$ zeros in $[\tau_5, b) = [\tau_5, \tau_9] \cup [\tau_9, b)$.

(b.3) Finally, consider $\Phi^{(N-2s_5-2)}(t)$, which is continuous in $[a, b] = [a, \tau_5] \cup [\tau_5, b]$.

First, consider $[a, \tau_5]$. For $v \in \{1, 2, 3, 4, 5\}$ (the indices of zeros belong to $[a, \tau_5]$), order in the decreasing sequence the values $N - 2s_v - 2$ so that the last is one which corresponds to the point of partition τ_5, and then consider the functions $\Phi^{(N-2s_v-2)}(t)$, respectively. Thus, in our case we consider $\Phi^{(N-2s_2-2)}(t)$, $\Phi^{(N-2s_3-2)}(t)$, $\Phi^{(N-2s_4-2)}(t)$, $\Phi^{(N-2s_1-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, respectively.

The function $\Phi^{(N-2s_2-2)}(t)$ is continuous in $[\tau_1, \tau_3] = [\tau_1, \tau_2] \cup [\tau_2, \tau_3]$ and has at most $(2s_2 + 2) + (2s_2 + 2)$ zeros in it.

Depending on $q \in \{3, 4, 5\}$, we put $p = q$ when $q = 3, 4$ and $p = 1$ when $q = 5$, and suppose that $\Phi^{(N-2s_p-2)}(t)$ has α_q zeros in $[\tau_1, \tau_q]$. Then, applying Rolle theorem, we conclude that $\Phi^{(N-2s_p-1)}(t)$ has at least $\alpha_q - 1$ zeros in (τ_1, τ_q), etc., $\Phi^{(N-2s_{q-1}-2)}(t)$ has at least $\alpha_q - (2s_p - 2s_{q-1})$ zeros in (τ_1, τ_q). Therefore, we have $\alpha_q - (2s_p - 2s_{q-1}) \leqslant \sum_{v=2}^{q-1} (2s_v + 2) + (2s_{q-1} + 2)$, i.e.,

$$\alpha_q \leqslant \sum_{v=2}^{q-1} (2s_v + 2) + (2s_p + 2) \quad (q = 3, 4, 5).$$

Further, for $q = 3$ and 4, in $[\tau_1, \tau_{q+1}] = [\tau_1, \tau_q] \cup [\tau_q, \tau_{q+1}]$ the function $\Phi^{(N-2s_q-2)}(t)$ is continuous and has at most $\alpha_q + (2s_q + 2)$ zeros, i.e., at most $(2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$ if $q = 3$, and $(2s_2 + 2) + (2s_3 + 2) + (2s_4 + 2) + (2s_4 + 2)$ if $q = 4$.

In $(a, \tau_5] = (a, \tau_1] \cup [\tau_1, \tau_5]$ the function $\Phi^{(N-2s_1-2)}(t)$ is continuous and has at most $(2s_1 + 2) + (2s_2 + 2) + (2s_3 + 2) + (2s_4 + 2)$ zeros.

Let in $(a, \tau_5]$ the function $\Phi^{(N-2s_5-2)}(t)$ have β_5 zeros. Then, applying Rolle theorem (with the conditions (4.8) for the point $t = a$), we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at least β_5 zeros in (a, τ_5). Therefore, in (a, τ_5) the function $\Phi^{(N-2s_5-2)}(t)$ has at most $\sum_{v=1}^{4} (2s_v + 2)$ zeros.

Taking into account the last conclusion for $\Phi^{(N-2s_5-2)}(t)$ in (a, τ_5) and the conclusion which we can obtain in an analogous way from (b.2) in (τ_5, b), we conclude that $\Phi^{(N-2s_5-2)}(t)$ has at most $\sum_{v=1}^{10} (2s_v + 2) - 2s_5 - 2 = N - 2s_5 - 2$ zeros in (a, b).

Further, in a similar way as in (a), we conclude that $\Phi(t)$ is positive on (a, b). On the basis of the previous considerations we have just proved the following statement:

Theorem 4.2. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a, b]$, the influence function $\Phi(t)$ defined by* (4.2) (*together with* (4.3) *and* (4.4)) *belongs to the class $C^{N-2s_k-2}[a, b]$ where $N - 2s_k - 2 = \min_{1 \leqslant v \leqslant n} (N - 2s_v - 2)$ and it is positive inside (a, b).*

Now, we can estimate the remainder in the formulas of the type (3.1), by using (4.1).

1° If $f \in AC^{N-1}[a,b]$ and $a, b \in \mathbb{R}$ we have

$$|R(f)| \leqslant \max_{a \leqslant t \leqslant b} \Phi(t) V_{N-1} = \Phi(\tau_0) V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ which is absolutely continuous on the interval $[a,b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a,b), then there exists $\tau_0 \in (a,b)$ such that $\max_{a \leqslant t \leqslant b} \Phi(t) = \Phi(\tau_0)$.

2° If $f^{(N)}(t)$ is bounded in $[a,b]$, i.e.,

$$M_N = \sup_{a<t<b} |f^{(N)}(t)|, \quad -\infty \leqslant a < b \leqslant \infty,$$

we have

$$|R(f)| \leqslant M_N \int_a^b \Phi(t)\,\mathrm{d}t.$$

3° If $f \in C^N[a,b], a, b \in \mathbb{R}$, because $\Phi(t) > 0$ on (a,b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t)\,\mathrm{d}t, \quad \xi \in (a,b).$$

5. A numerical method for computing of the zeros of s- and σ-orthogonal polynomials

An idea for finding s-orthogonal polynomials, i.e., their zeros τ_ν, solving the system of nonlinear equations (1.2) in unknowns $\tau_1, \tau_2, \ldots, \tau_n$, can be found in [4, pp. 214–226] (see also [5]). In this section, we use this idea in order to construct the corresponding s- and σ-orthogonal polynomials.

5.1. Numerical procedure

For a given sequence $\sigma = \sigma_n = (s_1, s_2, \ldots, s_n)$ we rewrite the orthogonality conditions (2.3) as the following system of nonlinear equations:

$$F_j(\tau_1, \tau_2, \ldots, \tau_n) \equiv \int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^{j-1}\,\mathrm{d}\lambda(t) = 0, \quad j = 1, \ldots, n \tag{5.1}$$

and put

$$\mathbf{t} = [\tau_1 \;\; \tau_2 \;\; \ldots \;\; \tau_n]^\top, \quad \mathbf{t}^{(k)} = [\tau_1^{(k)} \;\; \tau_2^{(k)} \;\; \ldots \;\; \tau_n^{(k)}]^\top, \quad k = 0, 1, \ldots$$

and

$$\mathbf{F}(\mathbf{t}) = [F_1(\mathbf{t}) \;\; F_2(\mathbf{t}) \;\; \ldots \;\; F_n(\mathbf{t})]^\top.$$

If $W = W(\mathbf{t})$ is the corresponding Jacobian of $\mathbf{F}(\mathbf{t})$, we can apply the Newton–Kantorovič method

$$\mathbf{t}^{(k+1)} = \mathbf{t}^{(k)} - W^{-1}(\mathbf{t}^{(k)})\mathbf{F}(\mathbf{t}^{(k)}), \quad k = 0, 1, 2, \ldots \tag{5.2}$$

for determining the zeros of the σ-orthogonal polynomial $\pi_{n,\sigma}$. If a sufficiently good approximation $\mathbf{t}^{(0)}$ is chosen, the convergence of the method (5.2) is quadratic. Notice that the Newton–Kantorovič method was used in [15] (see also [7]), but for the system in $2n$ unknowns.

The elements of the Jacobian

$$W = W(\mathbf{t}) = [w_{jk}]_{n\times n} = \left[\frac{\partial F_j}{\partial \tau_k}\right]_{n\times n},$$

we calculate by the formulas

$$w_{jk} = \frac{\partial F_j}{\partial \tau_k} = -(2s_k+1)\int_{\mathbb{R}} t^{j-1}(t-\tau_k)^{2s_k}\prod_{\substack{v=1\\ v\neq k}}^{n}(t-\tau_v)^{2s_v+1}\,\mathrm{d}\lambda(t), \tag{5.3}$$

where $j,k = 1,\ldots,n$.

All of the integrals in (5.1) and (5.3) can be calculated exactly, except for rounding errors, by using a Gauss–Christoffel quadrature formula with respect to the measure $\mathrm{d}\lambda(t)$ (see [11]),

$$\int_{\mathbb{R}} g(t)\,\mathrm{d}\lambda(t) = \sum_{v=1}^{M} A_v^{(M)} g(\tau_v^{(M)}) + R_M(g),$$

taking $M = n + \sum_{v=1}^{n} s_v$ nodes. This formula is exact for all polynomials g of degree at most

$$2M-1 = 2\left(n+\sum_{v=1}^{n} s_v\right) - 1 = 2(n-1)+1+2\sum_{v=1}^{n} s_v.$$

5.2. A choice of initial values

Let α_k, β_k $(k=0,1,\ldots)$ be coefficients in the three-term recurrence relation for a system $\{\pi_k\}_{k=0}^{+\infty}$ of (monic) orthogonal polynomials $\pi_k(\cdot) = \pi_k(\cdot;\mathrm{d}\lambda)$ relative to the measure $\mathrm{d}\lambda(t)$ on the real line $\mathbb{R}$, and let $\tau_1,\ldots,\tau_n$ be zeros of $\pi_n(t)$, i.e., the eigenvalues of the symmetric (tridiagonal) Jacobi matrix

$$J_n(\mathrm{d}\lambda) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathrm{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}.$$

In determining the initial value $\mathbf{t}^{(0)}$ for the method (5.2), we need to take into account the condition (2.2) which holds for all σ-orthogonal polynomials

$$\pi_{n,\sigma};\ n = 1,2,\ldots,\quad s_v = 0,1,2,\ldots\ (v = 1,2,\ldots,n).$$

Starting by the zeros of the (σ-) orthogonal polynomial for $\sigma = (0,0,\ldots,0)$, i.e., from $\mathbf{t}^{(0)} = [\tau_1\ \tau_2\ \ldots\ \tau_n]^\top$, by using (5.2) we obtain the zeros of σ-orthogonal polynomial for $\sigma = (1,0,\ldots,0)$. For small n (e.g., $n \leqslant 5$), such a choice of the initial values could be reasonable, but for bigger n

we need a better way for getting a sufficiently good approximation $\mathbf{t}^{(0)}$. Then, the convergence of the method (5.2) is quadratic. One of the simplest way to do it consists in introducing a "weight factor" $w_f(>0)$, such that $\mathbf{t}^{(0)} = w_f[\tau_1 \ \ \tau_2 \ \ \dots \ \ \tau_n]^\top$. Many numerical experiments show that w_f must be near 1. However, there is a much better way for modifying this starting vector.

According to (5.1) for the case $\sigma = (1, 0, \dots, 0)$, we can define a new measure $d\hat{\lambda}(t)$ by $d\hat{\lambda}(t) = (t - \tau_1)^2 \, d\lambda(t)$ and construct a sequence of (monic) polynomials $\hat{\pi}_k(\cdot) = \pi_k(\cdot; d\hat{\lambda})$ orthogonal with respect to this measure, such that

$$\int_{\mathbb{R}} (t - \tau_1)^2 \hat{\pi}_n(t) t^{j-1} \, d\lambda(t) = 0, \quad j = 1, \dots, n.$$

Then, the zeros $\hat{\tau}_1, \dots, \hat{\tau}_n$ of $\hat{\pi}_n(t)$, i.e., the eigenvalues of the corresponding Jacobi matrix $J_n(d\hat{\lambda})$, become very appropriate initial values. Thus, we take $\mathbf{t}^{(0)} = [\hat{\tau}_1 \ \ \hat{\tau}_2 \ \ \dots \ \ \hat{\tau}_n]^\top$.

An elegant algorithm for getting $J_n(d\hat{\lambda})$ consists in applying one QR step with the shift τ_1. Namely, if

$$J_{n+1}(d\lambda) - \tau_1 I_{n+1} = QR, \quad \text{then} \quad J_n(d\hat{\lambda}) = (RQ + \tau_1 I_{n+1})_{n \times n}.$$

Here, Q is an orthogonal matrix and R is upper triangular with nonnegative diagonal elements. Thus, discarding the last row and last column in the previous matrix of order $n+1$, we obtain $J_n(d\hat{\lambda})$. This algorithm is quite stable (cf. Gautschi [6]).

5.2.1. Construction of s-orthogonal polynomials

Starting by the zeros of the (σ-) orthogonal polynomial for $\sigma = (0, 0, \dots, 0)$ and using the previous procedure we determine the starting vector $\mathbf{t}^{(0)} = [\hat{\tau}_1 \ \ \hat{\tau}_2 \ \ \dots \ \ \hat{\tau}_n]^\top$. Now, applying the method (5.2) we obtain the zeros of σ-orthogonal polynomial for $\sigma = (1, 0, \dots, 0)$.

In each of following steps, we raise only one s_ν to $s_{\nu+1}$ via the following path:

$$\begin{aligned} &(1,0,0,\dots,0,0),\ (1,1,0,\dots,0,0),\dots,(1,1,1,\dots,1,1),\\ &(1,1,1,\dots,1,2),\ (1,1,1,\dots,2,2),\dots,(2,2,2,\dots,2,2),\\ &(3,2,2,\dots,2,2),\ (3,3,2,\dots,2,2),\dots,(3,3,3,\dots,3,3),\\ &(3,3,3,\dots,3,4),\ \dots \end{aligned} \tag{5.4}$$

until we get the desired s-orthogonal polynomial with $\sigma = \sigma_n = (s, s, s, \dots, s, s)$. In each step, except in the first one (when we use the described procedure), the initial value for each of the zeros we determine by Lagrange extrapolating polynomial by using the values, obtained in the previous steps, for the corresponding zero.

5.2.2. Construction of σ-orthogonal polynomials

Let $\sigma = \sigma_n = (s_1, s_2, \dots, s_n)$. If we put $\bar{s} = \max\{s_\nu \mid \nu = 1, \dots, n\}$, then in the first step we start by the zeros of the σ-orthogonal polynomial

$$\sigma = (\bar{s}, \bar{s}, \dots, \bar{s})$$

(i.e., the s-orthogonal polynomial with $s = \bar{s}$, which is constructed in above given way). By using (5.2) we obtain the zeros of the next σ-orthogonal polynomial.

Thus, in each step we reduce only one s_ν to $s_{\nu-1}$ via the following path:

$$\begin{aligned}
&(\bar{s},\bar{s},\ldots,\bar{s},\bar{s}),\ (\bar{s},\bar{s},\ldots,\bar{s},\bar{s}-1),\ldots,\ (\bar{s},\bar{s},\ldots,\bar{s},s_n),\\
&(\bar{s},\bar{s},\ldots,\bar{s}-1,s_n),\ldots,\ (\bar{s},\bar{s},\ldots,s_{n-1},s_n),\\
&\vdots\\
&(\bar{s},\bar{s}-1,\ldots,s_{n-1},s_n),\ldots,\ (\bar{s},s_2,\ldots,s_{n-1},s_n),\\
&(\bar{s}-1,s_2,\ldots,s_{n-1},s_n),\ldots,\ (s_1,s_2,\ldots,s_{n-1},s_n),
\end{aligned}$$

until we get the desired σ-orthogonal polynomial with $\sigma=(s_1,s_2,\ldots,s_{n-1},s_n)$. In each step, except in the first one, the initial values $\tau_\nu^{(0)}$, $\nu=1,\ldots,n$, we determine by Lagrange extrapolating polynomial by using the values, obtained in the previous steps, for the corresponding zero.

6. Numerical results

Using the procedures outlined in Section 5 for constructing s- and σ-orthogonal polynomials we prepared corresponding software with the following types of the basic polynomials $\pi_n(\cdot;\mathrm{d}\lambda)$ (identified by the integer `ipoly`):

```
c  ipoly - integer identifying the kind of polynomials:
c
c   0 = Nonclassical polynomials with given coefficients
c       in the three-term recurrence relation
c   1 = Legendre polynomials on [-1,1]
c   2 = Legendre polynomials on [0,1]
c   3 = Chebyshev polynomials of the first kind
c   4 = Chebyshev polynomials of the second kind
c   5 = Jacobi polynomials with parameters al=.5, be=-.5
c   6 = Jacobi polynomials with parameters al,be
c   7 = generalized Laguerre polynomials with parameter al
c   8 = Hermite polynomials
c   9 = generalized Gegenbauer polynomials with parameters al,be
c  10 = polynomials for the logistic weight
c       w(t)=e^{-t}/(1+e^{-t})^2 on the real line
c
c  al,be - parameters for Jacobi, generalized Laguerre
c       and generalized Gegenbauer polynomials
c  For ipoly=9, the weight function is given by
c       w(x)=|x|^mu(1-x^2)^al, where be=(mu-1)/2.
```

All computations were done using Fortran 77 in double precision arithmetic with machine precision $\approx 2.22\times10^{-16}$. The procedure is quite numerically stable. In order to illustrate the efficiency of the proposed procedure, we give a few numerical examples.

Example 6.1. The zeros τ_ν of Legendre s-orthogonal polynomials, for some s,n, in the case of the Legendre measure $\mathrm{d}\lambda(t)=\mathrm{d}t$ on $[-1,1]$, are given in Table 6.1. Using Q-arithmetic (with machine

Table 6.1

(n,s)	v	τ_v (D-arith.)	τ_v (Q-arith.)
$(9,20)$	$1,9$	∓ 0.983775235585653	$\mp 0.983775235585652911812417493937 17$
	$2,8$	∓ 0.864275591060013	$\mp 0.8642755910600128500396739003569 2$
	$3,7$	∓ 0.641102332236528	$\mp 0.6411023322365281814423105798303 9$
	$4,6$	∓ 0.341008917072948	$\mp 0.3410089170729484586480269250492 4$
	5	0	0
$(11,15)$	$1,11$	∓ 0.988926442955277	$\mp 0.9889264429552771370440372176896 6$
	$2,10$	∓ 0.907974418885766	$\mp 0.9079744188857660087495731456481 3$
	$3,9$	∓ 0.753899396730503	$\mp 0.7538993967305025039782822933675 4$
	$4,8$	∓ 0.539093528704772	$\mp 0.5390935287047719417336955396164 1$
	$5,7$	∓ 0.280859527675888	$\mp 0.2808595276758881435125856994857 9$
	6	0	0

Table 6.2

(n,s)	v	τ_{2v-1}	τ_{2v}
$(8,8)$	1	6.86581496611533(−1)	6.21833617332603(+0)
	2	1.74998124446690(+1)	3.50177309272737(+1)
	3	5.96612976637955(+1)	9.30479545060901(+1)
	4	1.38448571011771(+2)	2.04629999599374(+2)

precision $\approx 1.93 \times 10^{-34}$) we can obtain the corresponding results with much higher precision (the last column in the same table).

Notice that the algebraic degree of precision of the corresponding Gauss–Turán quadrature formula, for example for $n=9$ and $s=20$ is 377.

Vincenti [26] also applied his process to the Legendre case and gave numerical results in the following cases: $n=2,3, 1 \leqslant s \leqslant 10$; $n=4,5,\ 1 \leqslant s \leqslant 5$; $n=6,7, 1 \leqslant s \leqslant 3$; $n=8,9, 1 \leqslant s \leqslant 2$; $n=10,11, s=1$. When n and s increase, his process becomes numerically unstable.

Example 6.2. In Table 6.2 the zeros τ_v of Laguerre s-orthogonal polynomial $L_{8,8}^{-1/2}(t)$, in the case of the measure $\mathrm{d}\lambda(t)=t^{-1/2}\,\mathrm{e}^{-t}\,\mathrm{d}t$ on $(0,+\infty)$, are given. (Numbers in parentheses denote decimal exponents.)

The method presented in Section 5 for the construction of s-orthogonal polynomials (see the path (5.4)) can be used and for the construction of σ-orthogonal polynomials for σ which belong to (5.4). Table 6.3 shows the zeros one of such σ-orthogonal polynomial for

$$\sigma=\sigma_8=(3,3,3,4,4,4,4,4).$$

Table 6.3

σ	ν	$\tau_{2\nu-1}$	$\tau_{2\nu}$
σ_8	1	2.68359224301233(−1)	2.43080103060716(+0)
	2	6.85565845191951(+0)	1.45478471601133(+1)
	3	2.67958396826477(+1)	4.39580116979721(+1)
	4	6.77657348446215(+1)	1.02919750773582(+2)

Table 6.4

σ	ν	$\tau_{2\nu-1}$	$\tau_{2\nu}$
$(0,1,2,3,4,5,0,1,2,3)$	1	−9.8845093941627(−1)	−9.5318409624038(−1)
	2	−8.5235706959736(−1)	−6.3570636273369(−1)
	3	−2.6778094438363(−1)	2.2011058968623(−1)
	4	5.0890710522041(−1)	6.4647909455086(−1)
	5	8.1515358350296(−1)	9.5850334120945(−1)
$(1,4,1,4,1,4,1,4,1,4)$	1	−9.8259959744955(−1)	−8.8945500733345(−1)
	2	−7.1868364748596(−1)	−4.8483263059522(−1)
	3	−2.0833697591839(−1)	8.6581698385070(−2)
	4	3.7407536827518(−1)	6.2894329433030(−1)
	5	8.2884348076387(−1)	9.5625208963718(−1)
$(15,0,0,12,15,3,5,7,9,11)$	1	−9.5176299664704(−1)	−8.3100687977284(−1)
	2	−7.9153006951918(−1)	−6.2011996407615(−1)
	3	−1.8221672595688(−1)	1.5412267835982(−1)
	4	3.5083349688219(−1)	5.8018250575978(−1)
	5	8.0791382042706(−1)	9.6870250897253(−1)

The number of iterations in (5.2), which correspond to the elements of path (5.4), equal:

$$7,6,6,6,6,6,6,6;\ \ 4,6,6,6,6,7,7,10;\ \ 5,7,6,6,6,6,6,7;\ \ 4,6,6,7,7,$$

respectively.

Example 6.3. Table 6.4 shows the zeros τ_ν of σ-orthogonal polynomials $C^2_{n,\sigma}(t)$ in the case of Gegenbauer measure

$$\mathrm{d}\lambda(t)=(1-t^2)^{3/2}\,\mathrm{d}t \quad \text{on } [-1,1],$$

when $n=10$, and $\sigma=\sigma_{10}$ is given by

$$(0,1,2,3,4,0,1,2,3), \quad (1,4,1,4,1,4,1,4,1,4), \quad (15,0,0,12,15,3,5,7,9,11),$$

Table 6.5

σ	$(2,2,5)$	$(2,5,2)$	$(5,2,2)$
$\tau_1^{(3,\sigma)}$	−2.83566649051922	−2.79216254193118	−1.94743219873889
$\tau_2^{(3,\sigma)}$	−0.76005918718102	0	0.76005918718102
$\tau_3^{(3,\sigma)}$	1.94743219873889	2.79216254193118	2.83566649051922

Table 6.6

t	∓ 1	$\pm\tau_1$	∓ 0.6	∓ 0.5	∓ 0.4	∓ 0.3	∓ 0.2	∓ 0.1
$\Phi(t)$	0	2.12(−13)	2.54(−11)	1.57(−10)	5.35(−10)	1.23(−9)	2.11(−9)	2.88(−9)

respectively. Notice that the corresponding quadrature formula (2.1) for the last σ-sequence has the maximum degree of exactness $d_{\max} = 173$.

Example 6.4. As we mentioned before, σ-orthogonal polynomials are unique under (2.2) with the corresponding multiplicities $m_\nu = 2s_\nu + 1$, $\nu = 1, \ldots, n$. Otherwise, the number of distinct σ-polynomials is

$$\frac{n!}{k_1!k_2!\ldots k_q!}$$

for some q $(1 \leqslant q \leqslant n)$, where k_i is the number of nodes of multiplicity $m_j = i$, each node counted exactly once, and $\sum_{i=1}^{q} k_i = n$.

For example, in the case $n = 3$, with multiplicities $2, 2, 11$, we have three different Hermite σ-polynomials ($w(t) = \mathrm{e}^{-t^2}$ on $\mathbb{R}$), which correspond to $\sigma = (2,2,5)$, $(2,5,2)$, and $(5,2,2)$ (see Table 6.5).

Notice that in the symmetric case $(2,5,2)$ the zeros are symmetrically distributed with respect to the origin. Also, we can see that the σ-orthogonal polynomial in the third case ($\sigma = (5,2,2)$) can be obtained from the first one by changing variable $t := -t$.

Example 6.5. By using the numerical methods from Section 5, for calculating the nodes, and [19], for calculating the coefficients, in the Chakalov–Popoviciu quadrature formula (2.1), i.e., (3.1), we can tabulate the corresponding influence function. Consider the Legendre case with $w(t) = 1$ on $[-1, 1]$. Let $\sigma = (1, 0, 1)$. Therefore, we have a symmetric task. The results will show that the nodes of the corresponding quadrature are symmetrically distributed with respect to the origin, namely,

$$\tau_1 = -\tau_3 = -0.75531134455904, \quad \tau_2 = 0,$$

the influence function is even (see Table 6.6), and

$$\max_{-1 \leqslant t \leqslant 1} \Phi(t) = \Phi(0) = 3.18 \times 10^{-9}.$$

Similarly, in the case $\sigma = (2, 0, 2)$,

$$\max_{-1 \leqslant t \leqslant 1} \Phi(t) = \Phi(0) = 2.55 \times 10^{-14}.$$

Therefore, these results can be use in estimations given at the end of Section 4.

Acknowledgements

The authors are grateful to the referees for their helpful comments and suggestions.

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