Orthogonal polynomials relative to a generalized Marchenko–Pastur probability measure

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Abstract

The Marchenko–Pastur probability measure, of interest in the asymptotic theory of random matrices, is generalized in what appears to be a natural way. The orthogonal polynomials and their three-term recurrence relation for this generalized Marchenko–Pastur measure are obtained in explicit form, analytically as well as symbolically using Mathematica. Special cases involve Chebyshev polynomials of all four kinds. Supporting Matlab software is provided.

Keywords: Orthogonal polynomials, generalized Marchenko–Pastur measure, three-term recurrence relation

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1 Introduction

In 1967 (see [9]), the Ukrainian mathematicians Vladimir Alexandrovich Marchenko (b. 1922) and Leonid Andreevich Pastur (b. 1937), working on the

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asymptotic theory of large random matrices, came to consider a probability measure now known as the Marchenko–Pastur measure or the Marchenko–Pastur law. Here, entirely out of curiosity and without any applications in mind, we generalize the measure in the same way as Jacobi polynomials are a generalization of Chebyshev polynomials (of the second kind). We are interested in the orthogonal polynomials relative to this generalized measure and in their three-term recurrence relation. Both are obtained explicitly in Sections 3 and 4. In special cases the orthogonal polynomials are identified in Section 5 in terms of Chebyshev polynomials of all four kinds. Double-precision and variable-precision Matlab routines for generating the recurrence coefficients of the generalized Marchenko–Pastur measure are also provided.

2 The generalized Marchenko–Pastur measure

The Marchenko–Pastur measure, as formulated in [1], [8], [10], is supported on the interval $[a, b]$, where

$$a = (\sqrt{c}-1)^2, \quad b = (\sqrt{c}+1)^2, \quad c > 0, \tag{2.1}$$

and defined, if $c > 1$, by the density function

$$w(x) = \frac{1}{2\pi c}\, x^{-1}[(b-x)(x-a)]^{1/2} + \left(1 - \frac{1}{c}\right)\delta_0, \quad a \le x \le b, \tag{2.2}$$

where δ_0 is the Dirac delta function at $x = 0$ with mass 1, and, if $c < 1$, by

$$w(x) = \frac{1}{2\pi c}\, x^{-1}[(b-x)(x-a)]^{1/2}, \quad a \le x \le b. \tag{2.3}$$

A natural generalization is to replace the exponent $1/2$ of $b - x$ and $x - a$ by α and β, respectively, as in the case of Jacobi measures, assuming $\alpha > -1$, $\beta > -1$. The constant $(2\pi)^{-1}$ multiplying $x^{-1}[(b-x)(x-a)]^{1/2}$ in (2.2) happens to be a normalization factor when $c > 1$. Likewise, $(2\pi c)^{-1}$ is the normalization factor when $c < 1$ (cf. (5.5).) Thus, our generalization of the Marchenko–Pastur measure is

$$w(x;\alpha,\beta,c) = \begin{cases} \dfrac{1}{c\,\mu_0}\, x^{-1}\,(b-x)^\alpha (x-a)^\beta + \left(1-\dfrac{1}{c}\right)\delta_0 & \text{if } c > 1, \\ \dfrac{1}{\mu_0}\, x^{-1}\,(b-x)^\alpha (x-a)^\beta & \text{if } c < 1, \end{cases} \qquad a \le x \le b, \tag{2.4}$$

where

$$\mu_0 = \mu_0(\alpha, \beta, c) = \int_a^b x^{-1} \, (b-x)^\alpha (x-a)^\beta \, \mathrm{d}x, \quad c > 0. \tag{2.5}$$

The weight function w in (2.4) is clearly a probability measure, for $0 < c < 1$ as well as for $c > 1$, in the sense that $\int_a^b w(x; \alpha, \beta, c)\mathrm{d}x = 1$. The case $c = 1$ can be transformed to a Jacobi weight function with Jacobi parameters α, $\beta - 1$ (if $\beta > 0$). This case is classical and well known, and therefore will not be considered any further.

The monic polynomials orthogonal with respect to the weight function (2.4) will be denoted by $\pi_k(x)$, $k = 0, 1, 2, \ldots$. As is well known, they satisfy a three-term recurrence relation,

$$\begin{gathered} \pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, 2, \ldots, \\ \pi_{-1}(x) = 0, \quad \pi_0(x) = 1. \end{gathered} \tag{2.6}$$

Our interest is in the coefficients α_k, β_k. We use the convention $\beta_0 = \int_a^b w(x; ; \alpha, \beta, c)\mathrm{d}x$.

Notice that, when $c < 1$, the factor $1/\mu_0$ in (2.4) is unimportant, as regards orthogonal polynomials, since it has no effect on them whatsoever. Not so when $c > 1$, where the constant $1/(c\mu_0)$ multiplies only the continuous part of the measure. A change of that constant will therefore also change the orthogonal polynomials.

3 Orthogonal polynomials and recurrence coefficients for (2.4) when $c > 1$

3.1 Orthogonal polynomials on $[-1, 1]$

Associated with the weight function (2.4) is the inner product

$$(u, v) = \int_a^b \left[\frac{1}{c\mu_0} \, x^{-1}(b-x)^\alpha (x-a)^\beta + \left(1 - \frac{1}{c}\right) \delta_0 \right] u(x)v(x)\mathrm{d}x, \tag{3.1}$$

where μ_0 is given by (2.5). Changing variables,

$$x = 2\sqrt{c}\, t + c + 1, \tag{3.2}$$

mapping the interval $[-1, 1]$ onto $[a, b]$, the constant μ_0 and inner product (3.1) become

$$\mu_0 = (2\sqrt{c})^{\alpha+\beta} \int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t, \tag{3.3}$$

respectively

$$(u, v)^* = \frac{1}{c} \frac{\displaystyle\int_{-1}^{1} \frac{u(t)v(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t}{\displaystyle\int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t} + \left(1 - \frac{1}{c}\right) u(-g)v(-g), \tag{3.4}$$

where $w^{(\alpha,\beta)}(t)$ is the Jacobi weight function and

$$g = \frac{c+1}{2\sqrt{c}}\,. \tag{3.5}$$

Note that g is invariant with respect to the transformation $c \mapsto 1/c$ and $g > 1$ for all $c \neq 1$.

We denote the monic polynomials orthogonal with respect to the inner product (3.4) by π_k^*. Inserting (3.2) into (2.6) and using $\pi_k(2\sqrt{c}\,t + c + 1) = (2\sqrt{c})^k\, \pi_k^*(t)$ yields the recurrence relation for the π_k^*,

$$\pi_{k+1}^*(t) = (t - \alpha_k^*)\pi_k^*(t) - \beta_k^*\pi_{k-1}^*(t), \tag{3.6}$$

where

$$\alpha_k^* = \frac{\alpha_k - (c+1)}{2\sqrt{c}}\,, \quad \beta_k^* = \frac{\beta_k}{4c}\,, \quad k \geq 0. \tag{3.7}$$

For later use, we obtain $\alpha_0^* = \alpha_0/(2\sqrt{c}) - g$ in (3.7) more explicitly, noting that

$$\alpha_0 = \int_a^b x\, w(x; \alpha, \beta, c)\mathrm{d}x = \frac{1}{c\mu_0} \int_a^b (b-x)^\alpha (x-a)^\beta \mathrm{d}x, \tag{3.8}$$

the term with the Dirac function being zero because of the factor x in the first integral of (3.8). Applying the change of variables (3.2) to the second integral in (3.8) as well as to the integral in (2.5) representing μ_0 gives

$$\alpha_0 = \frac{2^{\alpha+\beta+2}}{\sqrt{c}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \left(\int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t \right)^{-1}. \tag{3.9}$$

The integral in large parentheses can be identified with Mathematica to be (3.10)

$$\int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t = \frac{2^{\alpha+\beta+1}}{g-1}\,B(\alpha+1,\beta+1)\,{}_2F_1(1,\beta+1;\alpha+\beta+2;-2/(g-1)),$$

where B is Euler's beta integral and ${}_2F_1$ the hypergeometric function. Therefore, by (3.7) for $k=0$,

$$\alpha_0^* = \frac{g-1}{c}\,\frac{1}{{}_2F_1(1,\beta+1;\alpha+\beta+2;-2/(g-1))} - g, \quad g = \frac{c+1}{2\sqrt{c}}\,. \tag{3.11}$$

Since π_k^* is orthogonal with respect to the inner product (3.4) to all polynomials of degree $k-1$, putting $u(t)=\pi_k^*(t)$ and $v(t)=v_j(t)=(t+g)^j$ in (3.4) gives

(3.12)

$$(\pi_k^*, v_j)^* = \frac{1}{c}\,\frac{\displaystyle\int_{-1}^{1} \pi_k^*(t)(t+g)^{j-1} w^{(\alpha,\beta)}(t)\mathrm{d}t}{\displaystyle\int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t} + \left(1-\frac{1}{c}\right)\pi_k^*(-g)\delta_{j,0} = 0,$$

$$j = 0,1,2,\ldots,k-1,$$

where $\delta_{j,0}$ is the Kronecker delta. The relations for $j=1,2,\ldots,k-1$ imply that

$$\int_{-1}^{1} \pi_k^*(t)(t+g)^{j-1} w^{(\alpha,\beta)}(t)\mathrm{d}t = 0, \quad j=1,2,\ldots,k-1,$$

that is, π_k^* is orthogonal with respect to the Jacobi weight function to all polynomials of degree $\le k-2$, hence

Theorem 1 *The monic orthogonal polynomials π_k^* relative to the inner product (3.4) are*

$$\pi_k^*(t) = \mathring{P}_k^{(\alpha,\beta)}(t) + \gamma_k \mathring{P}_{k-1}^{(\alpha,\beta)}(t), \quad k \ge 1, \tag{3.13}$$

where $\mathring{P}_k^{(\alpha,\beta)}(t)$ denote the monic Jacobi polynomials and γ_k are constants depending on α, β, c.

The constants γ_k can be determined from the orthogonality relation (3.12) with $j=0$,

$$\int_{-1}^{1} \frac{\pi_k^*(t)}{t+g}\, w^{(\alpha,\beta)}(t)\,\mathrm{d}t + (c-1)\pi_k^*(-g)\int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t = 0, \tag{3.14}$$

as will be shown in the next subsection.

3.2 Determination of the coefficients γ_k

Let α_k^J, β_k^J be the recurrence coefficients for the monic Jacobi polynomials,

$$\begin{aligned} &\mathring{P}_{k+1}^{(\alpha,\beta)}(t) = (t-\alpha_k^J)\mathring{P}_k^{(\alpha,\beta)}(t) - \beta_k^J \mathring{P}_{k-1}^{(\alpha,\beta)}(t). \quad k=0,1,2,\ldots, \\ &\mathring{P}_0^{(\alpha,\beta)}(t) = 1, \quad \mathring{P}_{-1}^{(\alpha,\beta)}(t) = 0, \end{aligned} \tag{3.15}$$

where [5, Table 1.1][1]
(3.16)

$$\begin{aligned} \alpha_k^J &= \frac{\beta^2-\alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}, \quad k\ge 0, \\ \beta_0^J &= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \quad \beta_k^J = \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2((2k+\alpha+\beta)^2-1)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k\ge 1, \end{aligned}$$

except in the case $k=0$ and $\alpha+\beta=0$, when $\alpha_0^J=-\alpha$, and in the case $k=1$ and $\alpha+\beta+1=0$, when $\beta_1^J=-2\alpha(\alpha+1)$.

Theorem 2 *The recurrence coefficients α_k^*, β_k^* in* (3.6)*, expressed in terms of the γ_k and the recurrence coefficients in* (3.16)*, are*

$$\begin{aligned} &\alpha_0^* = \alpha_0^J - \gamma_1, \quad \alpha_k^* = \alpha_k^J + \gamma_k - \gamma_{k+1}, \quad k\ge 1, \\ &\beta_k^* = \beta_k^J + \gamma_k(\alpha_{k-1}^J - \alpha_k^J) - \gamma_k(\gamma_k - \gamma_{k+1}), \quad k\ge 1. \end{aligned} \tag{3.17}$$

Alternatively, for $k\ge 2$,

$$\beta_k^* = \frac{\gamma_k}{\gamma_{k-1}}\,\beta_{k-1}^J. \tag{3.18}$$

[1]In the formula for β_0^J at the bottom of Table 1.1 of [5], the denominator should read $\Gamma(\alpha+\beta+2)$ instead of $\Gamma(\alpha+\beta+1)$.

Proof. Using (3.13) in (3.6), we have

$$\begin{aligned}\overset{\circ}{P}{}_{k+1}^{(\alpha,\beta)}(t) + \gamma_{k+1}\overset{\circ}{P}{}_{k}^{(\alpha,\beta)}(t) &= (t-\alpha_k^*)\left[\overset{\circ}{P}{}_{k}^{(\alpha,\beta)}(t) + \gamma_k \overset{\circ}{P}{}_{k-1}^{(\alpha,\beta)}(t)\right] \\ &\quad -\beta_k^*\left[\overset{\circ}{P}{}_{k-1}^{(\alpha,\beta)}(t) + \gamma_{k-1}\overset{\circ}{P}{}_{k-2}^{(\alpha,\beta)}(t)\right].\end{aligned} \tag{3.19}$$

Expressing $t\,\overset{\circ}{P}{}_{k}^{(\alpha,\beta)}$ and $t\,\overset{\circ}{P}{}_{k-1}^{(\alpha,\beta)}$ from (3.15) in terms of the monic Jacobi polynomials and putting the results into (3.19) yields

(3.20)

$$\begin{aligned}\left[\alpha_k^J - \alpha_k^* + \gamma_k - \gamma_{k+1}\right]\overset{\circ}{P}{}_{k}^{(\alpha,\beta)}(t) + \left[\beta_k^J - \beta_k^* + \gamma_k(\alpha_{k-1}^J - \alpha_k^*)\right]\overset{\circ}{P}{}_{k-1}^{(\alpha,\beta)}(t) \\ + \left[\gamma_k\beta_{k-1}^J - \gamma_{k-1}\beta_k^*\right]\overset{\circ}{P}{}_{k-2}^{(\alpha,\beta)}(t) \equiv 0.\end{aligned}$$

Since orthogonal polynomials are linearly independent, all coefficients in (3.20) must vanish, that is,

(3.21)

$$\left.\begin{aligned}&\alpha_k^* = \alpha_k^J + \gamma_k - \gamma_{k+1}, \\ &\beta_k^* = \beta_k^J + \gamma_k(\alpha_{k-1}^J - \alpha_k^*) = \beta_k^J + \gamma_k(\alpha_{k-1}^J - \alpha_k^J - \gamma_k + \gamma_{k+1})\end{aligned}\right\} \quad k \ge 1$$

$$\beta_k^* = \frac{\gamma_k}{\gamma_{k-1}}\,\beta_{k-1}^J, \quad k \ge 2.$$

This proves (3.17) for $k \ge 1$ and (3.18).

Noting that by (3.13) there holds $\pi_1^*(t) = \overset{\circ}{P}_1{}^{(\alpha,\beta)}(t) + \gamma_1$ and $\overset{\circ}{P}_1{}^{(\alpha,\beta)}(t) = t - \alpha_0^J$ by (3.15), we get $\pi_1^*(t) = t - \alpha_0^J + \gamma_1$. On the other hand, by (3.6), $\pi_1^*(t) = t - \alpha_0^*$, so that $\alpha_0^* = \alpha_0^J - \gamma_1$, which is the first relation in (3.17).

□

Now the first relation in (3.17), together with (3.11) and (3.16) for $k = 0$, yields

(3.22)

$$\gamma_1 = \frac{\beta-\alpha}{\alpha+\beta+2} - \frac{g-1}{c}\,\frac{1}{{}_2F_1(1,\beta+1;\alpha+\beta+2;-2/(g-1))} + g, \quad g = \frac{c+1}{2\sqrt{c}}.$$

Let

$$p_k = \int_{-1}^{1} \frac{\overset{\circ}{P}{}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t + (c-1)\overset{\circ}{P}{}_k^{(\alpha,\beta)}(-g) \int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{t+g}\,\mathrm{d}t, \quad k = 0,1,2,\dots, \quad c > 1, \tag{3.23}$$

where the integral on the far right is known; see (3.10). If in (3.12) with $j = 0$ we replace π_k^* by (3.13) and then solve for γ_k yields the remaining γ's,

$$\gamma_k = -\frac{p_k}{p_{k-1}}, \quad k = 2,3,\dots\ . \tag{3.24}$$

To obtain a recurrence relation for the p_k, note first of all that from (3.23) we have

$$p_0 = c\, I, \quad I = \int_{-1}^{1} \frac{w^{(\alpha,\beta}(t)}{t+g}\,\mathrm{d}t, \quad c > 1. \tag{3.25}$$

Next, divide both sides of (3.15) by $t+g$ and integrate from -1 to 1 with weight function $w^{(\alpha,\beta)}$. We obtain, using (3.23) with k replaced by $k-1$, k, $k+1$, and (3.15) with $t = -g$,

$$\begin{aligned} p_{k+1} &= \int_{-1}^{1} \frac{t\,\overset{\circ}{P}{}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t - \alpha_k^J p_k - \beta_k^J p_{k-1} \\ &\qquad + (c-1)I\,[\overset{\circ}{P}{}_{k+1}^{(\alpha,\beta)}(-g) + \alpha_k^J \overset{\circ}{P}{}_k^{(\alpha,\beta)}(-g) + \beta_k^J \overset{\circ}{P}{}_{k-1}^{(\alpha,\beta)}(-g)] \\ &= \int_{-1}^{1} \frac{t\,\overset{\circ}{P}{}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t - \alpha_k^J p_k - \beta_k^J p_{k-1} - (c-1)I\,g\overset{\circ}{P}{}_k^{(\alpha,\beta)}(-g) \end{aligned}$$

or, writing $t/(t+g) = 1 - g/(t+g)$,

$$p_{k+1} = \int_{-1}^{1} \overset{\circ}{P}{}_k^{(\alpha,\beta)}(t)\, w^{(\alpha,\beta)}(t)\mathrm{d}t - (g + \alpha_k^J)p_k - \beta_k^J p_{k-1}.$$

Therefore, by orthogonality,

$$p_{k+1} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,\delta_{k,0} - (g+\alpha_k^J)p_k - \beta_k^J p_{k-1}, \quad k = 0,1,2,\dots, \tag{3.26}$$

where $\delta_{k,0}$ is the Kronecker delta and $p_{-1} = 0$. If we restrict k to $k \geq 1$, the inhomogeneous term in (3.26) disappears, and we have

$$p_{k+1} + (g+\alpha_k^J)p_k + \beta_k^J p_{k-1} = 0, \quad k = 1,2,3,\dots\ . \tag{3.27}$$

3.3 The three-term recurrence relation (3.27)

In this subsection it doesn't matter whether $c > 1$ or $c < 1$ since the parameter g in (3.27) is invariant with respect to the transformation $c \mapsto 1/c$.

The difference equation (3.27) is of Poincaré type, i.e., its coefficients

$$g + \alpha_k^J \to g, \quad \beta_k^J \to 1/4$$

have finite limits as $k \to \infty$, and the characteristic polynomial is

$$z^2 + \frac{c+1}{2\sqrt{c}}\, z + \frac{1}{4}\,. \tag{3.28}$$

The two zeros

$$z_1 = \frac{-1}{2\sqrt{c}}\,, \quad z_2 = \frac{-\sqrt{c}}{2}\,,$$

having different moduli, implies (cf. [4, Theorem 2.2]) that there are two linearly independent solutions $p_k^{(1)}$, $p_k^{(2)}$ of (3.27) such that

$$\frac{p_{k+1}^{1)}}{p_k^{(1)}} \to \frac{-1}{2\sqrt{c}}, \quad \frac{p_{k+1}^{(2)}}{p_k^{(2)}} \to \frac{-\sqrt{c}}{2} \quad \text{as} \quad k \to \infty. \tag{3.29}$$

One is dominant and the other minimal (see [4, p. 25] for terminology).

Theorem 3 *Let*

$$f_k = \int_{-1}^{1} \frac{\mathring{P}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t, \quad g_k = \mathring{P}_k^{(\alpha,\beta)}(-g), \quad k = 0, 1, 2, \ldots\,. \tag{3.30}$$

Both f_k and g_k are solutions of the difference equation (3.27)*, the first being minimal and the other dominant.*

Proof. The fact that f_k and g_k are solutions of (3.27) is readily verified. To show minimality of f_k, it suffices to show that

$$\lim_{k\to\infty} r_k = 0, \tag{3.31}$$

where

$$r_k = \frac{1}{\mathring{P}_k^{(\alpha,\beta)}(-g)} \int_{-1}^{1} \frac{\mathring{P}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t. \tag{3.32}$$

Since normalization of the Jacobi polynomials in (3.32) is irrelevant, we may drop the circle on top of the P's. Using the Cauchy–Schwarz inequality, we then have

$$\begin{aligned} |r_k| &= \frac{1}{|P_k^{(\alpha,\beta)}(-g)|}\left|\int_{-1}^{1}\frac{P_k^{(\alpha,\beta)}(t)}{t+g}\,w^{(\alpha,\beta)}(t)\mathrm{d}t\right| \\ &\leq \frac{1}{|P_k^{(\alpha,\beta)}(-g)|}\,||P_k^{(\alpha,\beta)}||\,||(\,\cdot\,+g)^{-1}||, \end{aligned}$$

where $||u|| = \sqrt{\int_{-1}^{1} u^2(t)w^{(\alpha,\beta)}(t)\mathrm{d}t}$. Since $||P_k^{(\alpha,\beta)}|| = \mathrm{O}(1/\sqrt{k})$ (cf. [11, p. 132]), $||(\,\cdot+g)^{-1}||$ is a positive constant not depending on k, and $|P_k^{(\alpha,\beta)}(-g)| = P_k^{(\beta,\alpha)}(g) \to \infty$ if $g > 1$ (cf.[16, Theorem 8.21.7]), there indeed holds (3.31).

□

3.4 Recurrence coefficients

The three-term recurrence relation (2.6) of interest can now be obtained from (3.7),

$$\begin{aligned} &\alpha_k = 2\sqrt{c}\,\alpha_k^* + c + 1, \quad k \geq 0, \\ &\beta_0 = 1, \quad \beta_k = 4\,c\,\beta_k^*, \quad k \geq 1, \end{aligned} \tag{3.33}$$

with α_0^* given by (3.11) and α_k^*, β_k^*, $k \geq 1$, by (3.21).

Theorem 4 *The recurrence coefficients α_k, β_k satisfy*

$$\lim_{k\to\infty} \alpha_k = 1 + c, \quad \lim_{k\to\infty} \beta_k = c. \tag{3.34}$$

Proof. The coefficients α_k^*, β_k^* in (3.6) have limits 0 resp. 1/4 as $k \to \infty$ (see [15], [13], [3, Theorem 4]). Therefore, letting $k \to \infty$ in (3.33) yields (3.34).

□

4 Orthogonal polynomials and recurrence coefficients for (2.4) when $c < 1$

The analysis is essentially the same as in the previous section, once the delta function as well as the factor $1/c$ in (2.4) have been removed. This requires, however, a few adjustments. Specifically, delete the factor c multipying ${}_2F_1$ both in (3.22) and (3.11). Moreover, replace p_k in (3.23) by

$$p_k = \int_{-1}^{1} \frac{\mathring{P}_k^{(\alpha,\beta)}(t)}{t+g}\, w^{(\alpha,\beta)}(t)\mathrm{d}t, \quad k = 0,1,2,\dots, \quad c < 1, \tag{4.1}$$

which, as shown in Theorem 3, is a minimal solution of (3.27).

The coefficients γ_k, $k \geq 2$, in Theorem 1 are again determined by (3.24), but now with p_k given in (4.1).

Theorem 5 *The coefficients γ_k in Theorem 1 satisfy*

$$\lim_{k\to\infty} \gamma_k = \sqrt{c}/2, \quad c \neq 1. \tag{4.2}$$

Proof. If $c > 1$, the limit (4.2), by (3.24), is the limit

$$-\lim_{k\to\infty} \frac{p_{k+1}}{p_k} \tag{4.3}$$

for the dominant solution p_k of (3.27) in (3.23), that is, $\sqrt{c}/2$, and if $c < 1$, the limit (4.3) for the minimal solution in (4.1), that is, again $\sqrt{c}/2$.

□

Remarks to Theorem 5.

1. When $\alpha = \beta = 1/2$ and both $c > 1$ and $c < 1$, the limit in (4.2) is attained instantaneously, that is, $\gamma_1 = \gamma_2 = \gamma_3 = \cdots$. The same is true if $\alpha = -\beta = \pm 1/2$ and $c < 1$.

2. When $\alpha = \beta = -1/2$ and $c < 1$, the limit in (4.2) is attained almost instantaneously, that is, $\gamma_2 = \gamma_3 = \cdots$. (High-precision computation may be required to make this visible.)

Computation of the minimal solution p_k of the recurrence relation (3.27) in forward direction is unstable, more so the smaller c. A stable algorithm to compute p_k for $0 \le k \le N$ is as follows (cf. [4, Eqs. (3.9)]). Select an integer $\nu > N$ and apply the following backward/forward recursion,

$$
\begin{aligned}
&r_\nu^{(\nu)} = -\sqrt{c}/2, \quad r_{k-1}^{(\nu)} = \frac{-\beta_k^J}{g + \alpha_k^J + r_k^{(\nu)}}, \quad k = \nu, \nu-1, \dots 1, \\
&f_0^{(\nu)} = p_0, \quad f_k^{(\nu)} = r_{k-1}^{(\nu)} f_{k-1}^{(\nu)}, \quad k = 1, 2, 3, \dots, N,
\end{aligned}
\tag{4.4}
$$

where (cf. [4, Eq. (30)]) $r_\nu^{(\nu)}$ is taken to be the right-hand limit in (3.29). Moreover (cf. (3.10)),

$$
p_0 = \frac{2^{\alpha+\beta+1}}{g-1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \, {}_2F_1(1, \beta+1; \alpha+\beta+2; -2/(g-1)).
$$

Since $f_k^{(\nu)} \to p_k$, $k = 0, 1, 2, \dots, N$, as $\nu \to \infty$, one applies (4.4) for a sequence of increasing ν-values, $\nu > N$, until two successive $f_k^{(\nu)}$ agree to the desired precision. In the special cases of Chebyshev weight functions of all four kinds, this is not necessary since the $r_{k-1}^{(\nu)}$, owing to the essentially constant values of α_k^J, β_k^J in these cases do not depend on ν nor on k (except for the single value of r_0 in the case of the Chebyshev weight function of the first kind). Besides, the p_k are explicitly known in this case (cf. §5).

c	N	ν	c	N	ν	c	N	ν
.01	10	14	.1	10	19	.5	10	40
	20	24		20	28		20	49
	50	54		50	58		50	78
	100	103		100	107		100	126
.05	10	17	.3	10	27	.8	10	103
	20	26		20	36		20	112
	50	56		50	66		50	140
	100	105		100	115		100	187

Table 1: Value of ν required in (4.4) to obtain 13-digit accuracy when using Matlab double-precision arithmetic in the case of the Legendre weight function

When the algorithm (4.4) is run in Matlab double precision, the smallest value of ν guaranteeing 13-digit accuracy is shown in Table 1 for the case $\alpha = \beta = 0$ and for selected values of c and N.

In the case of c very close to 1 (hence ν very large), one may get away with forward recursion and only moderate increase, if any, of the working precision.

The procedure described in this and the previous section is implemented in the Matlab functions `GMP.m` and `sGMP.m` in double resp. variable precision; visit `https://www.cs.purdue.edu/archives/2002/wxg/codes/GMP.html`.

5 Examples

The general results of §§ 3 and 4 are here specialized to the cases α, $\beta = \pm 1/2$ and $\alpha = \beta = 0$. In the former case, the desired recurrence coefficients α_n, β_n are obtained by the procedures of §§ 3 and 4 in symbolic form as functions of n and c, making use of the symbolic capabilities of Mathematica. In the latter case, numerical values are provided for $0 \le n \le 9$, $c = 4$, and $c = 1/4$.

5.1 Chebyshev weight function of the first kind

Here, $\alpha = \beta = -1/2$. The hypergeometric function needed in (3.10), (3.11), and (3.22) is [14, Eq. 15.4.6]

$$ {}_2F_1(1, 1/2; 1; -x) = {}_2F_1(1/2, 1; 1; -x) = (1 + x)^{-1/2}, $$

and therefore, since $x = 2/(g - 1) = 4\sqrt{c}/(\sqrt{c} - 1)^2$,

$$ {}_2F_1(1, 1/2, ; 1; -2/(g - 1)) = \begin{cases} (\sqrt{c} - 1)/(\sqrt{c} + 1) & \text{if } c > 1, \\ (1 - \sqrt{c})/(1 + \sqrt{c}) & \text{if } c < 1. \end{cases} $$

Using (3.3) and (3.10), we get

$$ \mu_0 = \begin{cases} \pi/(c - 1) & \text{if } c > 1, \\ \pi/(1 - c) & \text{if } c < 1. \end{cases} \tag{5.1} $$

With regard to the recurrence relation (3.26), we have, by (3.25), (3.10), and the first paragraph of §4,

$$ p_0 = \begin{cases} 2\, c^{3/2}\pi/(c - 1) & \text{if } c > 1, \\ 2\sqrt{c}\,\pi/(1 - c) & \text{if } c < 1. \end{cases} $$

From (3.26) with $k=0$ and $k=1$, noting that $\alpha_k^J = 0$, $\beta_1^J = 1/2$, we get

$$p_1 = \begin{cases} -(1+c^2)\,\pi/(c-1) & \text{if } c>1, \\ -2\,c\,\pi/(1-c) & \text{if } c<1, \end{cases}$$

$$p_2 = \begin{cases} (c^3-c^2+c+1)\pi/(2\sqrt{c}(c+1)) & \text{if } c>1, \\ c^{3/2}\pi/(1-c), & \text{if } c<1. \end{cases}$$

Finally, since $\beta_k^J = 1/4$ for $k \geq 2$,

$$p_{k+1} = -\frac{c+1}{2\sqrt{c}}\,p_k - \frac{1}{4}\,p_{k-1}, \quad k=2,3,\dots\,, \tag{5.2}$$

which is a three-term recurrence relation with constant coefficients and characteristic polynomial (3.28). This allows us to explicitly obtain the solution p_k of (5.2) with the above starting values p_1, p_2. The result, when $c>1$, is
(5.3)

$$p_k = 2\pi\sqrt{c}\,(-1)^k\left[\left(\frac{\sqrt{c}}{2}\right)^k + \frac{c+1}{c-1}\left(\frac{1}{2\sqrt{c}}\right)^k\right], \quad c>1, \quad k=3,4,\dots\,,$$

and when $c<1$,

$$p_k = \frac{4\pi\sqrt{c}}{1-c}(-1)^k\left(\frac{\sqrt{c}}{2}\right)^k, \quad c<1, \quad k=3,4,\dots\,. \tag{5.4}$$

Once we have p_k, we get all the γ_k from (3.24) and (3.22), which in turn gives

$$\alpha_0^* = \begin{cases} -\,(c^2+1)/(2\,c^{3/2}) & \text{if } c>1, \\ -1/\sqrt{c} & \text{if } c<1 \end{cases}$$

from (3.11) and α_k^*, β_k^* for $k \geq 1$ from (3.21), and thus α_k, β_k from (3.33). All this can be done numerically as well as symbolically.

The symbolic results are displayed in Table 2.

It is evident again, as was proved in Theorem 4, that α_n and β_n have limits $c+1$ resp. c as $n \to \infty$, but what is noteworthy, and follows from Remark 2 to Theorem 5, is that in the case $c<1$ these limits are attained almost instantaneously.

	n	$[\alpha_n,\, \beta_n]$
$c > 1$	0	$\left[\dfrac{c-1}{c}\,,\, 1\right]$
	1	$\left[c+2+\dfrac{(c-1)^2}{c(c^2+1)}, \dfrac{(c-1)(c^2+1)}{c^2}\right]$
	> 1	$\left[c+1-\dfrac{(c-1)^3(c+1)c^{n-1}}{(c^n-c^{n-1}+c+1)(c^{n+1}-c^n+c+1)}\,,\right.$ $\left. c\,\dfrac{(c^{n-1}-c^{n-2}+c+1)(c^{n+1}-c^n+c+1)}{(c^n-c^{n-1}+c+1)^2}\right]$
$c < 1$	0	$[1-c, 1]$
	1	$[2\,c+1, 2\,c\,(1-c)]$
	> 1	$[c+1, c]$

Table 2: Recurrence coefficients for the measure (2.4) in the case of the Chebyshev weight function of the first kind

In order to affirm the validity of the data shown in Table 2 and in the subsequent Tables 3 – 5, we used the modified Chebyshev algorithm [5, §2.1.7] based on the modified moments $m_k = \int_a^b w(x;\alpha,\beta,c)(x-a)^k \mathrm{d}x$, $k = 0, 1, 2, \ldots$, and implemented in the first author's `SOPQ` package [6, §2.1] as well as in the second author's Mathematica package *Orthogonal polynomials* [2], [12]. When run in sufficiently high precision (to combat possible instabilities), they produced results in agreement with those obtained by evaluating the symbolic expressions in Table 2 for selected values of c, both with $c > 1$ and $c < 1$, and for n as large as 100.

5.2 Chebyshev weight function of the second kind

This is the original case $\alpha = \beta = 1/2$. The hypergeometric function needed here is [14, Eq. 15.4.17]

$$ {}_2F_1(1, 3/2; 3; -x) = 4\left(1 + \sqrt{1+x}\right)^{-2}, $$

where

$$ x = \frac{2}{g-1} = \frac{4\sqrt{c}}{(\sqrt{c}-1)^2}\,. $$

A short calculation yields

$$ {}_2F_1(1, 3/2; 3; -2/(g-1)) = \begin{cases} (\sqrt{c}-1)^2/c & \text{if } c > 1, \\ (\sqrt{c}-1)^2 & \text{if } c < 1. \end{cases} $$

Therefore, by (3.3) and (3.10),

$$ \mu_0 = \begin{cases} 2\pi & \text{if } c > 1, \\ 2\pi\, c & \text{if } c < 1. \end{cases} \tag{5.5} $$

The recurrence relation (5.2) in this case is

$$ p_{k+1} = -\frac{c+1}{2\sqrt{c}}\, p_k - \frac{1}{4}\, p_{k-1}, \quad k = 1, 2, 3, \ldots, \tag{5.6} $$

where

$$ p_0 = \sqrt{c}\,\pi, \quad p_1 = -c\,\pi/2, $$

both, for $c > 1$ and $c < 1$. Hence, similarly as in §5.1,

$$ p_k = \pi\,\sqrt{c}\,(-1)^k\left(\frac{\sqrt{c}}{2}\right)^k, \quad k = 2, 3, 4, \ldots. \tag{5.7} $$

The recurrence coefficients α_k^*, β_k^*, $k \geq 0$, and thus also the α_k, β_k, can now be obtained as in the previous subsection. Table 3 displays the symbolic results. Convergence in (3.34) is now truly instantaneous, not only in the case $c < 1$, but also when $c > 1$; cf. Remark 1 to Theorem 5.

	n	$[\alpha_n, \beta_n]$
$c > 1$	0	$[1, 1]$
	> 0	$[c+1, c]$
$c < 1$	0	$[1, 1]$
	> 0	$[c+1, c]$

Table 3: Recurrence coefficients for the measure (2.4) in the case of the Chebyshev weight function of the second kind

5.3 Chebyshev weight function of the third kind

In this case, $\alpha = -1/2$, $\beta = 1/2$, and the relevant hypergeometric function is [14, Eq. 15.4.18]

$$ {}_2F_1(1, 3/2; 2; -x) = \frac{2}{1 + x + \sqrt{1+x}}\,, $$

which, for $x = 2/(g-1) = 4\sqrt{c}/(\sqrt{c}-1)^2$ becomes

$$ {}_2F_1(1, 3/2; 2; -2/(g-1)) = \begin{cases} (\sqrt{c}-1)^2/(\sqrt{c}\,(1+\sqrt{c})) & \text{if } c > 1, \\ (\sqrt{c}-1)^2/(1+\sqrt{c}) & \text{if } c < 1. \end{cases} $$

Therefore, by (3.3) and (3.10),

$$ \mu_0 = \begin{cases} 2\pi/(1+\sqrt{c}) & \text{if } c > 1, \\ 2\pi\sqrt{c}/(1+\sqrt{c}) & \text{if } c < 1. \end{cases} \tag{5.8} $$

From (3.25), (3.10), and from (3.26) with $k = 0$, noting that $\alpha_0^J = 1/2$, we have

(5.9)

$$ p_0 = \begin{cases} 2\pi\, c/(1+\sqrt{c}) & \text{if } c > 1, \\ 2\pi\,\sqrt{c}/(1+\sqrt{c}) & \text{if } c < 1, \end{cases} \qquad p_1 = \begin{cases} \pi\,(1 - c - c^{3/2})/(1+\sqrt{c}) & \text{if } c > 1, \\ -\pi\, c/(1+\sqrt{c}) & \text{if } c < 1, \end{cases} $$

and

$$ p_{k+1} = -\frac{c+1}{2\sqrt{c}}\, p_k - \frac{1}{4}\, p_{k-1}, \quad k = 1, 2, 3, \ldots . \tag{5.10} $$

When $c > 1$, the solution of (5.10) with starting values (5.9) is
(5.11)
$$p_k = \frac{2\pi\sqrt{c}}{1+\sqrt{c}}\,(-1)^k \left[(1+\sqrt{c}\,)\left(\frac{\sqrt{c}}{2}\right)^k - \left(\frac{1}{2\sqrt{c}}\right)^k\right], \quad k = 2, 3, 4, \ldots ,$$

while for $c < 1$ it is

$$p_k = \frac{2\pi\sqrt{c}}{1+\sqrt{c}}\,(-1)^k \left(\frac{\sqrt{c}}{2}\right)^k, \quad k = 2, 3, 4, \ldots . \tag{5.12}$$

In the same way as before, knowledge of p_k eventually yields the recurrence coefficients α_k, β_k. They are shown, as functions of c, in Table 4.

	n	$[\alpha_n,\, \beta_n]$
$c > 1$	0	$\left[\dfrac{\sqrt{c}+1}{\sqrt{c}}\,, 1\right]$
	1	$\left[c+1+\dfrac{(\sqrt{c}+1)(c-1)^2}{(\sqrt{c}\,(c^{3/2}+c-1)}\,, \dfrac{(\sqrt{c}+1)(c^{3/2}+c-1)}{c}\right]$
	> 1	$\left[c+1+\dfrac{(\sqrt{c}+1)(c-1)^2 c^{n-1}}{(c^{n-1/2}+c^{n-1}-1)(c^{n+1/2}+c^n-1)}\,, \; c\,\dfrac{(c^{n-3/2}+c^{n-2}-1)(c^{n+1/2}+c^n-1)}{(c^{n-1/2}+c^{n-1}-1)^2}\right]$
$c < 1$	0	$[1+\sqrt{c}, 1]$
	1	$[c+1, c\,(1+\sqrt{c})]$
	> 1	$[c+1, c]$

Table 4: Recurrence coefficients for the measure (2.4) in the case of the Chebyshev weight function of the third kind

5.4 Chebyshev weight function of the fourth kind

This is the case $\alpha = 1/2$, $\beta = -1/2$, where [14, Eq. 15.4.17]

$$ {}_2F_1(1, 1/2; 2; -x) = {}_2F_1(1/2, 1; 2; -x) = 2\left(1 + \sqrt{1+x}\right)^{-1}, $$

and, with $x = 2/(g-1) = 4\sqrt{c}/(\sqrt{c}-1)^2$,

$$ {}_2F_1(1, 1/2; 2; -2/(g-1)) = \begin{cases} (\sqrt{c}-1)/\sqrt{c}, & \text{if } c > 1, \\ 1 - \sqrt{c}, & \text{if } c < 1. \end{cases} $$

By (3.3) and (3.10), there follows

$$ \mu_0 = \begin{cases} 2\pi/(\sqrt{c}-1), & \text{if } c > 1, \\ 2\pi\sqrt{c}/(1-\sqrt{c}), & \text{if } c < 1. \end{cases} \tag{5.13} $$

The recurrence relation (3.26), for $k \geq 1$, is

$$ p_{k+1} = -\frac{c+1}{2\sqrt{c}}\, p_k - \frac{1}{4}\, p_{k-1}, \quad k = 1, 2, 3, \dots\ , \tag{5.14} $$

where, as in §5.3, but with $\alpha_0^J = -1/2$,

$$ p_0 = \begin{cases} 2\pi\, c/(\sqrt{c}-1) & \text{if } c > 1, \\ 2\pi\sqrt{c}/(1-\sqrt{c}) & \text{if } c < 1, \end{cases} \quad p_1 = \begin{cases} -(c^{3/2} - c + 1)\pi/(\sqrt{c}-1) & \text{if } c > 1, \\ -c\,\pi/(1-\sqrt{c}) & \text{if } c < 1. \end{cases} $$

The explicit solution of (5.14), when $c > 1$, is

$$ p_k = 2\pi\,\sqrt{c}\,(-1)^k \left[\left(\frac{\sqrt{c}}{2}\right)^k + \frac{1}{\sqrt{c}-1}\left(\frac{1}{2\sqrt{c}}\right)^k\right], \quad k = 2, 3, 4, \dots\ , \tag{5.15} $$

and when $c < 1$,

$$ p_k = \frac{2\pi\,\sqrt{c}}{1-\sqrt{c}}(-1)^k\left(\frac{\sqrt{c}}{2}\right)^k, \quad k = 2, 3, 4, \dots\ . \tag{5.16} $$

The desired recurrence coefficients α_k, β_k can now be obtained as in the three previous subsections. They are displayed as functions of c in Table 5.

	n	$[\alpha_n,\, \beta_n]$
$c>1$	0	$\left[\dfrac{\sqrt{c}-1}{\sqrt{c}}\,,\,1\right]$
	1	$\left[c+1-\dfrac{(\sqrt{c}-1)(c-1)^2}{\sqrt{c}\,(c^{3/2}-c+1)},\ \dfrac{(\sqrt{c}-1)(c^{3/2}-c+1)}{c}\right]$
	>1	$\left[c+1-\dfrac{(\sqrt{c}-1)(c-1)^2\,c^{n-1}}{(c^{n-1/2}-c^{n-1}+1)(c^{n+1/2}-c^n+1)}\,,\ c\,\dfrac{(c^{n-3/2}-c^{n-2}+1)(c^{n+1/2}-c^n+1)}{(c^{n-1/2}-c^{n-1}+1)^2}\right]$
$c<1$	0	$[1-\sqrt{c}, 1]$
	1	$[c+1, c(1-\sqrt{c})]$
	>1	$[c+1, c]$

Table 5: Recurrence coefficients for the measure (2.4) in the case of the Chebyshev weight function of the fourth kind

5.5 Legendre weight function

Here, $\alpha=\beta=0$, and by (3.3),

$$\mu_0 = \log\left(\frac{\sqrt{c}+1}{\sqrt{c}-1}\right)^2, \quad c \neq 1. \tag{5.17}$$

The recurrence relation (3.26), for $k \geq 1$, is

$$p_{k+1} = -\frac{c+1}{2\sqrt{c}}\,p_k - \frac{k^2}{4k^2-1}\,p_0, \quad k=1,2,3,\dots\ , \tag{5.18}$$

where

$$p_0 = \log\left(\frac{\sqrt{c}+1}{\sqrt{c}-1}\right)^2 \times \begin{cases} c & \text{if } c>1, \\ 1 & \text{if } c<1, \end{cases} \qquad p_1 = 2-(c+1)/(2\sqrt{c})\,p_0,$$

and the desired recurrence coefficients α_k, β_k follow as described in §3.4: first, α_0^* and α_0 are obtained from (3.7) with $k = 0$ in combination with (3.9) (where the integral is equal to the above value of μ_0); this gives γ_1 from the first relation in (3.17), the remaining γ's coming from (3.24); finally α_k^*, β_k^* for $k \geq 1$ are obtained from (3.21), and thus α_k, β_k from (3.33).

An attempt to generate the recurrence coefficients α_n, β_n in symbolic form proved to be unfeasible because of the rapidly increasing complexity of the symbolic expressions for the α_n and β_n as n increases. We therefore restrict ourselves to obtaining numerical values of the first ten recurrence coefficients for $c = 4$ and $c = 1/4$ using the routine `GMP.m`. Convergence

n	α_n	β_n
0	.910239226626837	1.00000000000000
1	5.39383052925145	3.72266068344396
2	4.85035318634382	4.81975085119498
3	4.91545178723217	4.43942238158180
4	4.96405975959264	4.20474171054654
5	4.98480915733151	4.10065191759388
6	4.99299025623952	4.05587961485571
7	4.99635149648618	4.03506402338317
8	4.99786955634723	4.02419082363368
9	4.99863659426219	4.01782940485962

Table 6: Recurrence coefficients for the measure (2.4), $c = 4$, in the case of the Legendre weight function

is seen to be monotone increasing for α_n and monotone decreasing for β_n, except at the very beginning.

n	α_n	β_n
0	.910239226626837	1.00000000000000
1	1.32084322743042	.309263583593324
2	1.26054739936794	.263830285912563
3	1.25362002567957	.256207550577421
4	1.25168096856576	.253540027022905
5	1.25091993817511	.252292444551082
6	1.25055903507149	.251607291494291
7	1.25036545798308	.251190105642641
8	1.25025216974473	.250917040883073
9	1.25018139891252	.25072846766016

Table 7: Recurrence coefficients for the measure (2.4), $c = 1/4$, in the case of the Legendre weight function

Here, α_n and β_n both decrease monotonically.

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