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Shape Preserving Approximations by Polynomials and Splines

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Abstract—A review of shape preserving approximation methods and algorithms for approximating univariate functions or discrete data is given. The notion of 'shape' refers to the geometrical behavior of a function's or approximant's graph, and usually includes positivity, monotonicity, and/or convexity. But, in the recent literature, the broader concept of shape also includes symmetry, generalized convexity, unimodality, Lipschitz property, possessing peaks or discontinuities, etc. Special stress is put on shape preserving interpolation methods by polynomials and splines. Of course, this text has no pretensions to be complete.

Keywords—Shape preserving approximation, Approximation of univariate functions, Approximation of discrete data, Shape preserving interpolation, Polynomial, Spline, Positivity, Monotonicity, Convexity, Generalized convexity, Unimodality, Lipschitz property.

1. INTRODUCTION

In many different problems from engineering and science, one of demands is that approximation methods represent physical reality as accurately as possible. For example, one wants to represent some more complicated quantitative information A by the less complicated one B, so that B reproduces some qualitative properties of A. Typically, A are some data that are monotone and/or convex, and B is some simple function (polynomial, spline, rational) that fits the data and preserves their 'shape,' i.e., it is also monotone and/or convex. Usually the data are discrete data, as a result of measurement. This kind of approximation is referred to as a *shape preserving approximation* or (rarely) an *isogeometric approximation*. The problems of such type arose in chemistry, VLSI, CAD/CAM, robotic, etc. In this paper, we give a survey of some shape preserving approximation methods.

The outline of this paper is the following: interpolation by polynomials and splines that preserve monotonicity of data is presented in Section 2. Convexity preserving interpolation by splines is the topic of Section 3. Rational splines that preserve monotonicity and/or convexity are considered in Section 4. The various approximating methods, like approximation by positive linear operators, that have capability of preserving even generalized convexity or other more complex 'shape' of the data or function being presented in Section 5. It also includes some methods for preserving moments by splines.

In further text, we adopt the following notation. The real sequence $\{t_i\}$ is *convex of order* k (≥ 2) or *k-convex* if $\Delta^k t_i \geq 0$, where $\Delta^k = \Delta(\Delta^{k-1})$, $\Delta^1 t_i = \Delta t_i = t_{i+1} - t_i$. If $\Delta^k t_i \leq 0$,

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX the sequence is *concave of order k (k-concave)*. If the above inequalities are strict, we said that $\{t_i\}$ is *strictly k-convex (strictly k-concave)*. It is customary to use terms *(strictly) increasing/decreasing*, instead of (strictly) 1-convex/concave, and *(strictly) convex/concave* instead of (strictly) 2-convex/concave. If the sequence is increasing or decreasing it is called *monotone*. The sequence is *polynomial of order* $n-1$ if it is both n-convex and n-concave.

If the data $\{x_i,\, y_i\}_{i=0}^n$ are given, and the sequence $\{y_i\}$ is increasing (convex, etc.) we say that the data $\{x_i,\, y_i\}$ are increasing (convex, etc.).

2. MONOTONICITY PRESERVING INTERPOLATION

The existence of interpolating polynomials that are monotone in the same sense as the data being interpolated is established by Wolibner in 1949 and published in 1951 [1]. Independently, the same result was gained by Kammerer [2] and Young [3].

Theorem 2.1. *(See [1].) Let $\{(x_i, y_i)\}_{i=0}^n$ be a set of data such that $x_0 < x_1 < \cdots < x_n$ and $y_i \neq y_{i+1}$, $i = 0, 1, \dots, n-1$, then there exists an algebraic polynomial p with the following properties:*

$$p(x_i) = y_i, \qquad i = 0, 1, \dots, n, \tag{2.1}$$

$$\operatorname{sgn}\,(p'(x)) = \operatorname{sgn}\,(\Delta y_i), \qquad x \in [x_i,\, x_{i+1}], \quad i = 0, 1, \dots, n-1, \tag{2.2}$$

where $\Delta y_i = y_{i+1} - y_i$.

A polynomial with properties (2.1) and (2.2) is said to preserve monotonicity of the data. In this case, we speak about *piecewise monotone interpolation* (PMI), or if $\{y_i\}_{i=0}^n$ does not change monotonicity, about *monotonicity preserving (MP) interpolation*.

Let $\mathcal{P}_m$ be the set of all algebraic polynomials of degree at most m defined on $[a, b]$. For polynomials defined on $\mathbb{R}$, we put only $\mathcal{P}_n$. The following two statements are direct consequences of the Wolibner's Theorem.

Corollary 2.1. *(See [4].) Let $f \in C[a,\, b]$. For every $\varepsilon > 0$ there exists a polynomial p such that*

(a) $p(x_i) = f(x_i)$, $i = 0, 1, \dots, n$;
(b) $\|f - p\| < \varepsilon$;
(c) $\|p\| = \|f\|$.

Corollary 2.2. *(See [5].) Let $f \in C[a,\, b]$. There exists the number $m_0 \in \mathbb{N}$ and a polynomial $p \in \mathcal{P}_m$ that for all $m > m_0$;*

(a) $p(x_i) = f(x_i)$, $i = 0, 1, \dots, n$;
(b) $\|f - p\| \leq 2 \inf_{q \in \mathcal{P}_m} \|f - q\|$.

In both statements, $\|\cdot\|$ denotes the Chebyshev norm. Thus, the theorem of Wolibner leads to the important class of *norm preserving* polynomial interpolation. For further generalizations of these results see [6].

Wolibner's Theorem is an existence-type theorem, which means that it does not provide any information on the polynomial p (degree, coefficients) except its existence. On the other hand, it is clear that there must exist a polynomial with smallest degree ν which still preserves the monotonicity of the data $\{(x_i, y_i)\}_{i=0}^n$. The degree ν is called *degree of PMI*.

The first concrete result concerning ν is given by Rubinstein [7], but only for $n = 2$ and $y_0 < y_1 < y_2$. An estimation of the degree of PMI, ν is given in the following theorem [8].

Theorem 2.2. *(See [8].) If the sequence $\{y_i\}_{i=0}^n$ is strictly increasing, then there exists a constant K such that*

$$\nu \le K \frac{\max\limits_{0\le i\le n-1} |\Delta_i|}{\min\limits_{0\le i\le n-1} |\Delta y_i|},$$

where $\Delta_i = \Delta y_i/\Delta x_i$, $0 \le i \le n-1$.

This result is generalized by Passow in [9]. Another estimation in a more restricted case: $x_i = i/n$, $\Delta y_i > 0$, $y_0 = 0$, $y_n = 1$ is given in [10].

THEOREM 2.3. *(See [10].) If $\Delta y_i \ge cm^{-\alpha}$ with real α satisfying $1 < \alpha < m$ for $m > ce^4$ (c is a constant), then there exists the monotone interpolant $p \in \mathcal{P}_m$ with $m = O(\alpha n \ln n)$. This estimation cannot be improved.*

Let the mesh of knots $X = \{x_0, x_1, \dots, x_n\}$ be given. With $S^j_m = S^j_m(X)$, we denote the set of splines of degree m with deficiency $m - j$, $0 \le j \le m-1$. This means that if $f \in S^j_m$, then $f \in \mathcal{P}_m[x_i, x_{i+1}]$ and $f \in C^j[x_0, x_n]$. The question of existence and estimation of the degree of PMI approximation with splines has been a subject of many papers. The existence of PMI splines (also called PMI scheme) is established by Passow [9]. The improved version of his theorem is given by de Boor and Swartz [11].

THEOREM 2.4. *(See [9,11].) For a given data $\{(x_i, y_i)\}_{i=0}^n$, there always exists piecewise monotone interpolant $H_m \in S^m_{2m+1}$.*

In [12], the result of this theorem was extended to S^m_{2m} by adding extra knots $\hat{x}_i = (x_{i+1}+x_i)/2$, $\hat{y}_i = (y_{i+1} + y_i)/2$, $i = 1, \dots, n$.

Note that the class S^m_{2m+1} contains three important subclasses: S^0_1—linear, S^1_3—cubic, and S^2_5—quintic splines. Also, in [9], a special attention was payed to the particular spline interpolates, which are flat of order m at each interior interpolation knot, i.e.,

$$H^{(j)}_m(x_i) = 0, \qquad j = 1, \dots, m, \quad i = 1, \dots, n-1.$$

In [11], de Boor and Swartz gave a more general definition of piecewise monotone interpolation, and put their considerations in a more constructive frame.

DEFINITION 2.1. *(See [11].) Let $X = \{x_i\}_{i=0}^n$ and $Y = \{y_i\}_{i=0}^n$ be two sequences. Let X be strictly increasing and $I = [x_0, x_n]$. A map $p(Y, \cdot)$ from $\mathbb{R}^{n+1}$ into the linear space $M(I)$ of all bounded real-valued functions on I is a PMI scheme (for X) if*

(i) $p(Y, x_i) = y_i$, $i = 0, 1, \dots, n$;
(ii) $p(Y, \cdot)$ *is monotone on* $[x_{i-1}, x_i]$, $i = 1, \dots, n$.

THEOREM 2.5. *(See [11].) Let $I = [x_{-1}, x_{n+1}]$. A linear map $p(Y, \cdot) : \mathbb{R}^{k+1} \to M(I)$ is a PMI scheme if and only if $p(Y, x) = \sum_{i=0}^n y_i \psi_i$ for all $Y \in \mathbb{R}^{n+1}$, and for some $\{\psi_i\}_{i=0}^n$, $\psi_i \in M(I)$, with*

(i) supp $\psi_i \subseteq (x_{i-1}, x_{i+1})$, $i = 0, 1, \dots, n$;
(ii) ψ_i *is monotone increasing on* $[x_{i-1}, x_i]$, $i = 1, \dots, n$;
(iii) $\psi_{i-1}(x) + \psi_i(x) = 1$, $x \in [x_{i-1}, x_i]$, $i = 1, \dots, n$.

In particular, a linear PMI scheme is *local*, i.e., the form of interpolant over the subinterval $[x_i, x_{i+1}]$ depends on the small number of data that are located nearby the interval $[x_i, x_{i+1}]$. If, also $p(Y, \cdot)$ maps $\mathbb{R}^{n+1}$ into $C^\nu(I)$, then $\psi_i \in C^\nu$, $i = 0, 1, \dots, n$, hence, ψ_i vanishes $(\nu+1)$-fold at x_{i-1}, $i = 2, \dots, n$, and at x_{i+1}, $i = 0, 1, \dots, n-2$. This means that $p^{(j)}(Y, x_i) = 0$, for $j = 1, \dots, m$, $i = 1, \dots, n-1$. Thus, if $p(Y, \cdot)$ also maps $\mathbb{R}^{n+1}$ into $S^m_{2m+1}(X)$, then $p(Y, x)$ on $[x_i, x_{i+1}]$ agrees with the *unique* polynomial from $\mathcal{P}_{2\nu+1}$, which takes on the value y_i $(\nu+1)$-fold at x_j, $j = i, i+1$. Consequently, $p(Y, \cdot) = H_\nu$ on $[x_1, x_{n-1}]$, and in this case the PMI is uniquely defined on $[x_1, x_{n-1}]$. But, in $[x_0, x_1]$ and $[x_{n-1}, x_n]$, $p(Y, \cdot)$ is not uniquely defined.

On the other hand, if $p(Y, \cdot)$ is not linear mapping into $S^m_{2m+1}(X)$, there is no uniqueness, except on intervals $[x_i, x_{i+1}]$ for which $\Delta y_i = 0$. If, for example, $\nu = 2$ and for some i, $\Delta y_i > 0$, and $\Delta y_{i+1} > 0$, then we can replace H_ν on $[x_{i-1}, x_{i+1}]$, by any of the infinitely many piecewise cubics f which have a double knot at x_i, interpolates the points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i-1}, y_{i-1}), satisfy $f'(x_{i-1}) = f'(x_{i+1}) = 0$ and

$$0 \leq f'(x_i) \leq 3 \min\{\Delta_{i-1}, \Delta_i\}, \tag{2.3}$$

where $\Delta_i = \Delta y_i / \Delta x_i$. Inequalities (2.3) define the square area in the (α, β)-plane, with $\alpha = f'(x_{i-1})/\Delta_i$, $\beta = f'(x_i)/\Delta_i$, called *de Boor-Swartz box.*

Lets consider the degree of approximation of linear PMI schemes. As it is shown in [11], the spline interpolant $H_m = H_m(f)$ $(m \geq 1)$ fails to provide a good approximation of the function f, no matter how smooth it might be. Let $h = \max_i\{\Delta x_i\}$ and $y_i = f(x_i)$, $i = 0, 1, \ldots, n$. It is known that H_0 (piecewise linear function) provides $O(h^2)$-approximation to f from a Sobolev space $L^2_\infty(I)$. In fact, in this case

$$\|H_0(f) - f\| \leq \frac{h^2}{8} \|f''\|,$$

(see, for example, [13,14]), so the approximation is the *second-order accurate*. But the spline H_m $(m \geq 1)$ provides only the first-order accurate approximation, or more generally, the following theorem.

THEOREM 2.6. *(See [11].) Let $p(Y, \cdot)$ be a linear PMI scheme in sense of Definition 2.1, where $y_i = f(x_i)$, $i = 0, 1, \ldots, n$. Then, if $f \in L^1_\infty(I)$, then*

$$\|f - p(Y, \cdot)\| \leq \omega_f(\Delta x_i).$$

It follows that, for $m \geq 1$, $\|H_m(f) - f\| = O(h)$ also for smooth f. The failure is particularly striking when $m \to \infty$. With the notation $x_{i-1/2} = (x_{i-1} + x_i)/2$, $(x_i, x_{i+1/2})$ for $i = 0, 1/2, 1, \ldots, k - 1/2$ are half intervals, where H_m converges *monotonely* to the piecewise constant interpolant H_∞ given by (see [11])

$$H_\infty(x) = \begin{cases} y_i, & x_{i-1/2} < x < x_{i+1/2}, \\ y_{i-1/2}, & x = x_{i-1/2}. \end{cases}$$

It is easy to see that the spline H_m, defined over the interval $[x_0, x_n]$ is given by

$$H_m(x) = y_i \phi_m\left(\frac{x - x_i}{\Delta x_i}\right) + y_{i+1}\phi_m\left(\frac{x_{i+1} - x}{\Delta x_i}\right), \qquad x \in [x_i, x_{i+1}],$$

where $\phi_m(t) = (1-t)^{m+1} \sum_{i=0}^{m} \binom{m+1}{i} t^i$, is actually the special case of Hermite basis function.

The PMI method with H_m splines is called *zero-d* method (after zero-derivative, [15]). Beside its slow convergence to the function being interpolated, the scheme introduces the inflection points although the data do not suggest their existence. Such points are known as *extraneous inflection points*. But, zero-*d* method may find an application in computer-aided design, where in cooperation with interpolation by factual functions can be used for modelling mountain ranges of specific profiles as the following example illustrates.

EXAMPLE 2.1. The data

x_i	1	2	3	4	5	6	7
y_i	0.5	1.2	4.7	7.5	2.5	0.7	0.2

are interpolated by the cubic spline H_1, and by the spline of ninth degree H_4. The corresponding ϕ-functions are $\phi_1(t) = (1-t)^2(1+2t)$ and $\phi_4(t) = (1-t)^4(1 + 5t + 15t^2 + 35t^3 + 70t^4)$. The graphs of interpolates are displayed in Figure 1.

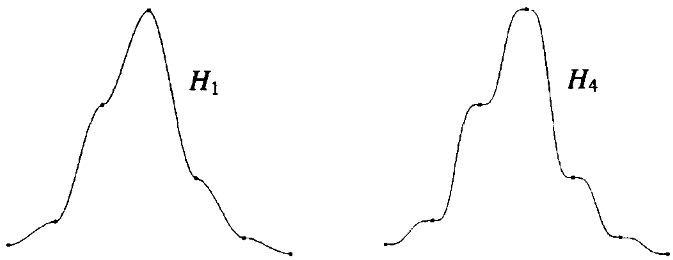


Figure 1. PMI splines $H_1 \in S_3^1$ and $H_4 \in S_9^4$.

Fitsch-Carlson Method

In [16], Manni gives the conditions for the existence of monotonicity preserving (MP) splines of arbitrary degree m with preassigned smoothness $k \geq 2$, where $3 \leq 3k \leq 2m - 1$.

But, the most interesting case for applications is the spline $H_1 \in S_3^1$, i.e., C^1-piecewise cubic spline interpolates. Fritsch and Carlson gave a class of algorithms that allow to calculate H_1 spline having the MP property.

The algorithm is based on the necessary and sufficient conditions for monotonicity of a C^1-cubic interpolant to the data $\{(x_i, y_i)\}_{i=0}^n$, which are monotone (i.e., $\Delta y_i \geq 0$ or ≤ 0, for all i) given by Ferguson and Miller [17], and independently, by Fritsch and Carlson [15,18].

Let the cubic segment be represented in the Hermite form for $x \in [x_i, x_{i+1}]$, i.e.,

$$p(x) = y_i h_0(t) + d_i h_1(t) + d_{i+1} h_2(t) + y_{i+1} h_3(t), \qquad t = \frac{x - x_i}{h_i}, \tag{2.4}$$

where $d_j = p'(x_j)$, $h_0(1-t) = h_3(t) = t^2(3-2t)$, $h_1(t) = -h_2(1-t) = t(1-t)^2$.

THEOREM 2.7. NECESSARY CONDITIONS. *Let p be an arbitrary monotone C^1 interpolant to the data $\{(x_i, y_i)\}$, $h_i = \Delta x_i$, and $\Delta_i = \Delta y_i / h_i$, $i = 0, 1, \dots, n-1$. Then*

$$\operatorname{sgn}(p'(x_i)) = \operatorname{sgn}(p'(x_{i+1})) = \operatorname{sgn}(\Delta_i). \tag{2.5}$$

Further, if $\Delta_i = 0$ then p is monotone (i.e., constant) if and only if $p'(x_i) = p'(x_{i+1}) = 0$.

THEOREM 2.8. SUFFICIENT CONDITIONS. *Let p be a cubic interpolant to the data $\{(x_i, y_i)\}$. Let*

$$\alpha_i = \frac{p'(x_i)}{\Delta_i}, \quad \beta_i = \frac{p'(x_{i+1})}{\Delta_i}, \qquad i = 0, 1, \dots, n-1, \tag{2.6}$$

where we use the convention $0/0 = 0$ and $a/0 = \infty$ for $a \neq 0$. If (2.4) is valid, and

$$\alpha_i^2 + \beta_i^2 + \alpha_i\beta_i - 6\alpha_i - 6\beta_i + 9 \leq 0, \tag{2.7}$$

$$\alpha_i + \beta_i \leq 3, \qquad i = 0, 1, \dots, n-1, \tag{2.8}$$

then p is monotone on $[x_0, x_n]$.

PROOF. (After [19].) Let $\mathcal{P}_3$ be the linear space of real polynomials of degree ≤ 3. Since any $p \in \mathcal{P}_3$ defined on $[x_i, x_{i+1}]$ can be mapped on $[0, 1]$ by a linear transformation, it is enough to consider $H \in \mathcal{P}_3$ such that $H = \{p \in \mathcal{P}_3 \mid p(0) = 0, p(1) = 1\}$. The *monotonicity region* M is defined as $\{p \in H \mid p'(t) \geq 0, t \in (0, 1)\}$. M is closed and convex; since $\sup_{0 \leq t \leq 1} |p(t)| = 1$, M is bounded, and hence, compact. Since the interior of M relative to H is $\{p \in H \mid p'(t) > 0\}$, if p is on the boundary of M relative to H, then either $p'(0) = 0$ or $p'(1) = 0$, or the discriminant of p' is zero. The monomial form of the cubic polynomial from H is $(\alpha + \beta - 2)t^3 + (3 - 2\alpha - \beta)t^2 + \alpha t$, where $\alpha = p'(0)$, $\beta = p'(1)$, and the discriminant of its derivative is $4(3 - 2\alpha - \beta)^2 - 12\alpha(\alpha + \beta - 2)$. Equaling it with zero, one gets

$$\alpha^2 + \beta^2 + \alpha\beta - 6\alpha - 6\beta + 9 = 0,$$

which is an ellipse in (α, β)-plane with the center at $(2,2)$ and the axes $\alpha = \beta$ and $\alpha + \beta = 4$. Since $\alpha > 0$ and $\beta > 0$, the only compact, convex set with a nonempty interior is the area obtained by the convex combination of this ellipse and the origin $(0,0)$ (see Figure 2 (left)). In other words, M is the union of the domains

$$\alpha^2 + \beta^2 + \alpha\beta - 6\alpha - 6\beta + 9 \le 0 \quad \text{(elliptical area)},$$

and

$$\alpha + \beta \le 3, \qquad \alpha \ge 0, \quad \beta \ge 0 \quad \text{(triangular area)},$$

which after inverse linear transformation from $[0,1]$ to $[x_i, x_{i+1}]$, becomes equivalent to (2.5), (2.7), and (2.8). ∎

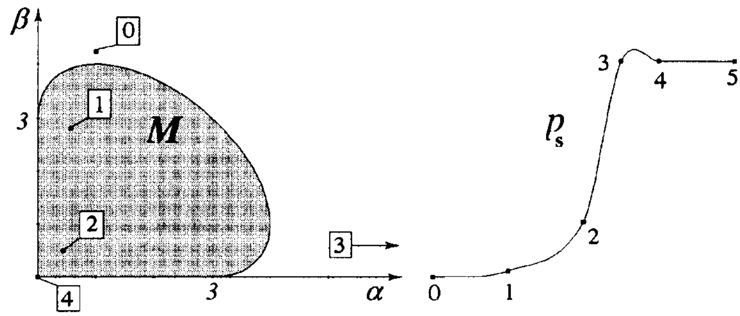


Figure 2. Monotonicity region and nonmonotone interpolant.

REMARK 2.1. So, the cubic p, given by (2.4) is monotone in $[x_i, x_{i+1}]$ if and only if $(\alpha_i, \beta_i) \in M$.

REMARK 2.2. General question of piecewise polynomial MP interpolant of degree r leads to interesting results. With freedom to vary the higher-order derivatives, the points (α_i, β_i) must be contained within one of a nested sequence of regions bounded by the coordinate axes and ellipses if r is odd, or line segments otherwise. More precisely, the monotonicity region for $r = 2k$ is triangular with vertices $(0,0)$, $(n^2+n,\, 0)$, $(0,\, n^2+n)$ and for $r = 2k+1$, $k \in \mathbf{N}$, it is the convex hull of the origin with the ellipse (see [19,20])

$$\left(k^2 - 1\right)\left(x + y - k^2\right)^2 + (x-y)^2 = 1.$$

Note that Theorems 2.7 and 2.8 are valid for both increasing or decreasing data. In the sequel, we shall consider only increasing data.

EXAMPLE 2.2. This example illustrates the relationship of a nonmonotone interpolant and the monotonicity region M. The data

x_i	10	11	12	12.5	13	14
y_i	0.42	0.55	1.52	4.64	4.64	4.64

are the subset of the interpolation data from Rice's book [21, p. 109]. These data are interpolated by C^1-piecewise cubic spline using Hermite osculatory interpolation. Cubic segments are joined at the knots, so to fit the first derivative estimated by *two-point difference formula* (2.9). For the above data, this estimation gives the sequence

$$\{d_i\}_{i=0}^{5} = \{0.13,\, 0.55,\, 2.7266, \dots,\, 3.12,\, 0,\, 0\},$$

of derivatives which yields the spline interpolant p_s as shown in Figure 2 (right). Note that p_s is not monotone. Actually, three pairs (α_i, β_i) out of five, lie inside the monotonicity region: $(\alpha_1, \beta_1) = (0.567, 2.8109)$, $(\alpha_2, \beta_2) = (0.4369, 0.5)$, and $(\alpha_4, \beta_4) = (0,0)$ (numbers are rounded to four digits), and two outside of it: $(\alpha_0, \beta_0) = (1,\ 4.2308)$ and $(\alpha_3, \beta_3) = (+\infty,\ 0)$. These points are marked by the index number in Figure 2 (left). Also, $\Delta_3 = 0$ but $d_3 = 3.12 \neq 0$. Therefore, neither the necessary nor sufficient conditions for monotonicity are fulfilled for segments 01 and 34, i.e., they are not monotone.

Based on Theorems 2.7 and 2.8, Fritsch and Carlson [15] gave an effective algorithm for constructing the monotone piecewise cubic interpolant to the data $\{(x_i,\ y_i)\}_{i=0}^n$.

ALGORITHM 2.1. (Fritsch and Carlson)

Step 1. Initialize derivatives d_i, $i = 0, 1, \dots, n$, such that $\operatorname{sgn}(d_i) = \operatorname{sgn}(d_{i+1}) = \operatorname{sgn}(\Delta_i)$. If $\Delta_i = 0$, set $d_i = d_{i+1} = 0$.

Step 2. For each interval $[x_i,\ x_{i+1}]$ in which $(\alpha_i,\ \beta_i) \notin M$, modify d_i and d_{i+1} to d_i^* and d_{i+1}^* such that $(\alpha_i^*,\ \beta_i^*) \in M$, where $\alpha_i^* = d_i^*/\Delta_i$ and $\beta_i^* = d_{i+1}^*/\Delta_i$. Then, PM interpolant is given by (2.4).

This kind of algorithm is known as *fit and modify* type of algorithm. In further elaboration, Fritsch and Carlson note that the mapping of the pair $(\alpha_i,\ \beta_i)$ to $(\alpha_i^*,\ \beta_i^*)$ is not an easy task due to interaction of parameters α and β in the adjacent intervals, i.e., $\beta_{i-1}\Delta_{i-1} = \alpha_i\Delta_i$. In other words, the algorithm is *nonlocal*. As a solution, they suggested selection of a subset $S \subset M$ with the following properties.

(a) If $(\alpha,\ \beta) \in S$, then $(\alpha^*,\ \beta^*) \in S$, whenever $0 \le \alpha^* \le \alpha$ and $0 \le \beta^* \le \beta$.
(b) If $(\alpha,\ \beta) \in S$, then $(\beta,\ \alpha) \in S$.

One such subset of M is just de Boor-Swartz box, given by (2.3), i.e., the square inscribed into M. Fritsch and Carlson denoted it by S_1 (Figure 4). Other subsets suggested by them are: S_2—a quarter of the disc; S_3—a right triangle, and S_4—the noncovex quadrilateral with vertices $(3, 0)$, $(1, 1)$, $(0, 3)$, and $(0, 0)$.

Now, Step 2 may be put in the more concrete form.

Step 2A. For each interval $[x_i,\ x_{i+1}]$ in which $(\alpha_i,\ \beta_i) \notin M$, modify d_i, and d_{i+1} to d_i^* and d_{i+1}^* such that $(\alpha_i^*,\ \beta_i^*) \in M$, where $\alpha_i^* = d_i^*/\Delta_i$ and $\beta_i^* = d_{i+1}^*/\Delta_i$.

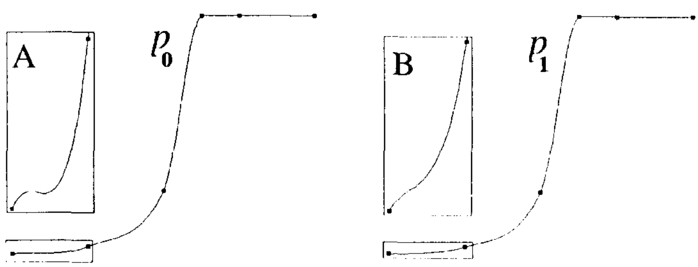


Figure 3. Two stages of Fitsch-Carlson algorithm.

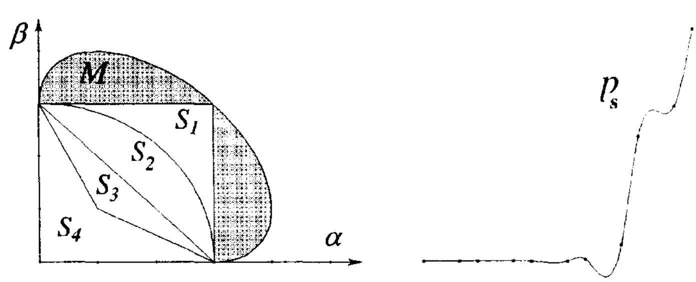


Figure 4. Subregions of monotonicity and spline interpolant to Akima data.

EXAMPLE 2.3. Let us apply the Fitsch-Carlson algorithm to the data from Example 2.2. After the first step, the necessary conditions for monotonicity are fulfilled. The new interpolant, p_0

is shown in Figure 3 (left). The segment 34 is now monotone and $(\alpha_3, \beta_3) = (0,0) \in M$. But, the point (α_0, β_0) is still outside of M so the first segment is fail to be monotone. Scaling the ordinate of p_0 by the factor 20 is done to make nonmonotonicity visible (frame A at Figure 3).

Step 2A of the Fitsch-Carlson algorithm modifies the point $(\alpha_0, \beta_0) = (1, 4.23)$ by projecting it onto S_1 so to get the new one $(\alpha_0^*, \beta_0^*) = (1, 3)$. This causes lowering the slope d_0 from 0.55 to 0.39, which changes α_1 from 0.567 to 0.402, thus, the point $(\alpha_1, \beta_1) = (0.402, 0.2809)$ does not leave the monotonicity area M. No other changes are made. The final interpolant p_1 is shown in Figure 3. Scaling the first segment (frame B) reveals that the monotonicity is reached.

EXAMPLE 2.4. Here we use the Akima data from [22], which are difficult for shape preserving interpolation.

x_i	0	2	3	5	6	8	9	11	12	14	15
y_i	10	10	10	10	10	10	10.5	15	50	60	85

The C^2-cubic spline interpolant p_s is displayed in Figure 4. The first step of Fitsch-Carlson algorithm gives C^1-spline interpolant p_0 (Figure 5). Using the subsets S_1, S_2, S_3, and S_4 for modifying (α, β) pairs, yields monotone interpolates of different shape: p_1, p_2, p_3, and p_4, respectively—see Figure 5 (first four segments are not shown).

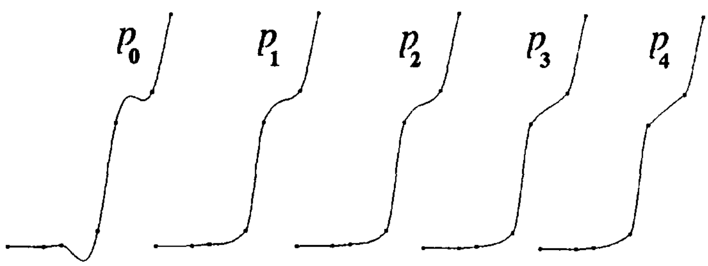


Figure 5. Fitsch-Carlson algorithm.

Note that the choice of S_1 produces the least change in the derivatives and the graph more closely resembles the graph of p_0. Using de Boor-Swartz box S_1 and the disc S_2 as the subdomains of projection is also considered by Hyman [23].

Also, note that $S_0 = \{(0,0)\}$, i.e., the origin point, used for the subregion will produce the zero-d interpolant, because all the pairs (α_i, β_i) (see formula (2.5)) are projected into the origin, which imply $d_i = 0$, for $i = 0, \dots, n$.

As Fritsch and Carlson notice in [15], the algorithm has three basic components:

(i) an initialization of derivatives $\{d_i\}_{i=0}^n$;
(ii) the choice of subregion $S \subset M$, satisfying properties (a) and (b);
(iii) the selection of mapping $(\alpha_i, \beta_i) \mapsto (\alpha_i^*, \beta_i^*)$.

Effects of the choice of different subregions of M are explained in Example 2.4, so we shall focus our attention to items (i) and (iii).

It is easy to see that *two-point difference formula*

$$d_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}, \qquad i = 1, \dots, n-1, \qquad d_0 = \frac{\Delta y_0}{h_0}, \qquad d_n = \frac{\Delta y_{n-1}}{h_{n-1}}, \tag{2.9}$$

approximates derivatives with accuracy $O(h)$. On the other hand, the *three-point (centered) difference formula*

$$d_i = \frac{h_i \Delta_{i-1} + h_{i-1} \Delta_i}{h_{i-1} + h_i}, \qquad i = 1, \dots, n-1, \tag{2.10}$$

has an accuracy $O(h^2)$, and was suggested in [15]. For the end derivatives, the *right*, i.e., *left noncentered* version of (2.10) is used

$$d_0 = \frac{(2h_0 + h_1)\Delta_0 - h_0\Delta_1}{h_0 + h_1}, \qquad d_n = \frac{(2h_{n-1} + h_{n-2})\Delta_{n-1} - h_{n-1}\Delta_{n-2}}{h_{n-2} + h_{n-1}}.$$

Formula (2.10) is also called the *parabolic formula* (see [23]) since it gives the slope at x_i of the parabola interpolating points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}). Both two- and three-point formulas are local.

Another local method, approximating derivatives with accuracy $O(h)$ is the Akima method (see [22])

$$d_i = \frac{a_i\Delta_{i-1} + b_i\Delta_i}{a_i + b_i}, \qquad i = 3, \dots, n-2,$$

where $a_i = |\Delta_{i+1} - \Delta_i|$, $b_i = |\Delta_{i-1} - \Delta_{i-2}|$. End points estimations d_0, d_1, d_{n-1}, d_n are calculated by adding two extra knots. For instance, for estimating d_n, two-points (x_{n+1}, y_{n+1}) and (x_{n+2}, y_{n+2}) are defined by the fact that they lie on the curve

$$\varphi(x) = c_0 + c_1(x - x_n) + c_2(x - x_n)^2,$$

where the constants c_i are determined by the property that φ interpolates the points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), and (x_n, y_n), and that abscissas x_{n+1} and x_{n+2} satisfy

$$x_{n+2} - x_n = x_{n+1} - x_{n+1} - x_{n-1} = x_n - x_{n-2}.$$

The Akima method reproduces shape of the data in a satisfactory manner, but do not have MP property.

If we need higher accuracy, the four-point approximation can be used

$$\begin{aligned} d_i = \Delta_{i-1} &+ \frac{\Delta_i - \Delta_{i-1}}{h_i + h_{i-1}} h_{i-1} \\ &+ \frac{(\Delta_i - \Delta_{i-1})/(h_i + h_{i-1}) - (\Delta_{i+1} - \Delta_i)/(h_{i+1} + h_i)}{h_{i+1} + h_i + h_{i-1}} h_{i-1}h_i, \end{aligned} \tag{2.11}$$

which is $O(h^3)$ accurate, and it is local.

The same accuracy, $O(h^3)$ can be achieved by using cubic spline interpolant $s(x)$ with one of the end conditions given in [13]. Contrary to the above methods, the spline method is *global*.

The influence of applying different methods for derivative initialization is shown in Figure 6 (left), where the labels 2, 3, and 4 denote two-, three-, and four-point difference formula, respectively.

The matter of mapping $(\alpha_i, \beta_i) \to (\alpha_i^*, \beta_i^*)$ is obviously the most subtle issue of the Fritsch-Carlson algorithm. For example, if we want to 'project' a point $(\alpha, \beta) \notin M$ onto S_1 (de Boor-Swartz box), we can use 'homothetic projection,' i.e., the intersection point (α^*, β^*) of the line $(0,0) - (\alpha, \beta)$ with ∂S_1 (boundary of S_1). Also, we can project (α, β) 'orthogonally' on ∂S_1, using the mappings $\alpha^* = \min\{3, \alpha\}$, $\beta^* = \min\{3, \beta\}$. The first method products an interpolant which graph is more taut, as it is seen in Figure 6 (right).

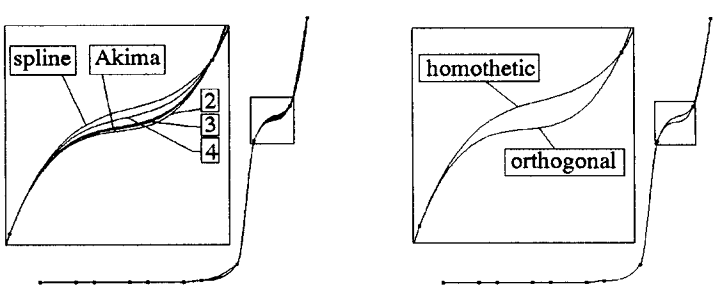


Figure 6. Influence of initial slopes and projection method.

Other Methods

After Fritsch and Carlson published their algorithm, many authors have tried to improve their results [8,23–28], etc., mainly by careful studying the way of (α, β)-projection onto M.

Let the subset S from Fritsch-Carlson algorithm is closed, and let T be the closed triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

Theorem 2.9. *(See [26].) Unless $(1,1) \in S$, the Fritsch-Carlson algorithm is at best first-order accurate. Unless $T \subset S$, the Fritsch-Carlson algorithm is at best second-order accurate.*

On the other hand, if $T \subset S$, then the Fritsch-Carlson algorithm is third-order accurate, as shown more precisely in Theorem 2.10.

Theorem 2.10. *(See [26].) Assume that $f \in C^3[a, b]$ is monotone increasing, the initial derivative approximations d_i of $f'(x_i)$ are second-order accurate, $T \subset S$ and the projection of (α_i, β_i) onto S satisfies $\alpha_i^* + \beta_i^* \geq 2$, then the Fritsch-Carlson algorithm is third-order accurate.*

In [25,26] Eisenstat, Jackson and Lewis gave two algorithms of fit and modify type, named *two-sweep* and *extended two-sweep* algorithm that are third-order whenever the initial derivatives are second-order accurate. They also proved that neither Fritsch-Carlson nor the two-sweep algorithm is a fourth-order method, while the extended two-sweep algorithm is fourth-order, provided that initial derivatives are third-order accurate, and $f \in C^4$ is monotone. This algorithm is based on decomposition of (α, β)-plane into six closed subsets A, B, C, D, E, and M—the known monotonicity subset, see Figure 7. Subsets A and E are curvilinear triangles, shaded in the figure.

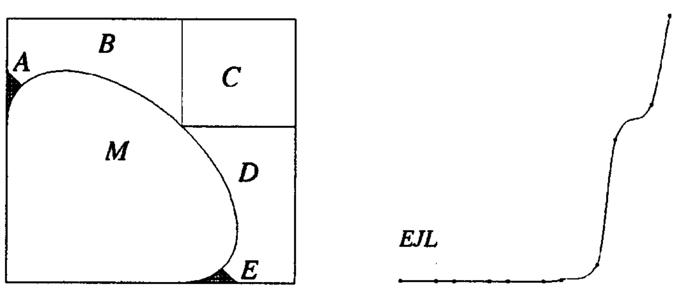


Figure 7. Method Eisenstat-Jackson-Lewis.

Algorithm 2.2. (See [26].)

Step 1. Compute the initial approximate derivative values $\{d_i\}$.
Step 2. Ensure that each d_i is nonnegative, i.e., $d_i := \max(d_i, 0)$, $i = 0, 1, \dots, n$.
Step 3. Modify $\{d_i\}$ so that each ordered pair $(\alpha_i, \beta_i) = (d_i/\Delta_i, d_{i+1}/\Delta_i) \in M$.

Forward sweep—modify the second component only unless $(\alpha_i, \beta_i) \in A$.

If $(\alpha_i, \beta_i) \in C$, then $\beta_i := 3$.
If $(\alpha_i, \beta_i) \in B$, then decrease β_i until $(\alpha_i, \beta_i) \in \partial M$.
If $(\alpha_i, \beta_i) \in A$, increase α_i until either
(a) (α_i, β_i) reaches ∂A, or
(b) $(\alpha_{i-1}, \beta_{i-1})$ reaches $\partial(M \cup D \cup E)$ $(i > 1)$.
If $(\alpha_i, \beta_i) \notin M$, then decrease β_i until $(\alpha_i, \beta_i) \in \partial M$.

Backward sweep—modify the first component only unless $(\alpha_i, \beta_i) \in E$.

If $(\alpha_i, \beta_i) \in D$, then decrease α_i until $(\alpha_i, \beta_i) \in \partial M$.
If $(\alpha_i, \beta_i) \in E$, then increase β_i until either

(a) $(\alpha_i, \beta_i) \in \partial E$, or

(b) $(\alpha_{i+1}, \beta_{i+1}) \in \partial M$ $(i < n-1)$.

If $(\alpha_i, \beta_i) \notin M$, then decrease α_i until $(\alpha_i, \beta_i) \in \partial M$.

The two-sweep algorithm applied to Akima data gives the MP interpolant which graph is shown in Figure 7.

For initializing derivatives, four-point approximation (2.11) is used.

Yan [29] develops a C^1 MP algorithm that gives fourth-order approximation to C^4 monotone functions. This is also fit and modify type algorithm. It inserts two extra knots (of multiplicity ≥ 1) in every subinterval within which the initial interpolant is not monotone.

ALGORITHM 2.3. (See [29].) Steps 1 and 2 are the same as in the Algorithm 2.2.

Step 3. If $(\alpha_i, \beta_i) \notin M$, then calculate

$$x^* := x_i + \frac{2\alpha_i + \beta_i - 3}{3(\alpha_i + \beta_i - 2)} h_i, \qquad \mu := x^* - x_i, \qquad \eta := x_{i+1} - x^*,$$

and then choose additional knots

$$\xi_i^1 := x_i + \frac{3\mu\Delta y_i}{\mu d_i + \eta d_{i+1}}, \qquad \xi_i^2 := x_{i+1} - \frac{3\eta\Delta y_i}{\mu d_i + \eta d_{i+1}}.$$

Then, the cubic segment

$$p(x) := \begin{cases} \dfrac{a_1\left(x - \xi_i^1\right)^3}{3} + b, & x \in \left[x_i, \xi_i^1\right], \\ b, & x \in \left[\xi_i^1, \xi_i^2\right], \\ \dfrac{a_2\left(x - \xi_i^2\right)^3}{3} + b, & x \in \left[\xi_i^2, x_{i+1}\right], \end{cases}$$

with $a_1 := d_i/(x_i - \xi_i^1)^2$, $a_2 := d_{i-1}/(x_{i+1} - \xi_i^2)^2$, and $b := y_i - d_i(x_i - \xi_i^1)/3$ is monotone in $[x_i, x_{i+1}]$.

For an example of Yan's MP interpolant, see Figure 9 (inserted knots are marked by black triangles).

Beatson and Wolkowicz [24] used *extended monotonicity region* E defined as union of M with the squares $[0,1] \times [3,4]$ and $[3,4] \times [0,1]$, and a projection on E, given by the following.

Approximate Projection onto E

Given (α, β) outside E;

if $\alpha \leq 1$ let $\beta := 4$;

else if $\beta \leq 1$, let $\alpha := 4$;

else project (α, β) to ∂E along the ray $(\alpha, \beta) - (1,1)$,

and if needs, a breakpoint is added according to the following.

Adding a Breakpoint

Let p be a C^1 piecewise cubic with knots $\{x_i\}$. Select $1 \leq \gamma \leq 2$ and suppose $(\alpha, \beta) \in E \setminus M$.

Step 1. Then calculate

$$\delta = \frac{h(2\alpha + \beta - 3)}{3(\alpha + \beta - 2)}, \qquad \varepsilon = \Delta_i \left[\frac{(2\alpha + \beta - 3)^2}{3(\alpha + \beta - 2)} - \alpha\right].$$

Step 2. If $\alpha < 1$, then insert a new knot at $\xi = x_i + 2\delta$ with $p(\xi) := p(\xi) + 4\gamma\varepsilon\delta/3$ and $p'(\xi)$ unchanged. If $\beta < 1$, then $\delta := h - \delta$ and $\xi = x_{i+1} - 2\delta$ with $p(\xi) := p(\xi) - 4\gamma\varepsilon\delta/3$ and $p'(\xi)$ unchanged.

The first algorithm of these authors is then given by the following.

ALGORITHM 2.4. (See [24].) Given monotone data $\{x_i, y_i\}_{i=0}^n$ and $\gamma \in [1, 2]$.

Step 1a. Fit C^2 cubic spline, s corresponding to one of three types of end-conditions (see [13]).
Step 1b. For $i := 0$ to n, if $d_i := p'(x_i)$ has the wrong sign, set $d_i := -d_i$.
Step 2a. For $i := 0$ to $n-1$ by twos, if $(\alpha_i, \beta_i) \notin E$, approximately project (α_i, β_i) onto E.
Step 2b. For $i := 1$ to $n-1$ by twos, if $(\alpha_i, \beta_i) \notin E$, approximately project (α_i, β_i) onto E.
Step 2c. For $i := 0$ to $n-1$, if $(\alpha_i, \beta_i) \notin M$, add a new knot in (x_i, x_{i+1}).

The second algorithm needs another kind of projection.

Relaxed Projection onto E

Given a function $g \in C[0,1]$ (called *relaxation function*) with the properties

(a) $g(x) \le x$, $x \in [0, 1]$,
(b) $(1 - g(x))/(1 - x)$ is bounded on $[0, 1)$ and $(\alpha, \beta) \notin M$.

Step 1. Calculate $\lambda > 0$ such that $(1, 1) + \lambda(\alpha - 1, \beta - 1) \in \partial M$.
Step 2. If $\alpha \le 1$, then $\beta := 1 + g(\lambda)(\beta - 1)$;
else if $\beta \le 1$, then $\alpha := 1 + g(\lambda)(\alpha - 1)$;
else $(\alpha, \beta) := (1, 1) + g(\lambda)(\alpha - 1, \beta - 1)$.

In [24], the following relaxation functions are mentioned

$$g_1(x) = \begin{cases} \dfrac{x}{2}, & x < \dfrac{2}{3}, \\ 2x - 1, & x \ge \dfrac{2}{3}, \end{cases} \qquad g_2(x) = \begin{cases} \dfrac{2x}{3}, & x < \dfrac{2}{3}, \\ \dfrac{(5x-2)}{3}, & x \ge \dfrac{2}{3}, \end{cases} \tag{2.12}$$

and $g_3(x) = x(x+1)/2$.

ALGORITHM 2.5. (See [24].) Given monotone data $\{x_i, y_i\}_{i=0}^n$, $\gamma \in [1, 2]$ and relaxation function g: Steps 1a and 1b are identical as in Algorithm 2.4; Steps 2a–2c are the same as in Algorithm 2.4, except the phrase 'approximate projection' which is to be replaced by 'relaxed projection.'

In [24], the choice

$$d_i := \min\{-d_i, \Delta_{i-1}, \Delta_i\},$$

is used as an alternative to the Step 1b.

EXAMPLE 2.5. The following table gives the PRN 14 radiochemical data [15].

x_i	7.99	8.09	8.19	8.7	9.2	10.0	12.0	15.0	20.0
y_i	0	$2.76\,10^{-5}$	$4.37\,10^{-2}$	0.169183	0.469428	0.94374	0.998636	0.999919	0.999994

These data are interpolated by C^2 cubic spline interpolant, Fritsch-Carlson interpolant and interpolant obtained by Eisenstat, Jackson and Levis's Algorithm 2.2. The graphs are shown in Figure 8. Three methods that use knot insertion, Algorithm 2.3 of Yan, Algorithms 2.4 and 2.5 of Beatson and Wolkowicz are presented in Figure 9. Black triangles point locations of new knots being inserted. The curve labeled BW1 is the graph of the interpolant derived by Algorithm 2.4 (approximate projection) with $\gamma = 1$. The curve BW2 corresponds to Algorithm 2.5 (relaxed projection) with $\gamma = 1$ and relaxation function $g = g_1$ from (2.12). In the last case, no extra knots has been required.

In geometric modelling and CAGD, it is important to have good methods for MP interpolation. It is also desirable to have one or several parameters that can influence the interpolant's shape in

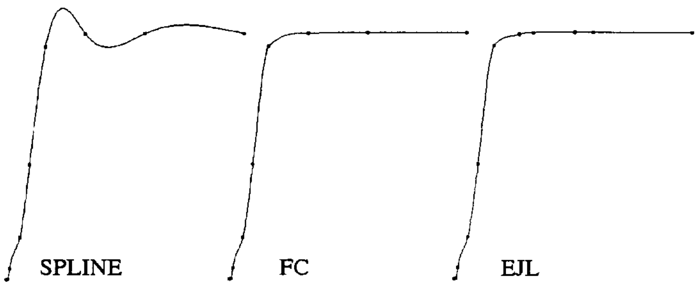


Figure 8.

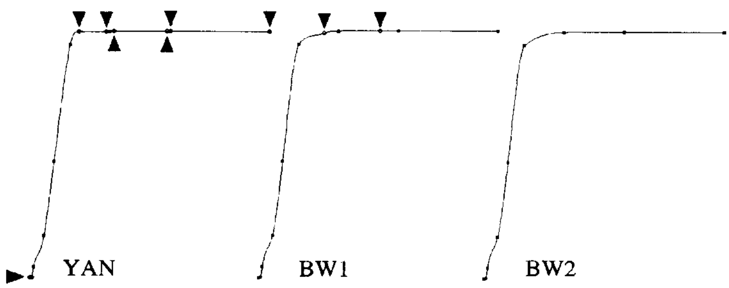


Figure 9.

a predictable way. One such method is given in [30]. The authors use parametric (2, 2)-rational Bézier curve

$$\mathbf{P}(t) = \frac{\mathbf{P}_0 b_0(t) + \omega \mathbf{P}_1 b_1(t) + \mathbf{P}_2 b_2(t)}{b_0(t) + \omega b_1(t) + b_2(t)}, \qquad t \in [0,\, 1], \quad \omega \in \left[-\frac{1}{2},\, +\infty\right),$$

where $\{b_i\}$ are Bernstein basis polynomials of degree 2, and $\mathbf{P}_i$ are points in $\mathbf{R}^2$: $\mathbf{P}_0 = (0,3)$, $\mathbf{P}_1 = (0,0)$, $\mathbf{P}_2 = (3,0)$. After simplifying, the above vector-valued expression reduces on the parametric form

$$x(t) = \frac{3t^2}{2(\omega - 1)t(1-t) + 1}, \qquad y(t) = \frac{3(1-t)^2}{2(\omega - 1)t(1-t) + 1}, \tag{2.13}$$

which is known [31] to be the arc of a conic section with endpoints in $\mathbf{P}_0$ and $\mathbf{P}_2$. For $\omega = -1/2$, the equation (2.13) describes the arc of the ellipse (2.7), i.e., it is just the border of the monotonicity region M. For $\omega = 0$, it is the line segment $(0,3) - (3,0)$, and defines the triangular area S_3 (see Figure 4). For $\omega > 0$, the curve (2.13) is a hyperbolic arc which adheres to the coordinate axes as much as ω is bigger. This allows to change the subregion of monotonicity in a continuous way, by simple changing value of ω. For $\omega < 0$, MP interpolant may be constructed by using modified Algorithm 2.2, and for $\omega \geq 0$ by the 'homothetic' projection. In this sense, ω plays the role of *shape parameter*. Increasing ω makes the monotone interpolant more resembling to zero-d interpolant.

For an approach that uses optimization technique to construct C^2-cubic MP spline, see [32].

Nonmonotone Data

All above algorithms preserve their accuracy when the data are monotone (increasing or decreasing). But, if the data change monotonicity the accuracy falls. The 'parabolic data' example [19], given in Figure 10, emphasizes this phenomenon.

Note that the Fritsch-Carlson algorithm (projection on de Boor-Swartz box) can handle both increasing or decreasing data, if it is formalized as follows:

$$d_i := \begin{cases} 0, & \Delta_{i-1}\Delta_i \leq 0, \\ \min\left[\max\left(0,\, d_i\right),\, 3\min\left(\Delta_{i-1},\, \Delta_i\right)\right], & \Delta_{i-1} \text{ and } \Delta_i > 0, \\ \max\left[\min\left(0,\, d_i\right),\, 3\max\left(\Delta_{i-1},\, \Delta_i\right)\right], & \Delta_{i-1} \text{ and } \Delta_i < 0. \end{cases}$$

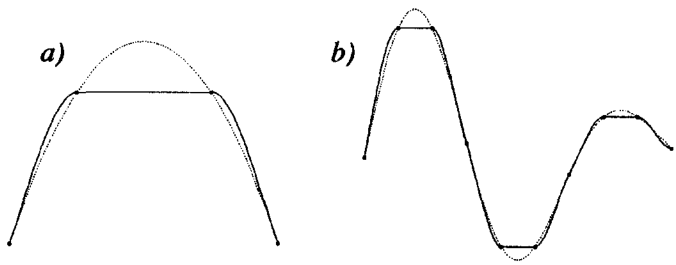


Figure 10. MP S_3^1 spline interpolates, formula (2.14).

Using the function $(x, y) \mapsto \text{minmod}(x, y)$, defined by

$$\text{minmod}(x, y) = \frac{1}{2}[\text{sgn}(x) + \text{sgn}(y)] \min(|x|, |y|),$$

the above algorithm can be put in the more compact form

$$s_i := \text{minmod}(\Delta_{i-1}, \Delta_i), \qquad d_i := \text{minmod}(d_i, 3s_i). \tag{2.14}$$

The algorithm (2.14), applied to the 'parabolic data,' $\{(x_i, y_i)\}_{i=1}^4$, where $y_i = f(x_i)$, $f(x) = x(1-x)$, $x_1 = 0$, $x_2 = 0.25$, $x_3 = 0.75$, and $x_4 = 1$, gives $d_2 = d_3 = 0$, which causes 'clipping' of the local extrema, and first-order accuracy for derivatives d_2 and d_3. On the other hand, the data carry the information about existence of the extreme in the interval $[x_2, x_3]$, since $(y_2 - y_1)(y_3 - y_4) > 0$. The resulting MP interpolant's graph is shown in Figure 10a with the solid line (the graph of f on $[0, 1]$ is shown by the dashed line).

In [23], Hyman extends the MP constraint (2.14) to

$$d_i := \text{sgn}(d_i) \min(|d_i|, 3|\Delta_{i-1}|, |\Delta_i|),$$

which leads to the *relaxed algorithm* [19].

Algorithm 2.6. (See [19].)

Step 1. Compute

$$p_i^{-1} = \frac{\Delta_{i-1}(2h_{i-1} + h_{i-2}) - \Delta_{i-2}h_{i-1}}{x_i - x_{i-2}}, \qquad 3 \le i \le n,$$

$$p_i^0 = \frac{\Delta_{i-1}h_i + \Delta_i h_{i-1}}{x_{i+1} - x_{i-1}}, \qquad 2 \le i \le n-1,$$

$$p_i^1 = \frac{\Delta_i(2h_i + h_{i+1}) - \Delta_{i+1}h_i}{x_{i+2} - x_i}, \qquad 1 \le i \le n-2.$$

Step 2. $M_i := 3\min(|\Delta_{i-1}|, |\Delta_i|, |p_i^0|)$, $2 \le i \le n-1$.

Step 3. If the numbers $i > 2$ and p_i^0, p_i^{-1}, $\Delta_{i-1} - \Delta_{i-2}$, and $\Delta_i - \Delta_{i-1}$ have the same sign, then

$$M_i := \max\left(M_i, \frac{3}{2}\min(|p_i^0|, |p_i^{-1}|)\right).$$

Step 4. If the numbers $i < n-1$ and $-p_i^0$, $-p_i^1$, $\Delta_i - \Delta_{i-1}$, and $\Delta_{i+1} - \Delta_i$ have the same sign, then

$$M_i := \max\left(M_i, \frac{3}{2}\min(|p_i^0|, |p_i^1|)\right).$$

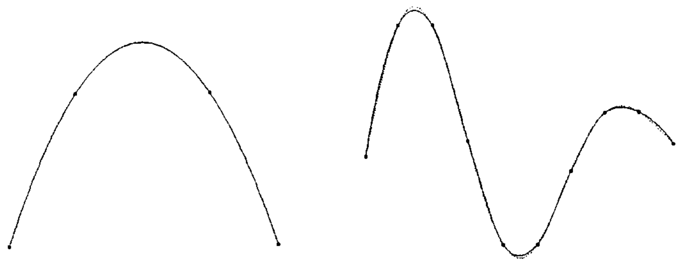
Step 5.

$$d_i := \begin{cases} (\text{sgn}\, d_i)\min(|d_i|, M_i), & \text{if } \text{sgn}\, d_i = \text{sgn}\, p_i^0, \\ 0, & \text{otherwise}, \end{cases} \qquad i = 2, \dots, n-1.$$

Step 6. If $\text{sgn}\, d_i = \text{sgn}\, \Delta_i$, then $d_i := (\text{sgn}\, d_i)\min(|d_i|, 3|\Delta_i|)$, otherwise $d_i := 0$; handle $i = n$ similarly.

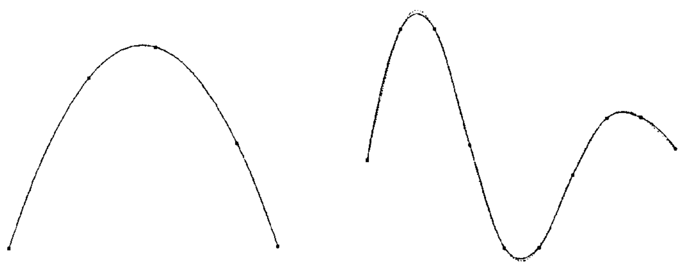
Algorithm 2.6 generates a third-order accurate (in $\mathbf{L}_\infty$) interpolant, if the original derivatives are second-order accurate (see [19]). The improvement is obvious if this algorithm is applied on the parabolic data, given above. The obtained interpolant restores the parabola (Figure 11a) no matter how many knots we have and if they are spaced nonuniformly (Figure 12a).

Huynh in [28] gives an exhaustive review of many MP methods, including (2.14). He compared eight such methods. Here, we will mention only MG3 (monotone general third-order) method.



(a) (b)

Figure 11. MP S_3^1 spline interpolates, Algorithm 2.6.



(a) (b)

Figure 12. MP S_3^1 spline interpolates, Algorithm 2.7.

ALGORITHM 2.7. (See [28].)

$$\begin{aligned} s_i &:= \text{minmod}\left(\Delta_i - \Delta_{i-1},\, \Delta_{i+1} - \Delta_i\right), \\ p_i^+ &:= \Delta_i - s_i, \\ p_i^- &:= \Delta_{i-1} + s_{i-1}. \end{aligned}$$

Compute

$$\begin{aligned} \alpha_i^L &= \min\left\{0,\, 3\Delta_{i-1},\, \frac{3}{2}p_i^-\right\}, & \beta_i^L &:= \max\left\{0,\, 3\Delta_{i-1},\, \frac{3}{2}p_i^-\right\}, \\ \alpha_i^R &= \min\left\{0,\, 3\Delta_i,\, \frac{3}{2}p_i^+\right\}, & \beta_i^R &:= \max\left\{0,\, 3\Delta_i,\, \frac{3}{2}p_i^+\right\}, \end{aligned}$$

then $\alpha_i := \max(\alpha_i^L, \alpha_i^R)$, $\beta_i := \min(\beta_i^L, \beta_i^R)$, and

$$d_i := d_i + \text{minmod}\left(\alpha_i - d_i,\, \beta_i - d_i\right).$$

The result of applying Algorithm 2.7 on the parabolic data gives the same result as the previous algorithm (Figures 11a and 12a).

EXAMPLE 2.6. Algorithms 2.6 and 2.7, and the algorithm represented by equation (2.14), are used to approximate the function

$$f(x) = e^{-2x^2} \sin(10x), \qquad x \in [0,\ 0.9095].$$

The interpolation data are supplied by sampling f on the mesh of $n = 10$ equidistant points, such that $x_1 = 0$ and $x_{10} = 0.9095$. Graphs of interpolates are displayed in Figures 10b, 11b, and 12b. It can be seen that the Fritsch-Carlson algorithm (2.14) results in clipping the strict extreme, while relaxed algorithm of Daugherty, Edelman and Hyman, or MG3 algorithm give almost identical interpolates.

The key operation in all above algorithms whose are of *fit and modify* type is correction of the initial values $d_i = p'(x_i)$. It is performed by a nonlinear averaging function $(s,t) \mapsto G(s,t)$, where $s = \Delta_{i-1}$ and $t = \Delta_i$, such that

$$d_i := G(s,\ t). \tag{2.15}$$

This gives an algorithm which must be independent on scaling coordinate axes, i.e., on the subclass of an affine transformation $(s,t) \mapsto (\lambda s,\ \lambda t)$, $\lambda \in \mathbb{R}$. Note that this transformation preserves monotonicity of the data. In fact, for $\lambda > 0$, the set of increasing (decreasing) sequence of data is closed under this transformation while, for $\lambda < 0$, it transforms increasing data in decreasing ones and vise versa So, if

$$G(\lambda s,\ \lambda t) = \lambda G(s,t), \tag{2.16}$$

the algorithm will equally efficiently handle the data before and after scaling. This makes the algorithm independent on the scale of measurement. As the consequence of (2.16), the function G is fully characterized by an one-variable function g defined by

$$g(r) = \frac{r}{s} G\left(s,\ \frac{s}{r}\right).$$

Both G and g are called *limiter functions*. It is desirable that G (and g) possesses the symmetry and averaging property

(a) $G(s,t) = G(t,s)$, or $rg(1/r) = g(r)$,
(b) $G(s,t) \in \operatorname{conv}\{s,t\}$, or $g(r) \in \operatorname{conv}\{1,r\}$.

The next two conditions provide stability and monotonicity of the algorithm (2.15):

(c) G or g is continuous,
(d) $G(s,t) \in \operatorname{conv}\{0, 3s_i\}$, or $g(r) \in \operatorname{conv}\{0,\ 3\operatorname{minmod}(1,r)\}$.

Stability of the algorithm means that a small change in the data may cause a large change in the interpolant. For example, in order to avoid the 'clipping' phenomenon described above, one may turn off the monotonicity constrain (2.14) near strict local extreme causing an instability of the method.

In [28], several examples of limiter functions are given. Here we reproduce three of them

$$g(r) = \frac{2r}{1+r}, \qquad g(r) = \begin{cases} \dfrac{3r}{1+2r}, & 0 < r \le 1, \\ \dfrac{3r}{2+r}, & r > 1, \end{cases} \qquad g(r) = \frac{3r(r+1)}{r^2+4r+1},$$

and in all cases, $g(r) = 0$ for $r \le 0$. As it is noticed by Huynh [28], it does not seem fruitful to improve accuracy by developing new limiter functions.

Finally, let us mention the conditions, the derivatives $d_i = p'(x_i)$ must obey so that the piecewise cubic p given by (2.4) preserve the positivity or negativity of the data if (see [19])

$$-\frac{3|y_i|}{h_i} \leq \operatorname{sgn}(y_i)\, d_i \leq \frac{3|y_i|}{h_{i-1}}.$$

For corresponding MP (also positivity preserving) methods for constructing C^2-piecewise quintic interpolates, see [19]. About quintic splines, see for example [33].

Some authors use optimization approach. Thus, in [34] Dauner and Reinsch discuss two algorithms for the construction of the cubic spline interpolant that preserve positivity or monotonicity of the data, by solving a finite-dimensional nonlinear minimization problem.

3. CONVEXITY PRESERVING INTERPOLATION

In this section, the CP refers to *convexity preserving* and MCP to *monotonicity and convexity preserving* interpolates.

The most popular definition of convexity of a twice differentiable function $f : I \to R$, where I is any nonempty segment of the real axis, is $f''(x) \geq 0$, $x \in I$. But, the definition of convexity need not any preassumption about continuity of the function. Actually, f is *convex* (on I) if for any $x, y \in I$, and for any $\lambda \in (0, 1)$, the *Jensen inequality* is valid

$$f\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda) f(x) + \lambda f(y). \tag{3.1}$$

If for $x \neq y$ in (3.1) $\leq$ is replaced by $<$, the function f is *strictly convex*. The function f is *concave* (*strictly concave*) if $-f$ is convex (strictly convex).

Convexity of f implies continuity except at the end points of I. The alternative definition to (3.1) is via second-order divided difference

$$[x,\, y,\, z;\, f] = \frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-x)(y-z)} + \frac{f(z)}{(z-x)(z-y)} \geq 0, \tag{3.2}$$

where x, y, and z are any three noncoincidental points from I. The 'discrete' version of the divided difference operator, defined for the data $\{(x_i,\, y_i)\}_{i\in \mathrm{N}}$ is

$$[x_i,\, x_{i+1},\, x_{i+2};\, y] = \frac{\Delta_{i+1} - \Delta_i}{x_{i+2} - x_i}.$$

The divided difference in (3.2) can be iterated as follows:

$$[x_i, \dots, x_{i+k};\, f] = \frac{[x_{i+1}, \dots, x_{i+k};\, f] - [x_i, \dots, x_{i+k-1}\,;\, f]}{x_{i+k} - x_i}, \qquad [x_i\,;\, f] = f(x_i).$$

Obviously, $\Delta_i = [x_i,\, x_{i+1} \,:\, y]$. The function f is *convex of order* $k \geq 0$, if for any set of noncoincidental points $\{x_0, \dots, x_r\}$ from I it is valid

$$[x_0, \dots, x_r;\, f] \geq 0. \tag{3.3}$$

Changing the sign $\geq$ into $>$ gives the definition of *strict convexity of order k*. Also, definitions of *(strict) concavity of order k* obtains from (3.3) by inverting these signs.

For other details about convex functions, see for example [35].

Let p be a polynomial from $\mathcal{P}_m$ $(m \geq 3)$ that interpolates the data $\{(x_i,\, y_i)\}_{i=0}^n$. Theorems 2.7 and 2.8 from the previous section (see, also, Remark 2.1), give the necessary and sufficient conditions for a polynomial to be monotone on some subinterval $[x_i,\, x_{i+1}]$. The monotonicity regions in the $(\alpha,\, \beta)$-plane are or triangular areas or convex combinations of characteristic ellipses with the origin. In the case of convexity, we have corresponding results.

THEOREM 3.1. NECESSARY CONDITIONS. *Let $p \in \mathcal{P}_m$ $(m \geq 1)$ be convex on $[x_i,\, x_{i+1}]$. Then*

$$\alpha_i < 1, \qquad \beta_i > 1, \quad \text{or} \quad \alpha_i = \beta_i = 1, \tag{3.4}$$

where α_i and β_i are given by (2.6).

Proof.

(i) Let $m \geq 2$. The derivative $p'(x_i)$ is the slope of the line of support ℓ_1 (see [35]) that touches p in x_i. Similarly, ℓ_2 is the line of support of p in x_{i+1}. Therefore, for $x \in (x_i,\, x_{i+1})$, p satisfies $p(x) > \max\{\ell_1(x),\, \ell_2(x)\}$. On the other hand, by the Jensen inequality, $p < \ell_0$ on $(x_i,\, x_{i+1})$, where ℓ_0 is the chord joining the points $(x_i,\, p(x_i))$ and $(x_{i+1},\, p(x_{i+1}))$, with the slope Δ_i. Therefore, graphs of ℓ_0, ℓ_1, and ℓ_2 form a triangle such that $\max\{\ell_1,\, \ell_2\} \leq \ell_0$ with equality at the end-points. Comparing slopes of triangle's sides results in $p'(x_i) < \Delta_i$ and $p'(x_{i+1}) > \Delta_i$, i.e., at the first two inequalities in (3.4).

(ii) Let $m = 1$. Then, obviously $p'(x_i) = p'(x_{i+1}) = \Delta_i$, so (3.4) is complete. ∎

Let $r_m = 2/(1 - a_{m-2})$ and $s_m = 2/(1 + a_{m-2})$, where a_m denotes the largest zero of the Jacobi polynomial $P_{\nu+1}^{(0,0)}$ if $m = 2\nu$, or $P_{\nu+1}^{(0,1)}$ if $m = 2\nu + 1$. Note that $1/r_m + 1/s_m = 1$, and we have the following theorem.

Theorem 3.2. *(See [20].)*

(a) *If $(\alpha_i,\, \beta_i) \in D$, where D is a wedge shaped region defined by the following inequalities:*

$$s_m(\alpha - 1) \leq \alpha - \beta \leq r_m(\alpha - 1), \qquad \alpha \leq 1, \quad \beta \geq 1,$$

with vertex $(1, 1)$ (Figure 13 (left)), then $p \in P_m$ $(m \geq 3)$ is convex on $[x_i,\, x_{i+1}]$. Note that ∂D intersects β-axis in the points s_m and r_m.

(b) *If $(\alpha_i,\, \beta_i) \in D_1$, where $D_1 \subset D$ is the triangular region with vertices $(1, 1)$, $(0,\, s_m)$, and $(0,\, r_m)$, then p is a monotone and convex polynomial.*

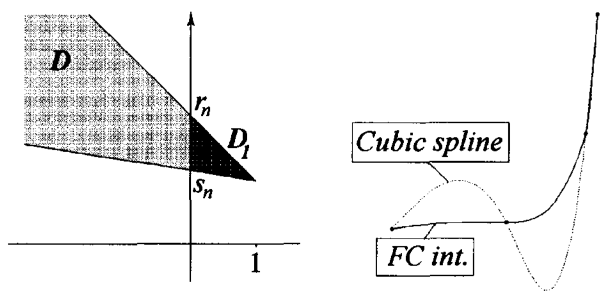


Figure 13.

Theorem 3.2 generalizes the result of Neuman [36], which states that the cubic spline p given by (2.4) is convex on $[x_i,\, x_{i+1}]$ if and only if $(\alpha_i,\, \beta_i) \in W$, where $W = \{(\alpha,\, \beta) \in \mathbb{R}^2 : 2\alpha + \beta \leq 3,\ \alpha + 2\beta \geq 3\}$.

The following theorem establishes the existence of the polynomial of the *best* approximation which preserves convexity of higher order. Let ω be the modulus of continuity of $f^{(2m-1)}$.

Theorem 3.3. *(See [37].) Let $m \geq 1$ and $f \in C^{2m-1}[-1,\, +1]$. Let $1 \leq k_1 < k_2 < \cdots < k_p < m$ be p fixed integers and $\varepsilon_i = \pm 1$, $i = 1, \ldots, p$ fixed signs. For each positive integer n, let $P_n \in \mathcal{P}_n$ be the polynomial of the best approximation to f on $[-1,\, +1]$. If $\varepsilon_i\, f^{(k_i)}(x) > 0$ on $[-1,\, +1]$ for $i = 1, \ldots, p$, and if*

$$\sum_{k=1}^{+\infty} \frac{1}{k}\, \omega\left(\frac{1}{k}\right) < +\infty,$$

then for n sufficiently large we have $\varepsilon_i P_n^{(k_i)}(x) \geq 0$ on $[-1,\, +1]$ for $i = 1, \ldots, p$.

In [38], Passow and Roulier have proved four negative theorems on the best convex approximation. We cite their Theorem 1.

THEOREM 3.4. *(See [38].) Let $f \in C[-1, 1]$ have bounded r^{th} order divided differences, and nonnegative $(r+1)^{st}$ order divided differences on $[-1, 1]$. Let p_n be the polynomial of best approximation of f on $[-1, 1]$. Assume that there is no $C > 0$ for which*

$$E_n(f) \le \frac{C}{(n+1)^{r+1}}, \qquad \text{for } n = 0, 1, \dots .$$

Then there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \ge 0$ on $[-1, 1]$.

As it has been proved by Erdős, the classical Markov inequality

$$\|p'\| \le \frac{2n^2}{b-a}\|p\|,$$

($p \in \mathcal{P}_n$, $\|\cdot\|$ is sup-norm) can be improved by replacing n^2 by $en/2$ for polynomials with only real zeros outside $[a, b]$. Myers and Roulier have considered the subclass of convex polynomials, and have proved the following negative result.

THEOREM 3.5. *(See [39].) Let $p \in \mathcal{P}_n$, and $\|\cdot\|$ be the sup-norm on $[a, b]$. Markov's inequality cannot be improved by replacing n^2 by some lower power of n for convex increasing interpolates.*

In the paper [40], Ivanov considered the problem of interpolation of k-convex (strictly k-convex) data $\{(x_i, y_i)\}_{i=0}^n$ by the function $f \in C[x_0, x_n]$ with absolutely continuous derivative $f^{(k-1)}$, $f^{(k)} \in L_p$ $(1 \le p \le \infty)$, such that $f^{(k)} \ge 0$. Moreover, the algorithm for finding f is given under the min $|f^{(k)}|_{L_p}$ constrain.

Let us turn to the more promising piecewise polynomial CP and MCP interpolation. It is well known that the classical interpolating splines, for example, parabolic or cubic spline, in general, do not preserve monotonicity and convexity. In the preceding section, we presented several algorithms for monotonicity (or positivity) preserving C^1 piecewise cubic interpolates. The algorithms for finding PM C^2-quintic piecewise interpolates are also known (see [19]). But, these interpolates do not preserve convexity as the following example shows.

EXAMPLE 3.1. Figure 13 (right) shows the data $\{x_i, f(x_i)\}_{i=0}^3$ taken from the convex function $f(x) = 1/x^2$, at the nodes $x_0 = -2$, $x_1 = -1$, $x_2 = -0.3$, $x_3 = -0.2$, used in [41]. Neither C^2-cubic spline (dotted line) nor C^1 MP interpolant obtained by Fritsch-Carlson method (solid line) preserves the convexity of the data.

The impossibility of constructing similar algorithms for CP piecewise cubic interpolates is shown in the following counterexample [19].

EXAMPLE 3.2. Let us consider the samples of $f(x) = |x|$ on the mesh $-1 = x_1 < x_2 < \cdots < x_n = 1$, which contains the point 0, say $x_j = 0$. Suppose that p is a C^1 piecewise cubic interpolant. The left derivative at x_j, $p'(x_j - 0)$ must be equal to $\Delta_{i-1} = -1$ to preserve convexity on $[x_{j-2}, x_j]$. The right derivative $p'(x_j + 0)$ must equal $\Delta_i = 1$ to preserve convexity on $[x_j, x_{j+2}]$. Thus, $p'(x_j - 0) \ne p'(x_j + 0)$ and p is not differentiable at x_j.

Accordingly, we must modify our demands, accepting either nondifferentiable splines, or splines of higher degree, or splines with additional knots or nonpolynomial splines (exponential, rational, etc.)

In the case of C^0-piecewise cubic spline, the following conditions on $d_i^- = p'(x_i - 0)$ and $d_i^+ = p'(x_i + 0)$ ensure that the cubic Hermite interpolant (2.8) preserves convexity or concavity of the data.

THEOREM 3.6. *(See [19,42].) Given the convex (concave) data $\{(x_i, y_i)\}_{i=0}^n$. Let*

$$\rho_i = \operatorname{sgn}[x_{i-1}, x_i, x_{i+1}; y], \qquad i = 1, \dots, n-1.$$

If

$$\rho_i \Delta_{i-1} \le \rho_i d_i^- \le \rho_i d_i^+ \le \rho_i \Delta_i \tag{3.5}$$

and

$$-2\rho_i\left(d_{i+1}^- - \Delta_i\right) \le \rho_i\left(d_i^+ - \Delta_i\right) \le -\frac{1}{2}\rho_i\left(d_{i+1}^- - \Delta_i\right),$$

then p is convex (concave).

We consider now again the set of splines $S_m^j = S_m^j(X)$ that corresponds to the mesh $X = \{x_0, x_1, \dots, x_n\}$, introduced in Section 2. Passow [43] proved the following theorem concerning interpolation with a quadratic C^1 spline.

THEOREM 3.7. *(See [43].)*

(i) *If $y_{i+1} - y_i > 0$, $\Delta_{i+1} - \Delta_i > 0$, $i = 0, 1, \dots, n-1$, then there exists an increasing interpolant in S_2^1.*

(ii) *If $\Delta_{i+1} - \Delta_i > 0$ and $\Delta_{i+2} - 2\Delta_{i+1} + \Delta_i > 0$, $i = 0, 1, \dots, n-2$, then there exists a convex interpolant in S_2^1.*

If the data $X = \{x_0,\ x_1,\ x_2,\ x_3\}$ are increasing and convex, then there exists an increasing and convex interpolant $p \in S_m^1$, where (see [39])

$$m^2 \ge \frac{\Delta_2}{\Delta_3 - \Delta_2 + \Delta_1}.$$

An exhaustive study of shape preserving interpolation with various classes of splines is given by Schmidt in [44,45] (for cubic splines see [46]). For example, he considers the conditions for existence of quadratic spline interpolant p from S_2^1, given by

$$p(x) = y_i + d_i h_i t + (\Delta_i - d_i) h_i t^2, \qquad t = \frac{x - x_i}{h_i} \in [0,\ 1]. \tag{3.6}$$

The necessary and sufficient condition for p to be convex, monotone, or positive on $[0,1]$ is, respectively,

$$d_i \le \Delta_i, \qquad i = 0, 1, \dots, n-1, \tag{3.7}$$

$$d_i \ge 0, \qquad i = 0, 1, \dots, n, \tag{3.8}$$

$$d_i \ge -\frac{2}{h_i}\left(y_i + \sqrt{y_i y_{i-1}}\right), \qquad i = 1, \dots, n. \tag{3.9}$$

Let the data, $\{(x_i,\ y_i)\}$ are interpolated by the spline p, given by (3.6). Let $\{\beta_i\}_{i=1}^n$ be given with $0 < \beta_i < 1$, such that $(1-\beta_i)d_{i-1} + \beta_i d_i = \Delta_{i-1}$, $i = 1, \dots, n$.

THEOREM 3.8. *(See [45].) Define the sequences $\{\gamma_i\}$ and $\{\delta_i\}$ by $\gamma_0 = 1$, $\delta_1 = \Delta_0$, and*

$$\gamma_i = \frac{\beta_i \gamma_{i-1}}{1 - \beta_i}, \quad \delta_{i+1} = \delta_i + (-1)^i \gamma_i \left(\Delta_i - \Delta_{i-1}\right), \qquad i = 1, \dots, n-1.$$

If $\delta' = \max\{\delta_{2j}\}$ and $\delta'' = \min\{\delta_{2j-1}\}$, $j = 1, 2, \dots$, and $\delta' \le \delta''$, then (3.7) is valid, i.e., the quadratic spline interpolant (3.6) preserves convexity of the data. If $d_0 \in [\delta',\ \delta'']$, then

$$d_i = \Delta_{i-1} - (-1)^{i+1}\frac{d_0 - \delta_{i+1}}{\gamma_i}, \qquad i = 1, \dots, n.$$

THEOREM 3.9. *(See [45].) Define $\{\gamma_i\}$ and $\{\delta_i\}$ by $\gamma_0 = 0$, $\delta_1 = 1$, and*

$$\gamma_i = \gamma_{i-1} + (-1)^{i-1}\frac{\delta_{i-1}\Delta_i}{1 - \beta_i}, \quad \delta_i = \frac{\beta_i \delta_{i-1}}{1 - \beta_i}, \qquad i = 1, \dots, n-1.$$

If $\delta' = \max\{\delta_{2j}\}$ and $\delta'' = \min\{\delta_{2j-1}\}$, $j = 1, 2, \dots$, and $\delta' \le \delta''$, then (3.8) is valid, i.e., the quadratic spline interpolant (3.6) preserves monotonicity of the data. If $d_0 \in [\delta',\ \delta'']$, then

$$d_i = (-1)^i \frac{d_0 - \gamma_i}{\delta_i}, \qquad i = 1, \dots, n.$$

THEOREM 3.10. *(See [45].) Define* $\{\gamma_i\}$ *and* $\{\delta_i\}$ *by* $\gamma_0 = 1$, $\delta_1 = 0$, *and*

$$\gamma_i = \frac{\beta_i \gamma_{i-1}}{1-\beta_i}, \quad \delta_{i+1} = \delta_i + (-1)^i \frac{\gamma_{i-1}\Delta_i}{1-\beta_i}, \qquad i = 1, \dots, n-1.$$

If $\delta' = \max\{\delta_{2j}\}$ *and* $\delta'' = \min\{\delta_{2j-1}\}$, $j = 1, 2, \dots$, *and* $\delta' \le \delta''$, *then (3.9) is valid, i.e., the quadratic spline interpolant (3.6) preserves positivity of the data. If* $d_0 \in [\delta', \delta'']$, *then*

$$d_i = \frac{\delta_{i+1} - d_0}{\gamma_i}, \qquad i = 1, \dots, n.$$

The shape preserving effect can be obtained also by using knot insertion technique, an idea elaborated by Schumaker in [47]. He considered the following problem.

PROBLEM 3.1. Find a function $s \in C^1[x_i, x_{i+1}]$ such that

$$s(x_j) = (y_j), \quad s'(x_j) = d_j, \qquad j = i, i+1,$$

where y_i, y_{i+1}, d_i, and d_{i+1} are given numbers.

This problem can be solved by a quadratic polynomial, if and only if

$$d_i + d_{i+1} = 2\Delta_i. \tag{3.10}$$

In this case, the solution is

$$s(x) = y_i + d_i(x - x_i) + \frac{(d_{i+1} - d_i)(x - x_i)^2}{2(x_{i+1} - x_i)}. \tag{3.11}$$

Otherwise, the solution can be a quadratic spline with one (simple) knot.

THEOREM 3.11. *(See [47].) For every* $t \in (x_i, x_{i+1})$, *there exists a unique quadratic spline* p *with a (simple) knot at* t *solving Problem 3.1. In particular,*

$$p(x) = \begin{cases} a_1 + b_1(x - x_i) + c_1(x - x_i)^2, & x_i < x < t, \\ a_2 + b_2(x - t) + c_2(x - t)^2, & t \le x < x_{i+1}, \end{cases}$$

with $a_1 = y_i$, $b_1 = d_i$, $c_1 = (d - d_i)/(2\alpha)$, $a_2 = a_1 + b_1\alpha + c_1\alpha^2$, $b_2 = d$, $c_2 = (d_{i+1} - d)/(2\beta)$, *where*

$$d = p'(t) = 2\Delta_i - \frac{\alpha d_i + \beta d_{i+1}}{x_{i+1} - x_i}, \qquad \alpha = t - x_i, \quad \beta = x_{i+1} - t.$$

Based on the test (3.10) and Theorem 3.11, different algorithms can be constructed. At the first stage of such an algorithm, we specify the interpolating data and define the set of slopes $\{d_i\}$ that ensure the quadratic interpolant is monotone or convex or both. Then, if the result of test (3.10), on j^{th} interval is positive, the unique quadratic polynomial (3.11) is the interpolant we need. Otherwise, we insert a knot t in $[x_i, x_{i+1}]$ such that the following conditions are valid.

Monotonicity:

$$2|\Delta y_i| \ge |(t - x_i)d_i + (x_{i+1} - t)d_{i+1}|.$$

Convexity:

$$x_i < t \le x_i + \frac{2\Delta x_i(d_{i+1} - \Delta_i)}{d_{i+1} - d_i}, \qquad \text{if } |d_{i+1} - \Delta_i| < |d_i - \Delta_i|,$$

$$x_i + \frac{2\Delta x_i(d_i - \Delta_i)}{d_{i+1} - d_i} \le t < x_{i+1}, \qquad \text{if } |d_{i+1} - \Delta_i| > |d_i - \Delta_i|.$$

This idea is further explored by Lahtinen in [48,49], where interesting applications can be found.

In the series of papers, McAllister, Passow and Roulier have elaborated the way of constructing spline interpolant of arbitrary degree that preserves shape of the data (see [41,50–52]). Their method is based on the concept of $\{\alpha\}$-admissibility of the sequence of numbers [51]. More precise, let $\{\alpha_i\}_{i=1}^n$ be a given sequence of numbers such that $0 < \alpha_i < 1$. Let $\{(x_i, y_i)\}_{i=0}^n$ be the data that are increasing (and/or convex) and $\bar{x}_i = x_{i-1} + \alpha_i \Delta x_i$. The set of numbers $\{t_i\}$ is said to be increasing (and/or convex) $\{\alpha_i\}$-*admissible* if the piecewise linear function L with the breakpoints

$$(x_0, y_0), (\bar{x}_1, t_1), (\bar{x}_2, t_2), \dots, (\bar{x}_n, t_n), (x_n, y_n)$$

passes through the points (x_i, y_i), $i = 1, \dots, n-1$, and is increasing (and/or convex).

In the case that there exist increasing (and/or convex) $\{\alpha_i\}$-admissible points for given data, the piecewise linear function L exists, and the spline interpolant p is constructed such that on $[x_i, x_{i+1}]$ it is a polynomial from $\mathcal{P}_k$, which Bernstein form is

$$p(x) = \frac{1}{h_i} \sum_{j=0}^{k_i} L\left(x_i + j\frac{h_i}{k_i}\right) b_j(x), \tag{3.12}$$

where $b_j(x) = \binom{k_i}{j}(x - x_i)^j (x_{i+1} - x)^{k_i - j}$ is the j^{th} basic Bernstein polynomial (see Section 5). It is well known (Theorem 5.2) that Bernstein polynomials preserve m^{th} order convexity of the generating function. Thus, if L is monotone (and/or convex) so is p. The integer numbers k_i, defining the degree of polynomial pieces are free parameters that may control the smoothness of the spline. The interpolating algorithm offered by authors of [41] is based on the following.

ALGORITHM 3.1. (See [41].) Let $\{\alpha_i\}_{i=1}^n$ with $0 < \alpha_i < 1$ be given. Define $m_0 = 0$ and $M_0 = \Delta_1$. Now, for $i = 1, 2, \dots, n-1$, define

$$m_i = \frac{\Delta_i - \alpha_i M_{i-1}}{1 - \alpha_i}$$

and

$$M_i = \min\left\{\Delta_{i+1}, \frac{\Delta_i - \alpha_i m_{i-1}}{1 - \alpha_i}\right\}.$$

THEOREM 3.12. *(See [41].) Given increasing and convex data $\{(x_i, y_i)\}$, $i = 0, 1, \dots, n$, and $\{\alpha_i\}_{i=1}^n$ with $0 < \alpha_i < 1$, $i = 1, \dots, n$. Then there exist $\{\alpha_i\}$-admissible points for this data, if and only if the alpha-algorithm can be completed with $m_i \le \Delta_{i+1}$ for $i = 1, \dots, n-1$.*

EXAMPLE 3.3. For the data taken from the function $f(x) = 1/x^2$, as in Example 3.1, the shape preserving algorithm McAllister-Passow-Roulier is illustrated in Figure 14. The leftmost graph shows the data points, as well as the inserted points $(\bar{x}_i, t_i)$. The middle graph represents two different C^1-cubic splines that are obtained for the following α-sequences: $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$ (graph 1) and $\alpha_1 = 0.1$, $\alpha_2 = 0.6$, and $\alpha_3 = 0.9$ (graph 2). The third graph shows two different interpolates for $k = 3$ (all pieces have the same degree) and for $k = 15$. All interpolates preserve monotonicity and convexity.

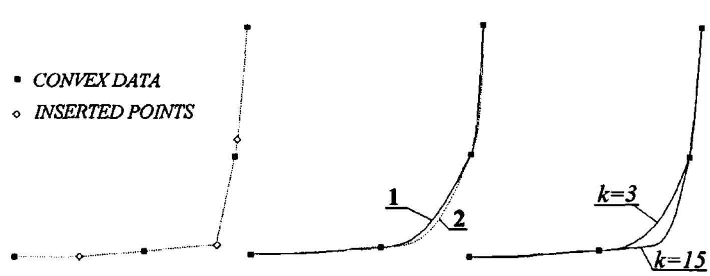


Figure 14. The algorithm McAllister-Passow-Roulier.

As it is noted in [12], the degree of various piecewise polynomials may be forced to be arbitrarily high by a suitable choice of data points. In other words, it is impossible to interpolate convex data by a convex polynomial spline of bounded degree for general data and knots. In [50] it is shown that in CP interpolation this undesirable property can occur for any choice of fixed knots. Further development of the idea of the shape preserving interpolation by splines of arbitrary degree, using the shape preserving property of Bernstein polynomials can be found in [53–55].

In [56], the algorithm is presented for calculating an osculatory MP and CP quadratic spline, that is consistent with the given derivatives at the data points.

As far as the most important class of cubic splines is concerning, Costantini and Morandi [57,58] have studied C^1-cubic splines which preserve both convexity and monotonicity. For other interesting results of the Italian group, concerning shape preserving interpolation, see [53–55,59–63].

The C^2-cubic spline CP interpolation is considered in [64] (also MP), [13,65].

For example, Miroshnichenko (see [64]) consider the cubic spline s from $C^2[x_0, x_n]$. Denote $M_i = s''(x_i)$, $i = 0, 1, \dots, n$, $\lambda_i = h_i/(h_{i-1} + h_i)$, $\mu_i = 1 - \lambda_i$, $i = 1, \dots, n-1$.

THEOREM 3.13. *(See [64].) Let $D_i = 6[x_{i-1}, x_i, x_{i+1}; y]$, $i = 1, \dots, n-1$, and $D_{-1} = D_{n+1} = 0$. Suppose that the constants $\lambda_0 < 4$, $\mu_n < 4$, $D_0 \ge 0$, and $D_n \ge 0$ are given. Let $s \in C^2[x_0, x_n]$ be a cubic spline with end conditions*

$$2M_0 + \lambda_0 M_1 = D_0, \qquad \mu_n M_{n-1} + 2M_n = D_n,$$

which interpolates the convex data $\{(x_i, y_i)\}_{i=0}^n$. If

$$2D_i - \lambda_i D_{i+1} - \mu_i D_{i-1} \ge 0, \qquad i = 0, 1, \dots, n,$$

then s is convex on $[x_0, x_n]$.

THEOREM 3.14. *(See [64].) Let the constants λ_n, μ_0, c_0, and c_n are given such that $|\lambda_n| \le 2$, $|\mu_0| \le 2$. Let $c_i = 3(\lambda_i \Delta_{i-1} + \mu_i \Delta_i)$, $i = 1, \dots, n-1$. Let $s \in C^2[x_0, x_n]$ be a cubic spline with end conditions*

$$2d_0 + \mu_0 d_1 = c_0, \qquad \lambda_n d_{n-1} + 2d_n = c_n,$$

which interpolates the increasing data $\{(x_i, y_i)\}_{i=0}^n$. If

$$\begin{aligned} c_1 \max\{0, \mu_0\} &\le 2c_0 \le 12\Delta_0, \\ c_{n-1} \max\{0, \lambda_n\} &\le 2c_n \le 12\Delta_{n-1}, \\ \lambda_i \Delta_{i-1} \le (1 + \lambda_i)\Delta_i, &\qquad i = 1, \dots, n-1, \\ \mu_i \Delta_i \le (1 + \mu_i)\Delta_{i-1}, &\qquad i = 1, \dots, n-1, \end{aligned}$$

then s is increasing on $[x_0, x_n]$.

For convex and monotone parabolic splines see [64]. Also, some other aspects of quadratic shape preserving approximation can be found in [66,67]. An algorithm that uses an optimization approach for constructing C^2 cubic spline is given by Dierckx [68].

Quite a different approach is used by de Boor in [13]. It is based on the concept of *extraneous* inflection point of the interpolant p, which is such inflection point from (x_i, x_{i+1}) so that $\delta_i \delta_{i+1} > 0$, where $\delta_i = \Delta_i - \Delta_{i-1}$, and as above, $\Delta_i = \Delta y_i / h_i$, $h_i = \Delta x_i$. Let the piecewise linear interpolant to the data $\{(x_i, y_i)\}$, $i = 0, 1, \dots, n$ is convex (concave) on the interval $[x_{r-1}, x_{s+1}]$, then the interpolant p is said to be *good in de Boor sense*, if it is convex (concave) on the interval $[x_{i-1}, x_{i+1}]$. Obviously, if $\delta_i = 0$, this definition recognizes a linear function on $[x_{i-1}, x_{i+1}]$ as an interpolant. In other words, good shape preserving interpolant should not introduce extraneous inflection points, because the existence of such points results in appearance of interpolant's oscillations.

Trying to eliminate extraneous inflection points, Schweikart [69] introduced the *spline in tension*. Each segment of such a spline is an exponential function, or, more precisely, a linear combination of the basis $\{1, x, e^{\lambda x}, e^{-\lambda x}\}$ for $\lambda > 0$, or the customary cubic spline for $\lambda = 0$. Increasing λ increases 'tension' of a spline curve, such that for $\lambda \to \infty$, it approaches to the piecewise linear interpolant. From this reason, λ is often refereed to as *tension (parameter)*. Clearly, increasing of tension may result in eliminating of extraneous inflection points. For some generalizations of Schweikart's model, see [70]. For shape preserving aspect of exponential splines see [71–75], while a good survey of the topic is given in [76].

The only shortcoming of the Schweikart-Späth spline interpolant is that the exponential functions are much more expensive for calculation than the cubic ones. This motivated a number of authors to try to find a cubic alternative to exponential splines under tension [77–80].

One good alternative is de Boor's *taut spline* [13]. It uses a knot inserting technique to place the extra knots that allow sharp bending of a cubic spline without breaking out into oscillations. The taut spline, for $x \in [x_i, x_{i+1}]$ has a form

$$p(x) = A_i + B_i u + C_i \phi(u;\, z) + D_i \phi(1-u;\, 1-z), \tag{3.13}$$

with

$$\phi(x;\, z) = \alpha x^3 + (1-\alpha)\left(\frac{x-\zeta}{1-\zeta}\right)_+^3,$$

$\alpha(z) = (1-\gamma/3)/\zeta$ and the additional knot $\zeta(z) = 1 - \gamma \min\{1-z,\, 1/3\}$, where $\gamma \in [0, 3]$. Also,

$$u(x) = \frac{(x-x_i)}{h_i},$$

$$z = \begin{cases} \dfrac{\delta_{i+1}}{(\delta_i + \delta_{i+1})}, & \text{if } \delta_i\delta_{i+1} \ge 0, \quad \delta_i + \delta_{i+1} \ne 0, \\ \dfrac{1}{2}, & \text{otherwise,} \end{cases}$$

where $\delta_i = \Delta_i - \Delta_{i-1}$, $i = 1, \dots, n-1$, and $z = 1/2$ for $i = 0, 1$. Note that γ is a free parameter, and increasing of γ made the interpolant to look more 'round.'

With the notation $M_i = p(x_i)$, the coefficients in (3.13) are

$$A_i = p(x_i) - D_i, \qquad B_i = h_i\Delta_i - C_i + D_i,$$
$$C_i = h_i^2 \frac{M_{i+1}}{\phi''(1;\, z)}, \qquad D_i = h_i^2 \frac{M_i}{\phi''(1;\, 1-z)}.$$

It remains to determine the sequence $\{M_i\}_{i=0}^n$ so that p belongs to $C^2[x_0, x_n]$, i.e., to solve the tridiagonal system

$$aM_{i-1} + bM_i + cM_{i+1} = \delta_i, \qquad i = 1, \dots, n-1, \tag{3.14}$$

where

$$a = \frac{h_{i-1}}{\phi(1;\, 1-z_{i-1})}, \qquad b = \frac{\phi'(1;\, z_{i-1}) - 1}{\phi''(1;\, z_{i-1})} h_{i-1} + \frac{\phi'(1;\, 1-z_i) - 1}{\phi''(1;\, 1-z_i)} h_i, \qquad c = \frac{h_i}{\phi''(1;\, z_i)}.$$

If $\alpha = 1$, no additional knots are introduced, and then (3.14) reduces to

$$\frac{h_{i-1}}{6} M_{i-1} + \frac{h_{i-1} + h_i}{3} M_i + \frac{h_i}{6} M_{i+1} = \delta_i,$$

which results in the customary cubic spline interpolant, which depends on the end-point conditions.

In the paper [81], the convexity preserving properties of a class of C^2 polynomial splines of nonuniform degree is studied, and the corresponding algorithm is given.

The interpolation splines of degree $k+1$, and l times continuously differentiable (i.e., the splines from the class S^l_{k+1}) that are both MP and CP are studied by Neuman ([82,83]). He uses a set of knots $\{t_i\}$ that is related with interpolation nodes $\{x_i\}$ in the following way:

$$\begin{aligned} x_0 = t_0 = \cdots = t_{m-1} < t_m = x_1 = t_{k+1} = \cdots = t_{k+m} < t_{k+m+1} \\ = \cdots = t_{2k+1} = x_2 = \cdots = x_{n-1} = t_{(n-1)(k+1)} = \cdots \\ = \cdots = t_{(n-1)(k+1)+m-1} < t_{(n-1)(k+1)+m} = \cdots = t_{n(k+1)-1} = x_n, \end{aligned}$$

where m is an arbitrary real number such that $1 \le m \le k$.

THEOREM 3.15. *(See [83].) Let k and m be arbitrary natural numbers such that $1 \le m \le k$. If for all $i = 0, 1, \dots, n-1$,*

$$\Delta_i \ge \frac{k+1}{m} \sum_{j=0}^{i-1} \left(1 - \frac{k+1}{m}\right)^{i-j-1} \Delta_i,$$

then there exists an increasing and convex spline $s \in S^{k+1-r}_{k+1}$, $r = \max\{m, k+1-m\}$ that interpolates increasing and convex data $\{(x_i, y_i)\}_{i=0}^n$.

4. RATIONAL SPLINES

An acceptable alternative to MP or CP interpolation processes by polynomials or polynomial splines are shape preserving rational splines. The rational spline is a smooth enough, piecewise (m, n)-rational function. A function is called (m, n)-rational if it is an m-degree polynomial divided by an n-degree one.

In [84], Delbourgo and Gregory have introduced piecewise rational quadratic function for which the necessary derivative condition for monotonicity is also sufficient (see also [85]). They have given a closed form solution to the monotonic interpolation problem. It results in C^1 monotonic (2, 2)-rational spline (or simply quadratic rational spline) with at most $O(h^4)$ accuracy.

Let $\{(x_i, y_i)\}_{i=0}^n$ be given monotone data ($x_0 < x_1 < \cdots < x_n$). Let s be the rational quadratic spline, defined for $x \in [x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$ by

$$s(x) = \begin{cases} \dfrac{P_i(t)}{Q_i(t)}, & \text{if } \Delta_i \ne 0, \\ y_i, & \text{if } \Delta_i = 0, \end{cases} \qquad \frac{t = (x - x_i)}{h_i}, \tag{4.1}$$

where

$$P_i(t) = \Delta_i y_{i+1} t^2 + (y_i d_{i+1} + y_{i+1} d_i) t(1-t) + \Delta_i y_i (1-t)^2, \tag{4.2}$$

$$Q_i(t) = \Delta_i t^2 + (d_{i+1} + d_i) t(1-t) + \Delta_i (1-t)^2. \tag{4.3}$$

As in previous sections, we set $\Delta_i = \Delta y_i / h_i$ and $d_i = s'(x_i)$.

THEOREM 4.1. *(See [84].) The spline $s(x)$, $x \in [x_0, x_n]$ has the following properties:*

(i) *if $\Delta_i \ne 0$, then $Q_i(t) \ne 0$ for all $t \in [0, 1]$;*
(ii) *$s(x_j) = y_j$, $s'(x_j) = d_j$, $j = i, i+1$;*
(iii) *$s \in C^1[x_0, x_n]$;*
(iv) *$s'(x) \ge 0$ for $x \in [x_0, x_n]$;*
(v) *$\lim_{\Delta_i \to 0} P_i(t)/Q_i(t) = y_i$.*

The convergence analysis is given in the following theorem.

THEOREM 4.2. *(See [84].) Let $f \in C^4[x_0, x_n]$, and suppose $|f'(x)| > 0$ on a compact set $K \subset [x_0, x_n]$. Let $y_i = f(x_i)$ and $f_i' = f'(x_i)$. Then for $x \in [x_i, x_{i+1}] \subset K$, $i = 0, 1, \dots, n-1$,*

(i) *there exists a constant c, independent of i, such that*

$$\min_{0 \le t \le 1} |Q_i(t)| \ge c > 0;$$

(ii) $|f(x) - s(x)| \le h_i A \max\{|f_i' - d_i|, |f_{i+1}' - d_{i+1}|\} + h_i^4 B_i$, *where*

$$A = \frac{1}{4c} \|f'\|, \qquad B_i = \frac{1}{384c} \left[\left\|f^{(4)}\right\| \|f'\| + \frac{2h_i}{3} \left\|f^{(3)}\right\|^2 + 2 \left\|f^{(2)}\right\| \left\|f^{(3)}\right\| \right],$$

and $\|\cdot\|$ denotes the uniform norm on $[x_0, x_n]$.

A rational cubic C^2-spline, or more precisely $(3,2)$-rational spline (on each subinterval it reduces on rational function with a cubic numerator and a quadratic denominator), which has MP and/or CP property is considered in [85].

In [86], Delbourgo develops the idea from [84] to use C^2 $(3,2)$-rational spline, defined in $x \in [x_i, x_{i+1}]$, by

$$s(x) = (1-t)y_i + ty_{i+1} - C_i h_i (1-t) t \frac{A_i(1-t) + B_i t}{C_i + t(1-t)h_i^2},$$

where $A_i = \Delta_i - d_i$, $B_i = d_{i+1} - \Delta_i$, and as above $t = (x - x_i)/h_i$. The numbers $1/C_i$ are tension parameters for the subinterval, i.e., when $C_i \to 0$, s approaches to the linear interpolant to (x_i, y_i) and (x_{i+1}, y_{i+1}), while $C_i \to \infty$ reduces s on the cubic segment. In [86], the consistency of a tridiagonal system is established, and it is shown how tension parameters can be used for adjusting convexity of the spline s.

Gregory [87] suggested a $(3,2)$-rational spline of the form

$$s(t) = \frac{(1-t)^3 y_i + t(1-t)^2 (r_i y_i + h_i d_i) + t^2 (1-t)(r_i y_{i+1} - h_i d_{i+1}) + t^3 y_{i+1}}{1 + t(1-t)(r_i - 3)}, \tag{4.4}$$

for $x \in [x_i, x_{i+1}]$, where $r_i > -1$ is the tension parameter in this interval. The case $r_i = 3$ is that of ordinary cubic spline approximation. The Gregory rational spline is C^2 if $r_i > 2$, $i = 1, \dots, n-1$. The spline is an MP interpolant if $r_i \ge (d_i + d_{i+1})/\Delta_i$, and a CP interpolant if

$$r_i \ge 1 + \frac{\max\{d_{i+1} - \Delta_i, \Delta_i - d_i\}}{\min\{d_{i+1} - \Delta_i, \Delta_i - d_i\}}.$$

The examples or Gregory rational spline are given on Figures 15 and 16 (Example 4.1).

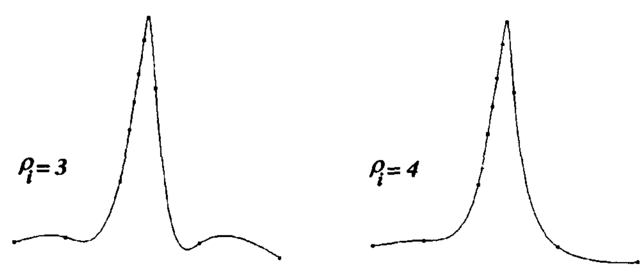


Figure 15. Gregory's rational spline uner tension.

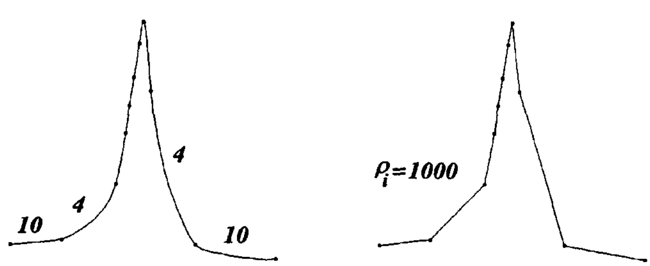


Figure 16. Gregory's rational spline uner tension.

A variation of (4.4) was used by Sarfraz [88] to obtain monotonicity preserving interpolation. His spline has the form

$$s(t) = \frac{(1-t)^3\alpha_i y_i + t(1-t)^2(2+\alpha_i)A_i + t^2(1-t)(2+\alpha_{i+1})B_i + t^3\beta_i y_{i+1}}{(1-t)^2\alpha_i + 2t(1-t) + t^2\beta_i},$$

where $A_i = y_i + \alpha_i/(2+\alpha_i)h_i d_i$, $B_i = y_{i+1} - \beta_i/(2+\beta_i)h_i d_{i+1}$, while α_i and β_i are the shape parameters. For $\alpha_i = \beta_i = 1$, s becomes the cubic interpolant.

The C^2 rational spline that for $x \in [x_i,\, x_{i+1}]$ takes the form

$$s(x) = (1-t)y_i + ty_{i+1} + C_i\left[\frac{t^3}{1+\varphi_i(t)} - t\right] + D_i\left[\frac{(1-t)^3}{1+\psi_i(t)} - (1-t)\right], \tag{4.5}$$

where φ_i and ψ_i are functions satisfying

(a) $\varphi_i,\ \psi_i \in C^2[0,\, 1]$;
(b) $\varphi_i(t) > -1$, $\psi_i(t) > -1$, $t \in [0,\, 1]$;
(c) $\varphi_i(1) = \psi_i(0) = 0$;
(d) $\varphi_i'(1) \le 0$, $\psi_i'(0) \ge 0$, $i = 0, 1, \dots, n-1$;

has been considered by Miroshnichenko [64,89]. Suppose that $y_i = f(x_i)$, where $f \in C^2[x_0,\, x_n]$. Let $m_i = s'(x_i)$, $M_i = s''(x_i)$, $i = 0, 1, \dots, n$, and for $i = 0, 1, \dots, n-1$,

$$\omega_i = 1 - [2 + \psi_i'(0)]\,[2 - \varphi_i'(1)],$$
$$u_i = \left\{6 - 6\varphi_i'(1) - \varphi_i''(1) + 2\,[\varphi_i'(1)]^2\right\}^{-1},$$
$$v_i = \left\{6 + 6\psi_i'(0) - \psi_i''(0) + 2\,[\psi_i'(0)]^2\right\}^{-1}.$$

Then, the constants C_i and D_i in (4.5) are given by

$$C_i = h_i^2 u_i M_{i+1}, \quad D_i = h_i^2 v_i M_i, \qquad i = 0, 1, \dots, n-1.$$

Let $\lambda_i = h_i/(h_{i-1} + h_i)$, $\mu_i = 1 - \lambda_i$, $i = 1, \dots, n-1$ and $P_i = (u_i\omega_i)^{-1}$, $Q_i = (v_i\omega_i)^{-1}$, $i = 0, \dots, n-1$. In [89], it has been proved that m_i satisfy the tridiagonal system

$$-\lambda_i P_{i-1} m_{i-1} - L_i m_i - \mu_i Q_i m_{i+1} = -g_i, \qquad i = 1, \dots, n-1,$$

where $L_i = \lambda_i P_{i-1}[2 + \psi_{i-1}'(0)] + \mu_i Q_i[2 - \varphi_i'(1)]$ and

$$g_i = \lambda_i P_{i-1}\left[3 + \psi_{i-1}'(0)\right]\Delta_{i-1} + \mu_i Q_i\,[3 - \varphi_i'(1)]\,\Delta_i.$$

THEOREM 4.3. *(See [89].) Let the rational spline s, given by (4.5) interpolate convex data $\{(x_i,\, y_i)\}_{i=0}^n$, $y_i = f(x_i)$, and satisfies the boundary conditions*

$$M_0 = f''(x_0), \qquad M_n = f''(x_n),$$

where $f''(x_0) \ge 0$ and $f''(x_n) \ge 0$. If

$$\psi_i(t) = \varphi_i(1-t), \qquad \varphi_i > -1, \qquad \Phi_i''(t) \ge 0, \qquad t \in [0,\, 1],$$
$$\varphi_i(1) = 0, \qquad \varphi_i'(1) \le 0, \qquad i = 0, 1, \dots, n-1,$$
$$2 - \varphi_i'(1) \ge \max\left\{\frac{\delta_i}{\lambda_i\delta_{i+1}},\, \frac{\delta_{i+1}}{\mu_i\delta_i}\right\}, \qquad i = 1, \dots, n-2,$$
$$6 - 6\varphi_0'(1) - \varphi_0''(1) + 2\,[\varphi_0'(1)]^2 \ge \frac{f''(x_0)}{\delta_1},$$
$$6 - 6\varphi_{n-1}'(1) - \varphi_{n-1}''(1) + 2\left[\varphi_{n-1}'(1)\right]^2 \ge \frac{f''(x_n)}{\delta_{n-1}},$$

where $\Phi_i(t) = t^3/[1 + \varphi_i(t)]$, $\delta_i = [x_{i-1},\, x_i,\, x_{i+1};\, f]$, then s is convex on $[x_0,\, x_n]$.

THEOREM 4.5. *(See [89].) Let the rational spline s, given by (4.5) interpolate increasing data $\{(x_i, y_i)\}_{i=0}^n$, $y_i = f(x_i)$, and satisfy boundary conditions*

$$m_0 = f'(x_0), \qquad m_n = f'(x_n),$$

with $f'(x_0) \geq 0$ and $f'(x_n) \geq 0$. If $P_i < 0$ and $Q_i < 0$, $i = 0, 1, \dots, n-1$, and

$$2 + \frac{2}{3}\psi_i'(0) \geq \frac{\Delta_{i-1}}{\Delta_i}, \qquad 2 - \frac{2}{3}\varphi_i'(1) \geq \frac{\Delta_{i+1}}{\Delta_i},$$
$$3t\left[1 + \varphi_i(t)\right] - t^2\varphi_i'(t) \leq \left[3 - \varphi_i'(1)\right]\left[1 + \varphi_i(t)\right]^2,$$
$$3(1-t)\left[1 + \psi_i(t)\right] + (1-t)^2\psi_i'(t) \leq \left[3 + \psi_i'(0)\right]\left[1 + \psi_i(t)\right]^2,$$

for $i = 0, 1, \dots, n-1$, $f'(x_0) = [x_{-1}, x_0; f]$, $f'(x_n) = [x_n, x_{n+1}; f]$, then s is increasing on $[x_0, x_n]$.

A special case of the rational spline (4.5), with $\varphi_i(t) = p_i(1-t)$ and $\psi_i(t) = q_i t$, $i = 0, 1, \dots, n-1$, where $\{p_i\}$ and $\{q_i\}$ are two nonnegative sequences or real numbers, actually they are tension parameters, is considered in [90]. Note that, in this case, (4.5) becomes a $(4, 2)$-rational spline. If put $p_i = q_i = \rho_i$, the spline of the form

$$s(x) = A_i(1-t) + B_i t + \frac{C_i(1-t)^3}{1 + \rho_i t} + \frac{D_i t^3}{1 + \rho_i(1-t)}$$

is obtained. This kind of rational spline was studied by Frost and Kinzel [91], where ρ_i being adjusted automatically.

EXAMPLE 4.1. The following data (see [91])

x_i	0.10	2.00	4.00	4.35	4.50	4.67	4.85	5.00	5.30	7.00	10.0
y_i	1.0	1.2	3.2	5.0	6.0	7.0	8.2	9.0	6.5	1.0	0.5

are interpolated by the Gregory's rational spline (4.4). The uniform tension $\rho_i = 3$, $i = 0, 1, \dots, 9$, results in the ordinary cubic spline (Figure 15 (left)) with unsatisfactory shape preserving property. Increasing tension to 4 gives better result (Figure 14 (right)), but the optimal effect will probably be gained by a nonuniform tension, like in the case shown in Figure 16 (left), where $\rho_0 = \rho_9 = 10$, $\rho_1 = \rho_8 = 4$, while the other tensions are equal to one. Very large tension leads to the almost piecewise linear interpolant (Figure 16 (right)).

At the end of this section, we describe $(2, 1)$-rational splines of Schmidt and Hess and their shape preserving properties that include positivity, convexity, and symmetry. Suppose the data $\{(x_i, y_i)\}_{i=0}^n$ are given such that $x_0 = 0 < x_1 < \cdots < x_n = 1$ and $y_i \geq 0$, $i = 0, 1, \dots, n$. Hess in [44] and Schmidt and Hess in [92] have considered $(2, 1)$-rational spline of the form

$$s(x) = y_i + \Delta_i h_i t + (d_i - \Delta_i)h_i \frac{t(1-t)}{1 + r_i t}, \qquad x \in [x_i, x_{i+1}], \tag{4.6}$$

where $t = (x - x_i)/h_i$, and $r_i \geq 0$ are parameters.

THEOREM 4.6. *(See [44,45].) The rational spline s given by (4.6) is in $C^1[0, 1]$, and it*

(a) *preserves nonnegativity of data if and only if*

$$d_i \geq -\frac{(2 + r_i)y_i + 2\sqrt{(1 + r_i)y_i y_{i+1}}}{h_{i-1}}, \qquad i = 0, 1, \dots, n-1;$$

(b) *preserves monotonicity of data if and only if*

$$d_i \geq 0, \qquad i = 0, 1, \dots, n;$$

(c) *preserves convexity of data if and only if*

$$d_i \leq \Delta_i, \qquad i = 0, 1, \dots, n-1.$$

The following theorem of Schmidt and Hess gives the conditions for preserving symmetry of the data.

THEOREM 4.7. *(See [45].) Suppose that the data set* $\{(x_i, y_i)\}_{i=-n}^{n}$ *is positive and symmetric,*

$$x_{-i} = -x_i, \quad y_{-i} = y_i > 0, \qquad i = 0, 1, \dots, n,$$

with $x_n = 1$. *Then, for sufficiently large parameters* $r_1, \dots, r_n$ *the* C^1*-spline interpolant (4.6) is nonnegative and symmetric on* $[-1, 1]$.

5. APPROXIMATIONS

Let $K_r[a, b]$, or $K_r(I)$ denote the class of functions convex of order r on $I = [a, b]$. The $K_r^+(I)$ will denote the class of strictly convex on I functions of order r.

An early result is given by Pal [93] (see also [94]).

THEOREM 5.1. *(See [93].)*

(a) *Every* $f \in K_2(I)$ *can be infinitely close approximated by the algebraic polynomials that are convex on* I.

(b) *Every* $f \in C(I)$ *is the uniform limit of a sequence of polynomials that are of the same convexity as* f *on* I.

The *Bernstein polynomials* $B_n(f; x)$ that are defined for any function $f \in C[a, b]$ by

$$B_n(f; x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} f\left(a + k\frac{b-a}{n}\right) \binom{n}{k} (x-a)^k (b-x)^{n-k}, \tag{5.1}$$

where $n \geq 0$ are known to converge uniformly towards f, i.e., $\lim B_n(f) = f$.

THEOREM 5.2. *(See [94].) If* f *is strictly convex, convex, or polynomial of order* r, *then* $B_n(f)$ *is also strictly convex, convex, or polynomial of order* r.

PROOF. Without loss of generality, we put $I = [0, 1]$. Then the Bernstein polynomials takes the more simple form

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad x \in [0, 1],$$

with Bernstein basis polynomials

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The following relation is well known (see [95])

$$B_n^{(r)}(f; x) = \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} \Delta^r f\left(\frac{k}{n}\right) p_{n-r,k}(x),$$

and as

$$\Delta^r f\left(\frac{k}{n}\right) = n^r \left[\frac{k}{n}, \dots, \frac{k+r}{n}; f\right],$$

we have

$$B_n^{(r)}(f; x) = \sum_{k=0}^{n-r} \alpha_k \left[\frac{k}{n}, \dots, \frac{k+r}{n}; f\right], \qquad \alpha_k = \frac{n! n^r}{(n-r)!} p_{n-r,k}(x), \tag{5.2}$$

where $\alpha_k \geq 0$ for all $x \in [0, 1]$. Thus, $f \in K_r[0, 1]$, implies $[k/n, \dots, (k+r)/n; f] \geq 0$, which together with (5.2) gives $B_n^{(r)}(f; x) \geq 0$, and therefore, $B_n(f) \in K_r[0, 1]$. Obviously, $f \in K_r^+[0, 1]$, implies the strict positivity of $B_n^{(r)}(f)$, etc. ∎

Relation (5.1) also defines the *Bernstein operator* $B_n : C(I) \to C(I)$. This operator is *linear* and *positive* [95]. As the consequence, if f is (strictly) concave by the above theorem, $B_n(f)$ is (strictly) concave too. The convexity preserving property of Bernstein polynomials has been used in statistics by Wegmüller [96].

Berens and DeVore [97] gave an interesting characterization of Bernstein operator. Namely, they noticed that its slow convergence is a consequence of shape preserving properties of Bernstein polynomials, which is given by

$$|f(x) - B_n(f;\, x)| \le \text{const} \cdot \omega\left(f; \frac{1}{\sqrt{n}}\right), \qquad x \in [0,1],$$

but the Bernstein operator has the biggest rate of convergence among the all polynomial, convexity preserving operators that preserve linear function. More precisely, the authors of [97] have introduced $\mathcal{L}_n$—the class of operators L_n defined by

$$\mathcal{L}_n : \begin{cases} L_n(f) \in \mathcal{P}_n, & \text{for all } f \in C[0,1], \\ L_n(\ell) = \ell, & \text{for all } \ell \in \mathcal{P}_1, \\ [L_n(f)]^{(j)} \ge 0, & \text{if } f^{(j)} \ge 0, \qquad j = 0, 1, \dots, n, \end{cases}$$

and have proved the following result.

THEOREM 5.3. *(See [97].) For any $L_n \in \mathcal{L}_n$,*

$$L_n\left((\cdot - x)^2;\, x\right) \ge B_n\left((\cdot - x)^2; x\right) = \frac{x(1-x)}{n},$$

with equality if and only if $L_n = B_n$.

The Bernstein operator uses the data $\{(x_i,\, f(x_i))\}$, where $x_i = a + k(b-a)/n$ are uniformly spaced knots, sometimes thereby referring as *interpolation-type* operators (see [98]). In spite of this, in general, the Bernstein polynomial $B_n(f)$ *do not* interpolate f at the internal knots $x_1, \dots, x_{n-1}$. On the contrary, the end-points $x_0 = a$ and $x_n = b$ are interpolated, i.e., $B_n(f;\, x_j) = f(x_j)$ for $j = 0, n$. The following definition introduces the concept of interpolation-type operators.

DEFINITION 5.1. *Each operator from the sequence $\{F_n\}$ is of interpolation type with the knots $x_0^n < x_1^n < \cdots < x_n^n \in I \subset \mathbb{R}$ if*

$$F_n(f;\, x) = \sum_{k=0}^{n} f\left(x_k^n\right) h_k^n(x), \tag{5.3}$$

where $h_k^n : I \to \mathbb{R}^+$, $k = 0, 1, \dots, n$ are given functions, and $f : I \to \mathbb{R}$ is an arbitrary function.

THEOREM 5.4. *(See [99]) Let the sequence $\{\phi_i\}$ of functions be defined through*

$$\phi_0 = \sum_{k=0}^{n} h_k^n, \quad \phi_i = \sum_{k=i}^{n} (x_k - x_{i-1})\, h_k^n, \qquad i = 1, \dots, n.$$

Operators F_n defined by (5.3) preserve convexity of the function f if and only if the functions ϕ_0 and ϕ_1 are linear on I, and ϕ_i $(i = 2, \dots, n)$ are convex on I.

Tzimbalario in [100] determined necessary and sufficient conditions for an arbitrary continuous operator (not necessarily positive) to preserve convexity of any order.

THEOREM 5.5. *(See [100].) Let $L : C[a,b] \to C[a,b]$ be a continuous linear operator. The necessary and sufficient conditions for the implication*

$$f \in K^r[a,b] \Rightarrow L(f) \in K^r[a,b],$$

are

(i) $p \in \mathcal{P}_{r-1} \Rightarrow L(p) \in \mathcal{P}_{r-1}$;
(ii) $L(\varphi_c^{r-1}) \in K^r$ *for every* $c \in [a,b]$, *where*

$$\varphi_c^r(x) = \begin{cases} 0, & x \in [a,c), \\ (x-c)^r, & x \in [c,b]. \end{cases}$$

REMARK 5.1. In fact, this is a special case of Tzimbalario's Theorem. The original theorem treats generalized convexity preserving problem—convexity with respect to Chebyshev system. In the present form, Theorem 5.5 is the consequence of the theorem of positivity of continuous linear operators of Popoviciu [101] and Vasić and Lacković [102]. On approximation of functions convex with respect to arbitrary Chebyshev systems see, also [103–105].

A general type of operators, defined by

$$T(f;\, x) = \int_Y K(x,y) f(y)\, dy, \qquad x \in X, \quad X, Y \subset \mathbb{R},$$

where the kernel K is defined on the rectangle $X \times Y$, X, and Y are real intervals, and $f \in C(Y)$ has been considered by Karlin [106].

THEOREM 5.6. *(See [106].) If the kernel K is a totally positive function of order three (see [107]) and*

$$\int_Y K(x,y)\, dy = 1, \quad \int_Y yK(x,y)\, dy = ax + b, \qquad x \in X, \quad a > 0,$$

then T preserves convexity of f, i.e., $f \in K(I) \Rightarrow T(f) \in K(I)$.

A large number of known operators preserve convexity (of order two) of generating function. The long list includes operators of Weierstrass (the proof of convexity preserving is given by Butzer and Nessel [108]), Favard-Szatz-Mirakyan (Lupaş, [109]), Mayer-König-Zeller (Lupaş, [109]), Baskakov (Lupaş, [110]), Ibragimov-Gadžijev [111], Jakimovski-Leviatan (Wood, [112]), Gamma-operators of Lupaş and Müller [113], etc.

As it is shown by Della Vecchia [114], the *Stancu operators* defined for all $f \in C[a,\, b]$ by

$$S_n^{[\alpha]}(f;\, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) w_{nk}^{\alpha}(x), \tag{5.4}$$

where

$$w_{nk}^{\alpha}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$$

preserves convexity of every order of the function f. Moreover, the generalized version of (5.4), studied by Mastroianni and Occorsio [115] $S_{n,\lambda}^{[\alpha]} = [E - (E - S_n^{[\alpha]})]^{\lambda}$, where E is an identity operator and $\lambda \in \mathbb{R}^+$ has the same property [114]. For $\lambda = 1$, $S_{n,\lambda}^{[\alpha]}$ reduces on $S_n^{[\alpha]}$, which for $\alpha = 0$ becomes the Bernstein operator.

The Durrmeyer [116] operator D_n, defined for every f integrable on $[0,1]$ by

$$D_n(f;\, x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt, \qquad x \in [0,1],$$

preserves convexity of every degree, as it is proved by Derriennic [117].

In [118], Andrica and Badea have proved that the following theorem.

Theorem 5.7. *(See [118].) The following statements are equivalent.*

(i) *Jensen's inequality for functions convex on I.*
(ii) *There is a sequence of approximating and convexity preserving positive linear polynomial operators which reproduce the affine functions.*
(iii) *Korovkin's Theorem in the space $C(I)$.*
(iv) *Jessen's inequality for positive linear functionals on $C(I)$.*

Proof. The proof in [118] follows from the implication scheme that connects assertions (i)–(iv): (i) $\Leftrightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv). ∎

The convexity-preserving properties of linear operators can be used for characterizing convex functions, see [119,120].

Clearly, the notion of *shape* cannot be reduced only on monotonicity or convexity (even of higher order). The shape of the graph of some real function may be also related to *variation* of this function (see [107,108]), or to other types of convexity.

A subclass of the class of convexity of order 2, $K_2(I) = K(I)$ is the class of *logarithmically convex* functions. The function is logarithmically convex, i.e., $f \in K_{\log}(I)$ if $\log f \in K(I)$. It is easy to show that $K_{\log}(I) \subset K(I)$. It is interesting if Bernstein polynomial operator (5.1) is closed over the class $K_{\log}(I)$. The positive answer is given by Goodman [121].

Extended convexity preserving property of Bernstein polynomials are studied in [103–105, 122,123]. In the last one, the preservation of the classes of *quasiconvexity* QC, *strict quasiconvexity* SQC, *strong quasiconvexity* SnQC, convexity C $= K(I)$, strict convexity SC $= K_+(I)$, and affinity AFF $= K(I) \setminus K_+(I)$ is considered. The classes of quasiconvexity, strict quasiconvexity, and strong quasiconvexity are defined by for $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ in the following way. Let $x, y \in I$ and $\lambda \in (0,\, 1)$. Then

$$\begin{aligned} \mathrm{QC} &= \{f \,:\, f\left[(1-\lambda)x + \lambda y\right] \le \max\{f(x),\, f(y)\}\}\,, \\ \mathrm{SQC} &= \{f \,:\, f\left[(1-\lambda)x + \lambda y\right] < \max\{f(x),\, f(y)\}\,,\, f(x) \ne f(y)\}\,, \\ \mathrm{SnQC} &= \{f \,:\, f\left[(1-\lambda)x + \lambda y\right] < \max\{f(x),\, f(y)\}\,,\, x \ne y\}\,, \end{aligned}$$

The relationship between these classes is given by Ponstein [124] and it is shown with the help of the graph at Figure 17, where $A \to B$ indicates that $A \subset B$.

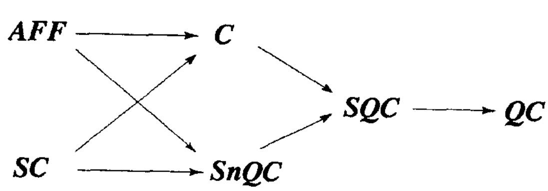


Figure 17. Graph of inclusion of different kinds of convexity.

A strongly quasiconvex function is also known under the name of *unimodal* function (a function with the only one minima). It has been proved in [123] that every variation diminishing operator L that maps a set of bounded functions into the set of continuous functions preserve quasiconvexity. Particularly, for the Bernstein operator, the following implications take place:

$$\begin{gathered} f \in \mathrm{QC} \Rightarrow B_n f \in \mathrm{SnQC}, \\ f \in \mathrm{SQC} \Rightarrow B_n f \in \mathrm{SnQC}, \\ f \in \mathrm{C} \Rightarrow B_n f \in \mathrm{SC}, \\ f \in \mathrm{AFF} \Rightarrow B_n f \in \mathrm{AFF}. \end{gathered}$$

Approximation with quasiconvex functions have been considered in [125].

The fact is that the Bernstein operator is reach with 'shape preserving' properties. Actually, it serves as a kind of model for testing some shape preserving properties. Many properties valid for Bernstein operator are also valid for larger classes of positive linear operators. For example, let $\mathrm{Lip}_M \mu$ denote the class of Hölder continuous functions with Lipschitz constant M and order μ, or such functions that satisfy

$$|f(x) - f(y)| \le M|x-y|^\mu, \qquad \text{for all } x, y \in I.$$

Then Brown, Elliot and Paget [126] have proved that for all $\mu \in (0,\, 1]$, the Bernstein operator is closed over $\mathrm{Lip}_M \mu$, i.e.,

$$f \in \mathrm{Lip}_M \mu \Rightarrow B_n f \in \mathrm{Lip}_M \mu.$$

Della Vecchia [127] has proved that this property can be spread out over many other important operators, like the operators of Favard-Szasz-Mirakyan, Stancu, Favard-Pethe-Jain, Baskakov, Weierstrass, Cauchy, Picard, Fejer-de la Vallée-Poussin, and Jackson.

Another generalization of convex function is *starshaped* function, i.e., the function $f : I \to \mathbb{R}$ such that

$$f(\lambda x) \le \lambda f(x), \qquad \text{for every } \lambda \in (0,1), \quad x \in I.$$

Let K^0 denote the set of functions convex on $[0,1]$ such that $f(0) = 0$, $f(x) \ge 0$, on $[0,1]$ and one-sided continuous at the points 0 and 1. If the set of functions starshaped on $[0,1]$ will be denoted by S, then $K^0 \subset S$ [128]. As it is shown by Lupaş [129], the Bernstein operator preserves starshapedness, i.e., $f \in S \Rightarrow B_n f \in S$. What is more, the operator of Hirschman-Widder and operator of Leviatan, both of them being generalizations of the Bernstein operator, have starshaped-preserving property as well [130].

Approximation of functions or discrete data by splines is also considered from the shape preserving point of view. Results obtained by Bojanić and Roulier [131] is generalized in Theorem 5.5.

Jackson-type estimations for approximation of convex functions by convex splines with equally spaced knots are given in [132] by Beatson.

Algorithms for computing shape preserving approximating splines with knots that coincide with the data points are given by Dodd and McAllister in [133].

Using the optimization theory, Andersson and Elfving [134] offer a Newton-type algorithm for the computation of the monotone spline approximant to noisy monotone data. The reacher spectra of 'shape' of the data being approximated and preserved by the smoothing spline is studied by Girard and Laurent [135]. It includes location of peaks or discontinuities, the value of period etc., and the authors reduce the problem on solving the minimization problem. In [136], Kocić uses Bernstein form of cubic to control 'nonstandard' shape features like peaks, tangent points, etc.

A lot of research has been done about the quantitative estimations of Jackson-type for *comonotone approximation* of a function by polynomials and splines. This means such approximations when an approximant changes monotonicity (convexity) at the same intervals as the function being approximated. The basic results goes back to Lorentz, Zeller and DeVore [137–140]. Unfortunately, the limited size of this review does not allow us to list these interesting results. The topic was further developed by Leviatan and Mhaskar [141,142], Shvedov [143–145], Beatson [146,147], and other authors. For the survey see [148], and for the recent results [149–152].

Also, we will mention a circle of authors that have considered problems concerning the best monotone approximation and the *Polya algorithm*. If f is Lebesgue measurable on $I = [a,\, b]$, let f_p denote the best L_p-*approximant* to f by nondecreasing functions, i.e.,

$$\|f - f_p\|_p = \inf_p \{\|f - g\|_p\},$$

where g is a nondecreasing function on I. It is known that f_p is uniquely determined (up to a.e. equivalence) if $1 < p < \infty$. It is known that $\lim_{p\to\infty} f_p = f_\infty$ exists uniformly on I provided that f is *quasiconvex*, which means that $\lim_{y\to x+0} f(y)$ exists at each $x \in [a, b)$ and $\lim_{y\to x-0} f(y)$ exists at each $x \in (a, b]$. The function f_∞ is known as a *best best L_∞-approximant*. The procedure for constructing f_∞ is known as Polya algorithm. For various results concerning this topic see, for example, [153–156].

Finally, a few words about *moment preserving* approximation by splines and its connection with quadratures.

Following earlier works [157,158] concerning some problems in physics, Gautschi [159] considered the problem of approximating a spherically symmetric function $t \mapsto f(t)$, $t = \|\mathbf{x}\|$, $0 \le t < \infty$ in $\mathbb{R}^d$, $d \ge 1$ by a piecewise constant function s_n, so that approximation preserves as many moments of f as possible. The problem was extended to spline approximation of arbitrary degree by Gautschi and Milovanović in [160], by considering a spline function $s_{n,m}$ of degree $m \ge 0$ on $[0, +\infty)$ vanishing at $t = +\infty$, with $n \ge 1$ positive knots τ_ν ($\nu = 1, \dots, n$), i.e.,

$$s_{n,m}(t) = \sum_{\nu=1}^{n} a_\nu (\tau_\nu - t)_+^m, \qquad a_\nu \in \mathbb{R}, \quad 0 \le t < +\infty, \tag{5.5}$$

so that it reproduces first $2n$ moments of f

$$\int_0^{+\infty} s_{n,m}(t) t^j \, dV = \int_0^{+\infty} f(t) t^j \, dV \qquad j = 0, 1, \dots, 2n-1. \tag{5.6}$$

Differential dV is the volume element depending on the geometry of the problem. We cite two theorems from [160].

THEOREM 5.8. *(See [160].) Let $f \in C^{m+1}[0, +\infty]$ and*

$$\int_0^{+\infty} t^{2n+m+1} \left| f^{(m+1)}(t) \right| dt < +\infty.$$

Then a spline function $s_{n,m}$ satisfies (5.6), exists and is unique if and only if the measure

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f(m+1)(t) \, dt, \qquad \text{on } [0, +\infty), \tag{5.7}$$

admits an n-point Gauss-Christoffel quadrature formula

$$\int_0^{+\infty} g(t) \, d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} g\left(\tau_\nu^{(n)}\right) + R_n(g; d\lambda), \tag{5.8}$$

with distinct positive nodes $\tau_\nu^{(n)}$, where $R_n(g; d\lambda) = 0$ for all $g \in \mathcal{P}_{2n-1}$. In that event, the knots τ_ν and weights a_n in (5.5) are given by

$$\tau_\nu = \tau_\nu^{(n)}, \qquad a_\nu = \tau_\nu^{-(m+1)} \lambda_\nu^{(n)} \qquad \nu = 1, \dots, n.$$

THEOREM 5.9. *(See [160].) Given f as in Theorem 5.8, assume that the measure $d\lambda$ in (5.7) admits the n-point Gauss-Christoffel quadrature formula (5.8) with distinct positive nodes $\tau_\nu = \tau_\nu^{(n)}$ and the remainder term $R_n(g; d\lambda)$. Then, for any $t > 0$, the error of the approximation by the spline $s_{n,m}$, given by (5.5) that satisfies (5.6) is*

$$f(t) - s_{n,m}(t) = R_n(\sigma_t; d\lambda),$$

where

$$\sigma_t(x) = x^{-(m+1)} (x - t)_+^m.$$

Further results are concerning moment preserving approximations by defective splines as well as approximation on finite intervals, see [161–164].

REFERENCES

1. W. Wolibner, Sur un polynôme d'interpolation, *Colloq. Math.* **2**, 136–137, (1951).
2. W.J. Kammerer, Polynomial approximations to finitely oscillating functions, *Math. Comp.* **15**, 115–119, (1961).
3. S.W. Young, Piecewise monotone polynomial interpolation, *Bull. Amer. Math. Soc.* **73**, 642–643, (1967).
4. F. Deutsch and P.D. Morris, On simultaneous approximation and interpolation which preserves the norm, *Bull. Amer. Math. Soc.* **75**, 812–814, (1969).
5. S. Paszkovski, On approximation with nodes, *Rozprawy Mat.* **14**, 1–61, (1957).
6. I. Mircea, Sur un théorème de W. Wolibner, *Anal. Numér. Théor. Approx.* **11**, 81–87, (1982).
7. Z. Rubinstein, On polynomial δ-type functions and approximation by monotonic polynomials, *J. Approx. Theory* **3**, 1–6, (1970).
8. E. Passow and L. Raymon, The degree of piecewise monotone interpolation, *Proc. Amer. Math. Soc.* **48**, 409–412, (1975).
9. E. Passow, An improved estimate of the degree of monotone interpolation, *J. Approx. Theory* **17**, 115–118, (1976).
10. M.G. Nikolcheva, Parametricheskoe priblizhenie funkcii i monotonnaya interpolyaciya, In *The Theory of Approximation of Functions*, Proc. Int. Conf. Kaluga 1975, Nauka, Moscow, (Edited by S.B. Stechkin and S.A. Telyakovskiĭ), pp. 272–275, (1977).
11. C. de Boor and B. Swartz, Piecewise monotone interpolation, *J. Approx. Theory* **21**, 411–422, (1977).
12. E. Passow and J.A. Roulier, Monotone and convex spline interpolation, *SIAM J. Numer. Anal.* **14**, 904–909, (1977).
13. C. de Boor, *A Practical Guide to Splines*, Applied Mathematical Sciences, Volume 27, Springer-Verlag, New York, (1978).
14. L.L. Schumaker, *Spline Functions: Basic Theory*, John Willey, New York, (1981).
15. F.N. Fritsch and R.E. Carlson, Monotone piecewise cubic interpolation, *SIAM J. Numer. Anal.* **17**, 238–246, (1980).
16. C. Manni, Sull'esistenza di funzioni spline interpolanti monotone, *Bollettino U. M. I.* **6** (5-A), 59–68, (1986).
17. J.C. Feguson and K. Miller, Characterization of shape in a class of third degree algebraic curves, TRW Report 5322-3-5, (1969).
18. R.E. Carlson and F.N. Fritsch, Monotone piecewise bicubic interpolation, *SIAM J. Numer. Anal.* **22**, 386–400, (1985).
19. R.L. Daugherty, A. Edelman and J.M. Hyman, Nonnegativity-, monotonicity, or convexity-preserving cubic and quintic Hermite interpolation, *Math. Comp.* **52**, 471–494, (1989).
20. A. Edelman and C.A. Micchelli, Admissible slopes for monotone and convex interpolation, *Numer. Math.* **51**, 441–458, (1987).
21. J.R. Rice, *Numerical Methods, Software and Analysis*, McGraw-Hill, (1983).
22. H. Akima, A new method of interpolation and smooth curve fitting based on local procedures, *J. Assoc. Comput. Mach.* **17**, 589–602, (1970).
23. J.M. Hyman, Accurate monotonicity preserving cubic interpolation, *SIAM J. Sci. Statist. Comput.* **4**, 645–654, (1983).
24. R.K. Beatson and H. Wolkowicz, Post-processing piecewise cubics for monotonicity, *SIAM J. Numer. Anal.* **26**, 480–502, (1989).
25. S.C. Eisenstat, K.R. Jackson and J.W. Lewis, The order of monotone piecewise cubic interpolation, Tech. Report 207, (Edited by Computer Science Department), Yale University, (1981).
26. S.C. Eisenstat, K.R. Jackson and J.W. Lewis, The order of monotone piecewise cubic interpolation, *SIAM J. Numer. Anal.* **22**, 1220–1237, (1985).
27. F.N. Fritsch and J. Butland, A method for constructing local monotone piecewise cubic interpolants, *SIAM J. Sci. Statist. Comput.* **5**, 300–304, (1984).
28. H.T. Huynh, Accurate monotone cubic interpolation, *SIAM J. Numer. Anal.* **30**, 57–100, (1993).
29. Z. Yan, Piecewise cubic curve fitting algorithm, *Math. Comp.* **49**, 203–213, (1987).
30. Lj.M. Kocić and N.V. Blagojević, Adaptive monotone interpolation, In *VIII Conference on Applied Mathematics*, Univ. of Montenegro, Podgorica, (Edited by M. Jaćimović), pp. 87–94, (1994).
31. G. Farin, Curves and surfaces for computer aided geometric design, In *A Practical Guide*, Academic Press, (1988).
32. M.L. Varas, A modified duel algorithm for the computation of the monotone cubic spline, In *Approx. Theory V*, (Proc. Symp. College Station, 1986), pp. 607–610, Academic Press, (1986).
33. P.J.Y. Wong and R.P. Agarwal, Explicit error estimates for quintic and biquintic spline interpolation, *Computers Math. Applic.* **18** (8), 701–722, (1989).
34. H. Dauner and C.H. Reinsch, An analysis of two algorithms for shape-preserving cubic spline interpolation, *IMA J. Numer. Anal.* **9**, 299–314, (1989).
35. A.W. Roberts and D.E. Varberg, *Convex Functions*, Academic Press, New York, (1973).
36. E. Neuman, Uniform approximation by some Hermite interpolating splines, *J. Comput. Appl. Math.* **4**, 7–9, (1978).
37. J.A. Roulier, Polynomials of best approximation which are monotone, *J. Approx. Theory* **9**, 212–217, (1973).

38. E. Passow and J.A. Roulier, Negative theorems on convex approximation, In *Approx. Theory*, (Proc. Int. Symp. Austin, TX, 1976), pp. 509–512, New York, (1976).
39. D.-C. Myres and J.A. Roulier, Markov-type inequalities and the degree of convex spline interpolation, *J. Approx. Theory* **28**, 267–272, (1980).
40. K.G. Ivanov, Shape-preserving interpolation and approximation, *Serdica* **11**, 398–413, (1985).
41. D.F. McAllister, E. Passow and J.A. Roulier, Algorithms for computing shape preserving spline interpolants to data, *Math. Comp.* **31**, 717–725, (1977).
42. J.M. Hyman, Accurate convexity-preserving cubic interpolation, Los Alamos Scientific Laboratory document, LA-UR-80-3700, Los Alamos, NM, (Nov. 1980).
43. E. Passow, Monotone quadratic spline interpolation, *J. Approx. Theory* **19**, 123–134, (1977).
44. J.W. Schmidt, 1-D shape preserving spline interpolation: Existence theorems and construction of optimal splines, Techn. Rep. 07-01-87, Extended version of lectures given at the Banach Center Warsaw, Spring 1986, Techn. Univ. Dresden, Dresden, (1987).
45. J.W. Schmidt and W. Hess, Quadratic and related exponential splines in shape preserving interpolation, *J. Comput. Appl. Math.* **15**, (1987).
46. J.W. Schmidt, Convex interval interpolation with cubic splines, *BIT* **26**, 377–387, (1986).
47. L.L. Schumaker, On shape preserving quadratic spline interpolation, *SIAM J. Numer. Anal.* **20**, 854–864, (1983).
48. A. Lahtinen, On the construction of monotony preserving taper curves, *Acta For. Fennica* **203**, 1–34, (1988).
49. A. Lahtinen, Shape preserving interpolation by quadratic splines, *J. Comput. Appl. Math.* **29**, 15–24, (1990).
50. D.F. McAllister and J.A. Roulier, Interpolation by convex quadratic splines, *Math. Comp.* **32**, 1154–1162, (1978).
51. E. Passow and J.A. Roulier, Shape preserving spline interpolation, In *Approx. Theory*, (Proc. Int. Symp. Austin, TX, 1976), pp. 503–507, New York, (1976).
52. J.A. Roulier and D.F. McAllister, Approximation by convex quadratic splines, In *Approximation Theory III*, pp. 757–761, Academic Press, New York, (1980).
53. P. Costantini, On monotone and convex spline interpolation, *Math. Comp.* **46**, 203–214, (1986).
54. P. Costantini, Co-monotone interpolating splines of arbitrary degree. A local approach, *SIAM J. Sci. Statist. Comput.* **8**, 1026–1034, (1987).
55. P. Costantini, An algorithm for computing shape-preserving interpolating splines of arbitrary degree, *J. Comput. Appl. Math.* **22**, 89–136, (1988).
56. D.F. McAllister and J.A. Roulier, An algorithm for computing a shape-preserving osculatory quadratic spline, *ACM Trans. Math. Software* **7**, 331–374, (1981).
57. P. Costantini and R.E. Morandi, Monotone and convex cubic spline interpolation, *Calcolo* **21**, 281–294, (1984).
58. P. Costantini and R.E. Morandi, An algorithm for computing shape preserving cubic spline interpolation to data, *Calcolo* **21**, 295–305, (1984).
59. C. Conti and R. Morandi, Parallel shape-preserving interpolation via Bézier curves, *Supercomputer* **10**, (1994) (to appear).
60. C. Conti and R. Morandi, Piecewise C^1 shape-preserving Hermite interpolation, *Computing* (to appear).
61. P. Costantini, A general method for constrained curves with boundary conditions, In *Multivariate Approximation: From CAGD to Wavelets*, (Edited by K. Jetter and F.I. Utreras), World Scientific, Singapore, (1993).
62. P. Costantini, Boundary valued shape-preserving interpolating splines, *ACM Trans. Math. Software* (to appear).
63. C. Manni, C^1 Comonotone Hermite interpolation via parametric cubics, *J. Comput. Appl. Math.* (to appear).
64. V.L. Miroshnichenko, Convex and monotone spline interpolation, In *Constructive Theory of Functions '84*, Proc. Int. Conf. Constructive Theory of Functions, Varna 1984, (Edited by B. Sendov *et al.*), pp. 610–620, Publishing House of the Bulgarian Academy of Sciences, Sofia, (1984).
65. J.-C. Fiorot and J. Tabka, Splines d'interpolation polynomiales de degré 3 de classe C^2 conservant la forme, *C.R. Acad. Sci. Paris* **307**, 133–138, (1988).
66. I. Oprea, Spline approximation functions with imposed shape, *Prepr. Babes-Bolyai Univ. Fac-Math. Res. Semin.* **7**, 201–206, (1986).
67. T. Wen, A keep-convex interpolation for monotonic smooth functions, *Math. Numer. Sinica* **2**, 299–306, (1980).
68. P. Dierckx, Algorithm/Algorithmus 42. An algorithm for cubic spline fitting with convexity constraints, *Computing* **24**, 349–371, (1980).
69. D.G. Schweikart, An interpolating curve using a spline in tension, *J. Math. Physics* **45**, 312–317, (1966).
70. H. Späth, *Spline Algorithms for Curves and Surfaces*, Utilitas Mathematica, Winnipeg, Manitoba, (1974).
71. R.J. Renka, Interpolatory tension splines with automatic selection of tension factors, *SIAM J. Sci. Statist. Comput.* **8**, 393–415, (1987).
72. N.S. Sapidis and P.D. Kaklis, An algorithm for constructing convexity and monotonicity-preserving splines in tension, *Comput. Aided Geom. Design* **5**, 127–137, (1988).
73. N.S. Sapidis, P.D. Kaklis and T.A. Loukakis, A method for computing the tension parameters in convexity-preserving spline-in-tension interpolation, *Numer. Approx.* **54**, 179–192, (1988).
74. J. Kozak, Shape preserving approximation, *Computers in Industry* **7**, 435–440, (1986).

75. Y.Y. Feng and J. Kozak, An approach to the interpolation of uniformly spaced data, *J. Comput. Appl. Math.* **23**, 169–178, (1988).
76. B.J. McCartin, Theory of exponential splines, *J. Approx. Theory* **66**, 1–23, (1991).
77. B.A. Barsky, Exponential and polynomial methods for applying tension to an interpolating spline curve, *Computer Vision, Graphics and Image Processing* **27**, 1–18, (1984).
78. T.A. Foley, Interpolation with interval and point tension controls using cubic weighted ν-splines, *ACM Trans. Math. Soft.* **13**, 68–96, (1987).
79. G.M. Nielson, Some piecewise polynomial alternatives to splines under tension, In *Computer Aided Geometric Design*, (Edited by R.E. Barnhill and R.F. Riesenfeld), pp. 209–235, Academic Press, (1974).
80. S. Pruess, Alternatives to the exponential spline in tension, *Math. Comp.* **33**, 1273–1281, (1979).
81. P.D. Kaklis and D.G. Pandelis, Convexity-preserving polynomial splines of non-uniform degree, *IMA J. Numer. Anal.* **10**, 223–234, (1990).
82. E. Neuman, Shape preserving interpolation by polynomial splines, Report Nr N 122, Institute of Computer Science, Univ. Wroclaw, Wroclaw, Poland, (May 1982).
83. E. Neuman, Convex interpolating splines of arbitrary degree, *BIT* **22**, 331–338, (1982).
84. R. Delbourgo and J.A. Gregory, Shape preserving piecewise rational interpolation, *SIAM J. Sci. Statist. Comput.* **6**, 967–976, (1985).
85. J.A. Gregory, Shape preserving rational spline interpolation, In *Rational Approximation and Interpolation*, (Edited by Graves-Morris, Saff and Varga), pp. 431–441, Springer-Verlag, (1984).
86. R. Delbourgo, Accurate C^2 rational interpolants in tension, *SIAM J. Numer. Anal.* **30**, 595–607, (1993).
87. J.A. Gregory, Shape preserving spline interpolation, *Comput. Aided Design* **18**, 53–57, (1986).
88. M. Sarfraz, Rational interpolation preserving the monotonic shape of the data, *Facta Univ. (Niš)* **8**, 87–95, (1993).
89. V.L. Miroshnichenko, Interpolation of functions with large gradients, In *Methods of Approximation and Interpolation*, (in Russian), pp. 98–107, Novosibirsk, (1980).
90. Yu.S. Zav'jalov, B.I. Kvasov and V.L. Miroshnichenko, *Methods of Spline Functions*, (in Russian), Nauka, Moscow, (1980).
91. C.E. Frost and G.L. Kinzel, An automatic adjustment procedure for rational splines, *Comput. Graph.*, 171–176, (1982).
92. J.W. Schmidt and W. Hess, Positive interpolation with rational quadratic splines, *Computing* **38**, 261–267, (1987).
93. J. Pal, Approksimation af konvekse Funktioner ved konvekse Polynomier, *Mat. Tidsskrift* **B**, 60–65, (1925).
94. T. Popoviciu, *Les Fonctions Convexes*, Exposés sur la théorie des fonctions, Hermann and C., Éditeurs, Paris, (1944).
95. G.G. Lorentz, *Bernstein Polynomials*, Univ. Toronto Press, Toronto, Ont., (1953).
96. W. Wegmüller, Ausgleichung Lursch Bernstein-Polynome, *Mitt. Verein. Schweiz. Versich, -Math.* **36**, 15–59, (1938).
97. H. Berens and R. DeVore, A characterization of Bernstein polynomials, In *Approximation Theory III*, (Proc. Int. Sympos. Austin 1980), (Edited by E.W. Cheney), pp. 213–219, Academic Press, New York, (1980).
98. R.A. DeVore, The approximation of continuous functions by positive linear operators, Lecture Notes in Math, Volume 293, Springer-Verlag, Berlin, (1972).
99. T. Popoviciu, Sur la conservation de l'allure de convexité d'une fonction par ses polynomes d'interpolation, *Mathematica (Cluj)* **3**, 311–329, (1961).
100. J. Tzimbalario, Approximation of functions by convexity preserving continuous linear operators, *Proc. Amer. Math. Soc.* **53**, 129–132, (1975).
101. T. Popoviciu, Notes sur les fonctions convexes d'ordre supérieur (IX), *Bull. Math. de la Soc. Roum. des Sciences* **43**, 85–141, (1942).
102. P.M. Vasić and I.B. Lacković, Notes on convex functions II: On continuous linear operators defined on a cone of convex functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. Fis. Fiz.* **602–633**, 53–59, (1978).
103. Lj.M. Kocić, Continuous liner operators defined on the cone of functions convex with respect to Chebyshev system, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. Fis.* **735–762**, 32–34, (1982).
104. Lj.M. Kocić, Approximation theorems for functions convex with respect to Chebyshev system, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. Fis.* **2**, 3–10, (1991).
105. Lj.M. Kocić and I.B. Lacković, Generalized monotonicity and approximations by linear operators, In *Numerical Methods and Approximation Theory II*, (Edited by D. Herceg), pp. 1–6, Univ. of Novi Sad, Novi Sad, (1985).
106. S. Karlin, Total positivity and convexity preserving transformations, In *Convexity*, Vol. VII, pp. 329–347, Proc. Symp. Pure Math., Amer. Math. Soc., Providence, RI, (1963).
107. S. Karlin, *Total Positivity*, Volume I, Stanford Univ. Press, Stanford, CA, (1963).
108. P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation*, One-Dimensional Theory, Volume 1, Birkhäuser Verlag, Basel, (1971).
109. A. Lupaş, Some prorerties of the linear positive operators I, *Mathematica (Cluj)* **9** (32), 77–83, (1967).
110. A. Lupaş, Some prorerties of the linear positive operators II, *Mathematica (Cluj)* **9** (32), 295–298, (1967).

111. I.I. Ibragimov and A.D. Gadžijev, On a sequence of linear positive operators, *Dokl. Akad. Nauk SSSR* **193**, 1222–1225, (1970).
112. B. Wood, Graphic behaviour of positive linear operators, *SIAM J. Appl. Math.* **20**, 329–335, (1971).
113. A. Lupaş and M.W. Müller, Approximation prorerties of the M_n-operators, *Aequationes Math. (Cluj)* **5**, 19–37, (1970).
114. B. Della Vecchia, A new class of linear positive operators of Bernstein type, *Anal. Numér. Théor. Approx.* **17**, 113–124, (1988).
115. G. Mastroianni and M.R. Occorsio, Una generalizzazione dell'operatore di Stancu, *Rend. Acad. Sci. M.F.N. Ser. IV* **XLV**, 495–511, (1978).
116. J.L. Durrmeyer, Une formule d'inversion de la transformée de Laplace. Applications á la théorie des moments, In *Thése de 3e cycle*, Fac. des Sciences de l'Université de Paris, Paris, (1967).
117. M.M. Derriennic, Sur l'approximation des fonctions d'une ou plusieurs variables par des polynômes de Bernstein modifiés et application au probléme des moments, In *Thése de 3e cycle*, Université de Rennes, Rennes, (1978).
118. D. Andrica and C. Badea, Jensen's and Jessen's inequality, convexity-preserving and approximating polynomial operators and Korovkin's theorem, Research Seminars, Seminar on Math. Anal., (Edited by Faculty of Mathematics), Preprint No. 4, pp. 7–16, Babes-Bolyai University Cluj-Napoca, (1986).
119. Lj.M. Kocić, Some new results for convex functions with applications in the approximation theory, Ph.D. Thesis, Univ. of Niš, Niš, (1984).
120. Lj.M. Kocić and I.B. Lacković, Convexity criterion involving linear operators, *Facta Univ. (Niš)* **1**, 13–22, (1986).
121. T.N.T. Goodman, Shape preserving representations, In *Mathematical Methods in CAGD*, (Edited by T. Lyche and L. Schumaker), pp. 333–351, Academic Press, Boston, (1989).
122. Lj.M. Kocić, On a conjecture concerning Bernstein polynomials, In *Conf. Inst. of Mathematics, Physics and Mechanics*, (Edited by Z. Bohte), pp. 67–71, Univ. of Ljubljana, Ljubljana, (1986).
123. Lj.M. Kocić, Extended shape preserving approximations, In *Constructive Theory of Functions*, (Edited by B. Sendov *et al.*), pp. 270–275, Bulgarian Academy of Sciences, Sofia, (1988).
124. J. Ponstein, Seven kinds of convexity, *SIAM Rev.* **9**, 115–119, (1967).
125. V.A. Ubhaya, Uniform approximation by quasi-convex and convex functions, *J. Approx. Theory* **55**, 326–336, (1988).
126. B.M. Brown, D. Elliot and D.F. Paget, Lipschitz constant for the Bernstein polynomials of a Lipschitz continuous functions, *JAT* **49**, 196–199, (1987).
127. B. Della Vecchia, On the preservation of Lipschitz constants for some linear operators, *Bolletino U. M. I.* **7** (3-B), 125–136, (1989).
128. A.M. Bruckner and E. Ostrow, Some function classes related to the class of convex functions, *Pacific J. Math.* **12**, 1203–1215, (1962).
129. L. Lupaş, A property of the S.N. Bernstein operator, *Mathematica (Cluj)* **9** (32), 299–301, (1967).
130. L. Lupaş, On starshapedness preserving properties of a class of linear positive operators, *Mathematica (Cluj)* **12** (35), 105–109, (1970).
131. R. Bojanić and J. Roulier, Approximation of convex functions by convex splines and convexity-preserving continuous linear operators, *Anal. Numér. Théor. Approx.* **3**, 143–150, (1974).
132. R.K. Beatson, Convex approximation by splines, *SIAM J. Math. Anal.* **12**, 549–559, (1981).
133. S.L. Dodd and D.F. McAllister, Algorithms for computing shape preserving spline approximations to data, *Numer. Math.* **46**, 159–174, (1985).
134. L.-E. Andersson and T. Elfving, Interpolation and approximation by monotone cubic splines, *J. Approx. Theory* **66**, 302–333, (1991).
135. D. Girard and P.J. Laurent, Splines and estimation of nonlinear parameters, In *Mathematical Methods in CAGD*, (Edited by T. Lyche and L. Schumaker), pp. 273–298, Academic Press, Boston, (1989).
136. Lj.M. Kocić, Affine shape control of cubics, *PU. M. A. Ser. B* **3**, 207–229, (1992).
137. R.A. DeVore, Monotone approximation by splines, *SIAM J. Math. Anal.* **8**, 891–905, (1977).
138. R.A. DeVore, Monotone approximation by polynomials, *SIAM J. Math. Anal.* **8**, 906–921, (1977).
139. G.G. Lorentz, Monotone approximation, In *Inequalities, III*, (Edited by O. Shisha), pp. 201–215, Acad. Press, New York, (1972).
140. G.G. Lorentz and K.L. Zeller, Degree of approximation by monotone polynomials I, *J. Approx. Theory* **1**, 501–504, (1968).
141. D. Leviatan and H.N. Mhaskar, Comonotone approximation by splines of piecevise monotone functions, *J. Approx. Theory* **35**, 364–369, (1982).
142. D. Leviatan, The degree of copositive approximation by polynomials, *Proc. Amer. Math. Soc.* **88**, 101–105, (1983).
143. A.S. Shvedov, Jackson's theorem in L^p, $0 < p < 1$, for algebraic polynomials and rates of comonotone approximations, (in Russian), *Mat. Zametki* **25**, 107–117, (1979).
144. A.S. Shvedov, Co-approximation rates of functions by algebraic polynomials, (in Russian), *Mat. Zametki* **29**, 117–130, (1981).
145. A.S. Shvedov, Co-approximation by piecewise-monotone functions by polynomials, (in Russian), *Mat. Zametki* **30**, 839–846, (1981).

146. R.K. Beatson, Restricted range of approximation by splines and variational inequalities, *SIAM J. Numer. Anal.* **19**, 372–380, (1982).
147. R.K. Beatson and D. Leviatan, On comonotone approximation, *Canad. Math. Bull.* **26**, 220–224, (1983).
148. J. Gilewicz and I.A. Shevchuk, Comonotone approximation, Techn. Report CPT-95/P.3203, Centre de Physique Théorique, Marseille, (May 1995).
149. G.A. Dzyubenko, J. Gilewicz and I.A. Shevchuk, Piecewise monotone pointwise approximation, Techn. Report CPT-94/P.3121, Centre de Physique Théorique, Marseille, (January 1995).
150. V.N. Konovalov and A.V. Pavlik, Positive, monotone and convex coapproximation of functions differentiable on a segment, In *Preprint Series*, (Edited by G.A. Dzyubenko *et al.*), Preprint 94.39, Inst. Math. NAS of Ukraine, Kyiv, (1994).
151. B. Lenze, On one-sided spline approximation operators, *Numer. Func. Anal. Optim.* **10**, 167–180, (1989).
152. I.A. Shevchuk, Approximation of monotone functions by monotone polynomials, (in Russian), *Mat. Sbornik* **183**, 63–78, (1992); English transl., *Russian Acad. Sci. Sb. Math.* **76**, 51–64, (1993).
153. D. Legg and D. Townsend, Best monotone approximation in $L_\infty[0, 1]$, *J. Approx. Theory* **42**, 30–35, (1984).
154. D. Legg, Best monotone approximation in $L_\infty[0, 1]$, *Contemporary Math.* **42**, 105–111, (1985).
155. R.B. Darst and R. Houtari, Best L_1-approximation of bounded, approximately continuous functions on [0, 1] by nondecreasing functions, *J. Approx. Theory* **43**, 178–189, (1985).
156. R.B. Darst and R. Houtari, Monotone approximation on an interval, *Contemporary Math.* **42**, 43–44, (1985).
158. A.C. Calder and J.G. Laframboise, Multiple-water-bag simulation of inhomogeneous plasma motion near an electrode, *J. Comput. Phys.* **65**, 18–45, (1986).
157. Laframboise and A.D. Stauffer, Optimum discrete approximation of the Maxwell distribution, *AIAA J.* **7**, 520–523, (1969).
159. W. Gautschi, Discrete approximation to spherically symmetric distributions, *Numer. Math.* **44**, 53–60, (1984).
160. W. Gautschi and G.V. Milovanović, Spline approximations to spherically symmetric distributions, *Numer. Math.* **49**, 111–121, (1986).
161. M. Frontini, W. Gautschi and G.V. Milovanović, Moment-preserving spline approximation on finite intervals, *Numer. Math.* **50**, 503–518, (1987).
162. M. Frontini and G.V. Milovanović, Moment-preserving spline approximation on finite intervals and Turán quadratures, *Facta Univ. Ser. Math. Inform.* **4**, 45–56, (1989).
163. C.A. Micchelli, Monosplines and moment preserving spline approximation, In *Numerical Integration III*, (Edited by H. Brass and G. Hámmerlin), pp. 130–139, Birkhäuser, Basel, (1988).
164. G.V. Milovanović and M.A. Kovačević, Moment-preserving spline approximation and Turán quadratures, In *Numerical Mathematics*, (Singapore, 1988), (Edited by R.P. Agarwal, Y.M. Chow and S.J. Wilson), ISNM Vol. 86, pp. 357–365, Birkhäuser, Basel, (1988).