

ON AN INTERPOLATION PROCESS OF LAGRANGE–HERMITE TYPE

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ABSTRACT. We consider a Lagrange–Hermite polynomial, interpolating a function at the Jacobi zeros and, with its first $(r-1)$ derivatives, at the points ± 1 . We give necessary and sufficient conditions on the weights for the uniform boundedness of the related operator in certain suitable weighted L^p -spaces, $1 < p < \infty$, proving a Marcinkiewicz inequality involving the derivative of the polynomial at ± 1 . Moreover, we give optimal estimates for the error of this process also in the weighted uniform metric.

1. Introduction

Let us denote by $L_{m,r}(v^\alpha, f)$ the polynomial of Lagrange–Hermite type based on the Jacobi zeros $x_k = x_{m,k}(v^\alpha)$ related to the weight $v^\alpha(x) = (1-x^2)^\alpha$ and whose j th order derivatives at ± 1 are equal to $f^{(j)}(\pm 1)$, $j = 0, 1, \dots, r-1$, i.e.,

$$\begin{aligned} L_{m,r}(v^\alpha, f, x_k) &= f(x_k), \quad k = 1, \dots, m, \\ L_{m,r}(v^\alpha, f)^{(j)}(\pm 1) &= f^{(j)}(\pm 1), \quad j = 0, 1, \dots, r-1, \end{aligned}$$

where $f^{(0)} \equiv f$.

This interpolation process is sometimes useful in the numerical treatment of differential equations with boundary conditions. The authors had already took into consideration a similar procedure obtaining some results that the reader can find in [3, pp. 260, 272]. In the present paper we are going to study the behaviour

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of the sequence $\{L_{m,r}(v^\alpha, f)\}_m$ in certain suitable weighted L^p -spaces and give necessary and sufficient conditions on the weights for the uniform boundedness of $\{L_m(v^\alpha)\}_m$. Optimal estimates of the error will be given and a Marcinkiewicz inequality involving the derivatives of the polynomial at ± 1 will be proved. The results of this paper cover the ones available in literature.

In Section 2 we will state our main results and in Section 3 we will prove them.

2. Main Results

In the following C denotes a positive constant which may have different values in different formulas. We will write $C \neq C(a, b, \dots)$ to say that C is independent of the parameters a, b, \dots . If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant C independent of the parameters of A and B , such that $B/C \leq A \leq CB$.

Now we introduce some function spaces, related to a Jacobi weight of the form

$$(2.1) \quad v^\gamma(x) = (1 - x^2)^\gamma, \quad \gamma \geq 0, \quad x \in (-1, 1).$$

Letting L^p , $1 \leq p < \infty$, denote the space of all measurable functions f with $\|f\|_p^p = \int_{-1}^1 |f|^p$, we say $f \in L_{v^\gamma}^p$ if $fv^\gamma \in L^p$, i.e., $\|f\|_{L_{v^\gamma}^p}^p = \int_{-1}^1 |fv^\gamma|^p < \infty$. For $p = \infty$ and $\gamma > 0$, we set $L_{v^\gamma}^\infty = C_{v^\gamma} = \{f \in C^0(-1, 1) : \lim_{|x| \rightarrow 1} (fv^\gamma)(x) = 0\}$ and $C_{v^0} \equiv C^0[-1, 1]$. Moreover, we set

$$C_r^0 = \{f \in C^0(-1, 1) : f \text{ is } (r - 1) - \text{times differentiable at } \pm 1\},$$

where $r \geq 1$ is an integer number. Of course, $C_r^0 \subset C_{v^\gamma}$, $\gamma \geq 0$, and $C_r^0 \supset C^{r-1}[-1, 1]$, where $C^{r-1}[-1, 1]$ is the collection of all functions whose $(r - 1)$ th derivative is continuous on $[-1, 1]$.

The Sobolev type spaces are defined as follows

$$W_p^s = W_p^s(v^\gamma) = \{f \in L_{v^\gamma}^p : f^{(s-1)} \in AC(-1, 1) \text{ and } \|f^{(s)}\varphi^s v^\gamma\|_p < \infty\},$$

where $\varphi(x) = \sqrt{1 - x^2}$, $AC(-1, 1)$ is the set of the absolutely continuous functions in every compact of $(-1, 1)$, $1 \leq p \leq \infty$ and $s \geq 1$ is an integer.

Let $v^\alpha(x) = (1 - x^2)^\alpha$, $\alpha > -1$, and $\{p_m(v^\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. For every function $f \in C_r^0$, $r \geq 1$, an expression of the polynomial $L_{m,r}(v^\alpha, f, x)$, $\alpha > -1$, is given by

$$(2.2) \quad \begin{aligned} L_{m,r}(v^\alpha, f, x) &= \sum_{k=1}^m v^r(x) \frac{l_k(x)}{v^r(x_k)} f(x_k) \\ &+ (1 - x)^r p_m(v^\alpha, x) \sum_{i=0}^{r-1} \frac{(1 + x)^i}{i!} \left(\frac{f}{(1 - \cdot)^r p_m(v^\alpha)} \right)^{(i)} (-1) \\ &+ (1 + x)^r p_m(v^\alpha, x) \sum_{i=0}^{r-1} \frac{(1 - x)^i}{i!} \left(\frac{f}{(1 + \cdot)^r p_m(v^\alpha)} \right)^{(i)} (1) \\ &=: A(x) + B_1(x) + B_2(x), \end{aligned}$$

where $x_k, k = 1, \dots, m$, are zeros of $p_m(v^\alpha)$ and

$$l_k(x) = \frac{p_m(v^\alpha, x)}{p'_m(v^\alpha, x_k)(x - x_k)}, \quad k = 1, \dots, m.$$

We complete the definition of $L_{m,r}(v^\alpha, f)$ setting $L_{m,0}(v^\alpha, f) = L_m(v^\alpha, f)$.

Finally, letting \mathbb{P}_m be the space of all polynomials of degree at most m , we denote by $E_m(f)_{v^\alpha, p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)v^\alpha\|_p, 1 \leq p \leq \infty$, the error of best polynomial approximation in $L^p_{v^\alpha}$.

Now we are able to study the behaviour of the sequence $\{L_{m,r}(v^\alpha)\}_m, r \geq 1$, in the above introduced function spaces.

THEOREM 2.1. *Let v^α and v^γ be two Jacobi weight functions defined in (2.1), with $\gamma \geq 0$ and $\alpha > -1$. Then, for every $f \in C^0_r, r \geq 1$, we have*

$$(2.3) \quad \|L_{m,r}(v^\alpha, f)v^\gamma\|_\infty \leq C \left\{ \|fv^\gamma\|_\infty \log m + \frac{1}{m^{2\gamma}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j! m^{2j}} \right\},$$

where $C \neq C(m, f)$, if and only if

$$(2.4) \quad \frac{\alpha}{2} + \frac{1}{4} \leq \gamma + r \leq \frac{\alpha}{2} + \frac{5}{4}.$$

Moreover, if $f \in C^{r-1}[-1, 1]$, the condition (2.4) implies

$$(2.5) \quad \|[f - L_{m,r}(v^\alpha, f)]v^\gamma\|_\infty \leq CE_{m+2r-1}(f)_{v^\alpha, \infty} \log m, \quad C \neq C(m, f).$$

The above theorem includes some special cases that are well-known in literature. For example, for $\gamma = 0$ and $r \geq 1$ we get Theorem 4.2.5 in [3, p. 260]. In the case $r = 0$ and $\gamma > 0$ we obtain Theorem 2.2 in [4] (see also [3, p. 272]), and for $\gamma = r = 0$ we get Theorem 14.4 in [9, p. 335].

THEOREM 2.2. *Let $1 \leq p < \infty, \gamma \geq 0$, and $\alpha > -1$. Then, for every function $f \in C^0_r, r \geq 1$, there exists a constant $C \neq C(m, f)$ such that*

$$(2.6) \quad \|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leq C \left\{ \|fv^\gamma\|_\infty + \frac{1}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j! m^{2j}} \right\}$$

if and only if

$$(2.7) \quad \frac{v^{\gamma+r}}{\sqrt{v^\alpha \varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} \in L^1$$

i.e.,

$$-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < 1.$$

Letting

$$(2.8) \quad \sigma_m(f) = \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j! m^{2j}},$$

it is easy to deduce from the proof of the above theorem that the only condition $v^{\gamma+r}/\sqrt{v^\alpha\varphi} \in L^p$ is necessary and sufficient to obtain the bound

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leq \mathcal{C}\|f\|_\infty + \frac{\sigma_m(f)}{m^{2\gamma+2/p}}, \quad 1 \leq p < \infty,$$

which, for $r = 0$, follows from a well-know theorem of P. Nevai [7, p. 680].

The following theorem is a refinement of the previous one and implies some interesting consequences.

THEOREM 2.3. *Let $1 < p < \infty$, $\gamma \geq 0$, and $\alpha > -1$. Then, for every function $f \in C_r^0$, $r \geq 1$, there exists a constant $\mathcal{C} \neq \mathcal{C}(m, f)$ such that*

$$(2.9) \quad \|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leq \mathcal{C} \left\{ \left(\sum_{k=1}^m \Delta x_k |f v^\gamma|^p(x_k) \right)^{1/p} \right.$$

$$(2.10) \quad \left. + \frac{1}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j! m^{2j}} \right\}$$

if and only if

$$(2.11) \quad \frac{v^{\gamma+r}}{\sqrt{v^\alpha\varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}} \in L^q, \quad q^{-1} + p^{-1} = 1,$$

i.e.,

$$-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q}.$$

Now we state some estimates of the error $f - L_{m,r}(v^\alpha, f)$ for f varying in the previously introduced spaces.

PROPOSITION 2.1. *For any $f \in C^{r-1}[-1, 1]$, $r \geq 1$, under the assumptions (2.7) we have*

$$(2.12) \quad \|[f - L_{m,r}(v^\alpha, f)]v^\gamma\|_p \leq \mathcal{C}E_{m+2r-1}(f)_{v^\gamma, \infty},$$

where $1 \leq p < \infty$ and $\mathcal{C} \neq \mathcal{C}(m, f)$. Moreover, if $f \in W_p^r$, $r \geq 1$, and $1 < p < \infty$, then the conditions (2.11) imply

$$(2.13) \quad \|[f - L_{m,r}(v^\alpha, f)]v^\gamma\|_p \leq \frac{\mathcal{C}}{m^r} \|f^{(r)}\varphi^r v^\gamma\|_p, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Note that (2.13) shows that, if $f \in W_p^r$, with $r \geq 1$ and $1 < p < \infty$, the polynomial $L_{m,r}(v^\alpha, f)$ converges with the order of the best polynomial approximation in $L_{v^\gamma}^p$. Therefore, in the usual way, we can establish the next corollary that shows the uniform boundedness (with respect to m) of the operator $L_{m,r}(v^\alpha)$ in Sobolev spaces (under the assumptions (2.11)).

COROLLARY 2.1. *Under the conditions (2.11), for every $f \in W_p^r$ with $r \geq 1$ and $1 < p < \infty$, we have*

$$(2.14) \quad \sup_m \|L_{m,r}(v^\alpha, f)\|_{W_p^r} \leq \mathcal{C}\|f\|_{W_p^r}, \quad \mathcal{C} \neq \mathcal{C}(f).$$

Coming back to Theorem 2.3, the estimate (2.9) with the notation (2.8) can be written as

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leq C \left\{ \left(\sum_{k=1}^m \Delta x_k |f v^\gamma|^p(x_k) \right)^{1/p} + \frac{\sigma_m(f)}{m^{2\gamma+2/p}} \right\} =: \Gamma_m(f).$$

Of course, if f is a polynomial P of degree $m + 2r - 1$, the inequality

$$\Gamma_m(P) \geq C \|L_{m,r}(v^\alpha, P)v^\gamma\|_p = C \|Pv^\gamma\|_p, \quad 1 < p < \infty,$$

is equivalent to the conditions (2.11).

Moreover, it is easy to prove that for arbitrary $\alpha > -1$, $\gamma \geq 0$, and $r \geq 1$, the inverse inequality $\Gamma_m(P) \leq C \|Pv^\gamma\|_p$ holds true for $1 \leq p < \infty$. In fact, the bound

$$(2.15) \quad \left(\sum_{k=1}^m \Delta x_k |Pv^\gamma|^p(x_k) \right)^{1/p} \leq C \|Pv^\gamma\|_p$$

is well-known (see, for example, [7, p. 675]). In order to obtain

$$\frac{\sigma_m(P)}{m^{2\gamma+2/p}} \leq C \|Pv^\gamma\|_p$$

it suffices to apply the inequalities of Markov, Schur and Nikol'skiĭ.

Therefore, we can state a new Marcinkiewicz inequality involving the derivatives of a polynomial at ± 1 .

COROLLARY 2.2. *Let $x_k, k = 1, \dots, m$, be the zeros of the m th Jacobi polynomial $p_m(v^\alpha)$ and let $1 < p < \infty$. Then, for every polynomial $P \in \mathbb{P}_{m+2r-1}$, the following equivalence*

$$(2.16) \quad \|Pv^\gamma\|_p \sim \left(\sum_{k=1}^m \Delta x_k |Pv^\gamma|^p(x_k) \right)^{1/p} + \frac{\sigma_m(P)}{m^{2\gamma+2/p}}$$

holds true, with the constants in “ \sim ” independent of m and P , if and only if

$$-\frac{1}{p} < r + \gamma - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q} \quad (p^{-1} + q^{-1} = 1).$$

Finally, we want to observe that if we introduce the m th Christoffel function of the weight $v^{\gamma p}$,

$$\lambda_m(v^{\gamma p}, x) = \left[\sum_{k=0}^{m-1} p_k^2(v^{\gamma p}, x) \right]^{-1} \sim v^{\gamma p}(x) \frac{\sqrt{1-x^2}}{m},$$

then the sum in (2.16) can be replaced by

$$\left(\sum_{k=1}^m \lambda_m(v^{\gamma p}, x_k) |P(x_k)|^p \right)^{1/p}.$$

3. Proofs

In this section we will frequently use the Remez-type inequality in the following form

$$(3.1) \quad (\forall P_m \in \mathbb{P}_m) \quad \|P_m v^\gamma\|_p \leq C \|P_m v^\gamma\|_{L^p(A_m)},$$

where $A_m = [-1 + am^{-2}, 1 - am^{-2}]$, with $a > 0$ fixed, and $C \neq C(m, P_m)$.

If I is a subinterval of $(-1, 1)$, the Hilbert transform $H(f, t)$ is defined as follows

$$H(f, t) = \int_I \frac{f(x)}{x-t} dx, \quad t \in I,$$

where the integral is understood in the Cauchy principal value sense. For $1 < p < \infty$, the following property is well-known:

$$\int_I gH(f) = - \int_I fH(g), \quad f \in L^p \text{ and } g \in L^q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Moreover, with $v^\sigma(x) = (1 - x^2)^\sigma$ and $1 < p < \infty$, one has

$$\|(Hf)v^\sigma\|_p \leq C \|fv^\sigma\|_p \quad \text{if and only if} \quad -\frac{1}{p} < \sigma < \frac{1}{q}.$$

Now, recalling (2.2) with the notation (2.8), we can state the following lemma:

LEMMA 3.1. *With the notation (2.2), we have*

$$\|(B_1 + B_2)v^\gamma\|_\infty \leq \frac{C}{m^{2\gamma}} \sigma_m(f), \quad C \neq C(f, m),$$

if and only if $\frac{\alpha}{2} + \frac{1}{4} \leq \gamma + r \leq \frac{\alpha}{2} + \frac{5}{4}$. Moreover, for $p \in [1, \infty)$, we get

$$\|(B_1 + B_2)v^\gamma\|_p \leq \frac{C}{m^{2\gamma+2/p}} \sigma_m(f), \quad C \neq C(f, m),$$

if and only if $-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q}$.

PROOF. We estimate only the norm $\|B_1 v^\gamma\|_p$, $1 \leq p \leq \infty$, since the estimate of $\|B_2 v^\gamma\|_p$ is similar. Using the Remez inequality (3.1) and letting

$$\bar{A}_i = \frac{1}{i!} \left(\frac{f}{(1-\cdot)^r p_m(v^\alpha)} \right)^{(i)} (-1),$$

we can write

$$(3.2) \quad \begin{aligned} \|B_1 v^\gamma\|_p &\leq C \|B_1 v^\gamma\|_{L^p(A_m)} \\ &\leq \sum_{i=0}^{r-1} |\bar{A}_i| \|(1-x)^r (1+x)^i (1-x^2)^\gamma p_m(v^\alpha, x)\|_{L^p(A_m)} =: \sum_{i=0}^{r-1} |\bar{A}_i| b_i. \end{aligned}$$

Of course, we have

$$\begin{aligned} b_i &\leq \|(1-x)^r (1+x)^i (1-x^2)^\gamma p_m(v^\alpha, x)\|_{L^p(-1+a/m^2, 0)} \\ &\quad + \|(1-x)^r (1+x)^i (1-x^2)^\gamma p_m(v^\alpha, x)\|_{L^p(0, 1-a/m^2)} \\ &:= I_1 + I_2. \end{aligned}$$

Moreover, by virtue of the estimate $|p_m(v^\alpha, x)| \leq C v^{-\frac{\alpha}{2} - \frac{1}{4}}(x)$, $|x| \leq 1 - a/m^2$, we have

$$I_1 + I_2 \leq C \left\{ \|(1+x)^{\gamma+i-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p(-1+a/m^2, 0)} + \|(1-x)^{\gamma+r-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p(0, 1-a/m^2)} \right\}.$$

Now, under the assumptions on the parameters α, γ and r (and only in this case), the first summand is dominated by $C(m^{-2})^{\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p}}$, with $\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p} \leq 0$, $1 \leq p \leq \infty$, while the second summand is bounded. In particular, for $p < \infty$ we have $\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p} < 0$, while for $p = \infty$ we have $\gamma+i-\frac{\alpha}{2}-\frac{1}{4} = 0$ only in the case $i = r - 1$. In any case, since $|p_m(v^\alpha, \pm 1)| \sim m^{\alpha+1/2}$ (see for instance [3, p. 251, formula (4.2.10)]) we conclude that

$$I_1 + I_2 \leq \frac{C}{m^{2\gamma+2/p}} \frac{|p_m(v^\alpha, -1)|}{m^{2i}}, \quad 1 \leq p \leq \infty,$$

taking into account that $2\gamma + 2i = \alpha + 1/2$ for $i = r - 1$ and $p = \infty$. Therefore, recalling (3.2), we have

$$\|B_1 v^\gamma\|_p \leq C \frac{|p_m(v^\alpha, -1)|}{m^{2\gamma+2/p}} \sum_{i=0}^{r-1} \frac{\bar{A}_i}{m^{2i}}.$$

It remains to estimate $\bar{A}_i, i = 0, 1, \dots, r - 1$. We have

$$|\bar{A}_i| \leq \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} |f^{(j)}(-1)| \left| \left(\frac{1}{(1-x)^r} \frac{1}{p_m(v^\alpha, x)} \right)^{(i-j)} (-1) \right|$$

and, taking into account that [7, p. 674, formula (23)]

$$\left(\frac{1}{p_m(v^\alpha, x)} \right)^{(k)} (-1) \leq C \frac{m^{2k}}{|p_m(v^\alpha, -1)|},$$

we obtain

$$\left| \left(\frac{1}{(1-x)^r} \frac{1}{p_m(v^\alpha, x)} \right)^{(i-j)} (-1) \right| \leq C \frac{m^{2i-2j}}{|p_m(v^\alpha, -1)|}.$$

Hence we get

$$|\bar{A}_i| \leq \frac{C}{i! |p_m(v^\alpha, -1)|} \sum_{j=0}^i \binom{i}{j} |f^{(j)}(-1)| m^{2i-2j}$$

and, for $1 \leq p \leq \infty$,

$$\begin{aligned} \|B_1 v^\gamma\|_p &\leq \frac{C}{m^{2\gamma+2/p}} \sum_{i=0}^{r-1} \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} \frac{|f^{(j)}(-1)|}{m^{2j}} \\ &= \frac{C}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)|}{m^{2j}} \sum_{i=j}^{r-1} \frac{1}{i!} \binom{i}{j} \\ &\leq \frac{C}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)|}{j! m^{2j}}, \end{aligned}$$

which completes the proof. □

PROOF OF THEOREM 2.1. In Lemma 3.1, we proved that

$$\|(B_1 + B_2)v^\gamma\|_\infty \leq \frac{\mathcal{C}}{m^{2\gamma}}\sigma_m(f).$$

Then, it remains to prove that $\|Av^\gamma\|_\infty \leq \mathcal{C}\|fv^\gamma\|_\infty \log m$. But, the latter inequality can be found in [3, p. 262] with a minor change. So, the proof of (2.3) is complete.

Concerning the estimate of the error (2.5), we refer to the proof of Proposition 2.1. \square

We are going to prove Theorem 2.3 before Theorem 2.2.

PROOF OF THEOREM 2.3. Taking into account Lemma 3.1, to prove the theorem, it suffices to show that the inequality

$$(3.3) \quad \|Av^\gamma\|_p^p \leq \mathcal{C} \sum_{k=1}^m \Delta x_k |fv^\gamma|^p(x_k), \quad 1 < p < \infty,$$

is equivalent to the conditions (2.11).

We first prove that (2.11) implies (3.3). To this end, using (3.1) and, letting $g = v^{\gamma(p-1)}|A|^{p-1} \operatorname{sgn} A$ in the interval A_m , we can write

$$\|Av^\gamma\|_{L^p(A_m)}^p = \int_{A_m} \sum_{k=1}^m v^{\gamma+r}(x) \frac{l_k(x)f(x_k)}{v^r(x_k)} g(x) dx,$$

where

$$l_k(x) = \frac{p_m(v^\alpha, x)}{p'_m(v^\alpha, x_k)(x - x_k)},$$

whence we deduce

$$\begin{aligned} \|Av^\gamma\|_{L^p(A_m)}^p &= \sum_{k=1}^m \frac{f(x_k)v^\gamma(x_k)}{p'_m(v^\alpha, x_k)v^{\gamma+r}(x_k)} \int_{A_m} \frac{v^{\gamma+r}(x)p_m(v^\alpha, x)}{x - x_k} g(x) dx \\ &\leq \mathcal{C} \sum_{k=1}^m \frac{\Delta x_k |fv^\gamma|(x_k)}{v^{\gamma+r-\frac{\alpha}{2}-\frac{1}{4}}(x_k)} \left| \int_{A_m} v^{\gamma+r}(x)p_m(v^\alpha, x) \frac{g(x)}{x - x_k} dx \right|, \end{aligned}$$

since, with $\Delta x_k = x_{k+1} - x_k$, $1/|p'_m(v^\alpha, x_k)| \sim \Delta x_k v^{\frac{\alpha}{2}+\frac{1}{4}}(x_k)$. Denoting by $G(x_k)$ the absolute value of the integral at the right-hand side and using the Hölder inequality, we get

$$\|Av^\gamma\|_{L^p(A_m)}^p \leq \left(\sum_{k=1}^m \Delta x_k |fv^\gamma|^p(x_k) \right)^{1/p} \left(\sum_{k=1}^m \Delta x_k \left[\frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}}(x_k) G(x_k) \right]^q \right)^{1/q}.$$

It remains to prove that the L^q norm is bounded by $\mathcal{C}\|Av^\gamma\|_{L^p(A_m)}^{p-1}$. We note that for an arbitrary polynomial $Q \in \mathbb{P}_m$, we can write

$$G(x_k) = \left| \int_{A_m} \frac{p_m(v^\alpha, x)Q(x) - p_m(v^\alpha, x_k)Q(x_k)}{x - x_k} v^{\gamma+r}(x) \frac{g(x)}{Q(x)} dx \right|.$$

Therefore, $G(t)$ is a polynomial of degree $2m - 1$. Then, using the Marcinkiewicz inequality (2.15), the L^q norm is dominated by a positive constant \mathcal{C} times the norm

$\left\| \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} G \right\|_{L^q(A_m)}$ that is bounded under the assumption $\frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} \in L^q$. Moreover, denoting by H the Hilbert transform defined on the interval A_m , we can write

$$\begin{aligned} \left\| \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} G \right\|_{L^q(A_m)} &\leq \left\| \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} H(p_m(v^\alpha)v^{\gamma+r}g) \right\|_{L^q(A_m)} \\ &\quad + \left\| \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} Q p_m(v^\alpha) H\left(\frac{gv^{\gamma+r}}{Q}\right) \right\|_{L^q(A_m)}. \end{aligned}$$

Taking also into account the assumption $\frac{v^{\gamma+r}}{\sqrt{v^\alpha \varphi}} \in L^p$, the Hilbert transform is a bounded operator and the first norm is dominated by

$$\left\| \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} p_m(v^\alpha)v^{\gamma+r}g \right\|_{L^q(A_m)} \leq \mathcal{C} \|g\|_{L^q(A_m)} = \|Av^\gamma\|_{L^p(A_m)}^{p-1},$$

since $|p_m(v^\alpha, x)\sqrt{(v^\alpha \varphi)(x)}| \leq \mathcal{C}$.

In order to prove the estimate of the second norm at the right-hand side, we choose a polynomial Q such that $Q(x) \sim v^{\gamma+r}(x)$ for $x \in A_m$ (see [2]). Consequently, the second norm is dominated by

$$\left\| H\left(\frac{gv^{\gamma+r}}{Q}\right) \right\|_q \leq \mathcal{C} \left\| \frac{gv^{\gamma+r}}{Q} \right\|_q \leq \mathcal{C} \|g\|_q = \|Av^\gamma\|_{L^p(A_m)}^{p-1}.$$

Then (2.11) implies (3.3).

Now, we prove that (2.11) is a consequence of (3.3). To this end, for any $f \in C_r^0$, we consider a piecewise linear function F_m such that

$$\begin{cases} F_m^{(i)}(\pm 1) = 0, & i = 0, 1, \dots, r-1, \\ F_m(x_k) = 0, & \text{for } x_k \in [-a, a], \text{ with } a < \frac{1}{4} \text{ fixed,} \\ F_m(x_k) = |f(x_k)| \operatorname{sgn}\{-x_k p'_m(v^\alpha, x_k)\}, & \text{for } x_k \notin [-a, a]. \end{cases}$$

Taking into account that $\operatorname{sgn}(-x_k) = \operatorname{sgn}(x - x_k)$ for $x \in [-a, a]$ and $x_k \notin [-a, a]$, we get

$$\begin{aligned} |L_{m,r}(v^\alpha, F_m, x)v^\gamma(x)| &= |v^{\gamma+r}(x)p_m(v^\alpha, x)| \sum_{k=1}^m \frac{|F_m v^\gamma|(x_k)}{|p'_m(v^\alpha, x_k)v^{\gamma+r}(x_k)||x - x_k|} \\ &\geq \mathcal{C} \left| \frac{v^{\gamma+r}(x)}{2} p_m(v^\alpha, x) \right| \sum_{k=1}^m \Delta x_k v^{\alpha + \frac{1}{4} - \gamma - r}(x_k) |F_m v^\gamma|(x_k), \end{aligned}$$

since $|p'_m(v^\alpha, x_k)|^{-1} \sim \Delta x_k v^{\frac{\alpha}{2} + \frac{1}{4}}(x_k)$ and $|x - x_k| \leq 2$. Moreover, by virtue of a result in [6], we have

$$\liminf_{m \rightarrow \infty} \|v^{\gamma+r} \chi_a p_m(v^\alpha)\|_p \geq \mathcal{C} \left\| \frac{v^{\gamma+r} \chi_a}{\sqrt{v^\alpha \varphi}} \right\|_p \geq \mathcal{C},$$

being χ_a the characteristic function of $[-a, a]$.

Then, collecting the previous inequalities, by (3.3), we obtain

$$(3.4) \quad \begin{aligned} \sum_{k=1}^m \Delta x_k \frac{\sqrt{v^{\alpha}\varphi}}{v^{\gamma+r}}(x_k) |F_m v^{\gamma}|(x_k) &\leq \|L_{m,r}(v^{\alpha}, F_m) \chi_a v^{\gamma}\|_{L^p(A_m)} \\ &\leq \mathcal{C} \left(\sum_{k=1}^m \Delta x_k |F_m v^{\gamma}|^p(x_k) \right)^{1/p}. \end{aligned}$$

Now, letting

$$a_k = (\Delta x_k)^{1/q} \frac{\sqrt{v^{\alpha}\varphi}}{v^{\gamma+r}}(x_k), \quad c_k = (\Delta x_k)^{1/p} |F_m v^{\gamma}|(x_k), \quad \|\bar{c}\|_p^* = \left(\sum_{k=1}^m |c_k|^p \right)^{1/p},$$

where $\bar{c} = (c_1, c_2, \dots, c_m)$, we can write (3.4) as $\sum_{k=1}^m a_k c_k \leq \mathcal{C} \|\bar{c}\|_p^*$ and, since $\mathcal{C} \neq \mathcal{C}(m, F_m)$,

$$\sup_m \sup_{\bar{c}} \sum_{k=1}^m a_k \frac{c_k}{\|\bar{c}\|_p^*} \leq \mathcal{C}.$$

Hence, we get $\sup_m (\sum_{k=1}^m |a_k|^q)^{1/q} \leq \mathcal{C}$, i.e.,

$$\sup_m \left(\sum_{k=1}^m \Delta x_k \left(\frac{\sqrt{v^{\alpha}\varphi}}{v^{\gamma+r}}(x_k) \right)^q \right)^{1/q} \leq \mathcal{C}.$$

The latter inequality is equivalent to $\left\| \frac{\sqrt{v^{\alpha}\varphi}}{v^{\gamma+r}} \right\|_q < \infty$ which is, therefore, a consequence of (3.3).

Finally, we prove that (3.3) implies also $\left\| \frac{v^{\gamma+r}}{\sqrt{v^{\alpha}\varphi}} \right\|_p < \infty$. To this end, since (3.3) holds true for every $f \in C^0(-1, 1)$, letting $g(x) = f(x)v^r(x)$, we have

$$\|Av^{\gamma}\|_{L^p(A_m)} = \|L_m(v^{\alpha}, f)v^{\gamma+r}\|_{L^p(A_m)} \leq \mathcal{C} \|fv^{\gamma+r}\|_{\infty},$$

i.e.,

$$\sup_m \sup_{\|fv^{\gamma+r}\|_{\infty}=1} \|L_m(v^{\alpha}, f)v^{\gamma+r}\|_{L^p(A_m)} \leq \mathcal{C}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Therefore, using [5], we get $\sup_m \|p_m(v^{\alpha})v^{\gamma+r}\|_p \leq \mathcal{C}$, i.e., $\frac{v^{\gamma+r}}{\sqrt{v^{\alpha}\varphi}} \in L^p$, and the proof is complete. \square

PROOF OF THEOREM 2.2. We first show that (2.7) implies (2.6). Taking into account Lemma 3.1, it remains to estimate only the quantity $\|Av^{\gamma}\|_{L^p(A_m)}^p$, where A is given by (2.2). Using the same argument of the previous proof, we have

$$\|Av^{\gamma}\|_{L^p(A_m)}^p \leq \mathcal{C} \|fv^{\gamma}\|_{\infty} \sum_{k=1}^m \Delta x_k v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}(x_k) |G(x_k)|$$

and

$$\begin{aligned} G(x_k) &= \int_{A_m} v^{\gamma+r}(x) \frac{p_m(v^{\alpha}, x)}{x - x_k} g(x) dx \\ &= \int_{A_m} \frac{p_m(v^{\alpha}, x)Q(x) - p_m(v^{\alpha}, x_k)Q(x_k)}{x - x_k} v^{\gamma+r}(x) \frac{g(x)}{Q(x)} dx, \end{aligned}$$

where, as in the proof of Theorem 2.3, $Q \in \mathbb{P}_m$ is equivalent to the weight $v^{\gamma+r}$ in the interval A_m . Then $G(t)$ is a polynomial of degree $2m - 1$ and, using a Marcinkiewicz inequality, we get

$$\|Av^\gamma\|_{L^p(A_m)}^p \leq C \|fv^\gamma\|_\infty \|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}G\|_{L^1(A_m)}$$

and the L^1 norm is bounded under our hypotheses. In fact, expressing G by means of the Hilbert transform, we have

$$\begin{aligned} \|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}G\|_{L^1(A_m)} &\leq \|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}H(p_m(v^\alpha)v^{\gamma+r}g)\|_{L^1(A_m)} \\ &\quad + \left\| v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}p_m(v^\alpha)QH\left(\frac{v^{\gamma+r}g}{Q}\right) \right\|_{L^1(A_m)}. \end{aligned}$$

Concerning the second summand at the right-hand side, using the Hölder inequality, the boundedness of H and $Q \sim v^{\gamma+r}$, we get

$$\left\| v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}p_m(v^\alpha)QH\left(\frac{v^{\gamma+r}g}{Q}\right) \right\|_{L^1(A_m)} \leq \left\| H\left(\frac{v^{\gamma+r}g}{Q}\right) \right\|_1 \leq C \|g\|_q.$$

In order to estimate the first summand, we note that the function under the sign of the Hilbert transform is bounded and the one outside is $L(\log^+ L)$ (see [8]). Therefore, with $\Gamma = \text{sgn } H(p_m(v^\alpha)v^{\gamma+r}g)$ and $\varrho = \gamma + r - \alpha/2 - 1/4$, we can write

$$\begin{aligned} \|v^{-\varrho}H(p_m(v^\alpha)v^{\gamma+r}g)\|_{L^1(A_m)} &\leq \|p_m(v^\alpha)v^{\gamma+r}gH(v^{-\varrho}\Gamma)\|_{L^1(A_m)} \\ &\leq C \|v^\varrho gH(v^{-\varrho}\Gamma)\|_{L^1(A_m)} \\ &\leq C \|g\|_q \|v^\varrho H(v^{-\varrho}\Gamma)\|_{L^p(A_m)} \\ &\leq C \|g\|_q, \end{aligned}$$

since the L^p -norm is bounded (see, for example, [7, p. 676]). Therefore, (2.7) implies (2.6).

In order to prove that (2.6) is a consequence of (2.7) it suffices use the same arguments of the proof of Theorem 2.3 (the part dealing with the necessary condition (2.9) \Rightarrow (2.11)) replacing p by ∞ and q by 1. So, the theorem is completely proved. \square

PROOF OF PROPOSITION 2.1. The proof is based on the following result due to Gopengauz [1]: “For every function $f \in C^s$, $s \geq 0$, there exists a polynomial $q \in \mathbb{P}_{m+2s-1}$ such that, for $i = 0, 1, \dots, s$, one has $q^{(i)}(\pm 1) = f^{(i)}(\pm 1)$ and

$$|(f^{(i)}(x) - q^{(i)}(x))| \leq C \left(\frac{\sqrt{1-x^2}}{m}\right)^{s-i} \omega\left(f^{(i)}, \frac{\sqrt{1-x^2}}{m}\right)_\infty, \quad |x| \leq 1,$$

where $C \neq C(m, f, x)$ and $\omega(\cdot, \cdot)_\infty$ is the ordinary modulus of smoothness” (in the uniform norm).

Then, if $f \in C^{r-1}$, we have

$$v^\gamma(x) |f(x) - L_{m,r}(v^\alpha, f, x)| = |f(x) - q(x)|v^\gamma(x) + \left| v^{\gamma+r}(x)L_{m,r}\left(v^\alpha, \frac{f-q}{v^r}, x\right) \right|,$$

whence, using (2.6),

$$\|[f - L_{m,r}(v^\alpha, f)]v^\gamma\|_p \leq C \|(f - q)v^\gamma\|_\infty.$$

Therefore, for every polynomial P of degree $m + 2r - 1$, we get

$$\| [f - L_{m,r}(v^\alpha, f)]v^\gamma \|_p \leq \mathcal{C} \| [(f - q) - P]v^\gamma \|_\infty$$

and, assuming the infimum on P , the estimate (2.12) follows.

Now we prove (2.13). Using the polynomial q of Gopengauz and (2.9), we get

$$(3.5) \quad \| [f - L_{m,r}(v^\alpha, f)]v^\gamma \|_p \leq \mathcal{C} \| (f - q)v^\gamma \|_p + \mathcal{C} \left(\sum_{k=1}^m \Delta x_k \left[\omega \left(f, \frac{\varphi(x_k)}{m} \right)_\infty v^\gamma(x_k) \right]^p \right)^{1/p}.$$

Now, we have

$$\begin{aligned} |f(x) - q(x)|v^\gamma(x) &\leq \mathcal{C}v^\gamma(x) \omega \left(f, \frac{\varphi(x)}{m} \right)_\infty \\ &\leq \mathcal{C}v^\gamma(x) \int_{x - \frac{\varphi(x)}{m}}^{x + \frac{\varphi(x)}{m}} |f'(t)| dt \leq \frac{\mathcal{C}}{m} \frac{m}{\varphi(x)} \int_{x - \frac{\varphi(x)}{m}}^{x + \frac{\varphi(x)}{m}} |f' \varphi v^\gamma|(t) dt, \end{aligned}$$

since $1 \pm x \sim 1 \pm t$ if $|x - t| \leq \mathcal{C} \frac{\varphi(x)}{m}$ for $x, t \in [x_1, x_m]$.

Then, using the maximal function of $f' \varphi v^\gamma$, the first summand in (3.5) is dominated by

$$\frac{\mathcal{C}}{m} \left(\int_{-1}^1 \left(\frac{m}{\varphi(x)} \int_{x - \frac{\varphi(x)}{m}}^{x + \frac{\varphi(x)}{m}} |f' \varphi v^\gamma|(t) dt \right)^p dx \right)^{1/p} \leq \frac{\mathcal{C}}{m} \|f' \varphi v^\gamma\|_p.$$

Concerning the sum in (3.5), for a sufficiently large s , we have

$$\begin{aligned} \omega \left(f, \frac{\varphi(x_k)}{m} \right)_\infty v^\gamma(x_k) &\leq s \omega \left(f, \frac{\varphi(x_k)}{sm} \right)_\infty v^\gamma(x_k) \\ &\leq \mathcal{C} s \int_{x_k - \frac{\varphi(x_k)}{sm}}^{x_k + \frac{\varphi(x_k)}{sm}} |f'(t)|v^\gamma(t) dt \leq \mathcal{C} \int_{x_{k-1}}^{x_{k+1}} |f' v^\gamma|(t) dt. \end{aligned}$$

Then, using the Hölder inequality in the latter integral, we get

$$\Delta x_k \left[\omega \left(f, \frac{\varphi(x_k)}{m} \right)_\infty \right]^p \leq \mathcal{C} (\Delta x_k)^p \int_{x_{k-1}}^{x_{k+1}} |f' v^\gamma|^p(t) dt \leq \frac{\mathcal{C}}{m^p} \int_{x_{k-1}}^{x_{k+1}} |f' \varphi v^\gamma|^p(t) dt,$$

for $k = 1, \dots, m$ and $x_0 = -1$. Adding up on k , we obtain that also the second term in (3.5) is dominated by $\frac{\mathcal{C}}{m} \|f' \varphi v^\gamma\|_p$. Consequently, in a usual way, we deduce

$$\| [f - L_{m,r}(v^\alpha, f)]v^\gamma \|_p \leq \frac{\mathcal{C}}{m} E_{m+2r-2}(f')_{v^\gamma, \varphi, p}.$$

Iterating the latter relation, (2.13) follows. \square

PROOF OF COROLLARY 2.1. The bound (2.14) is a consequence of (2.13) and the well-known estimate

$$\| (f - L_{m,r}(v^\alpha, f))^{(r)} v^\gamma \varphi^r \|_p \leq \mathcal{C} m^r \| (f - L_{m,r}(v^\alpha, f))v^\gamma \|_p + \| f^{(r)} \varphi^r v^\gamma \|_p$$

which holds for any $f \in W_p^r$, $1 < p < \infty$, and $r \geq 1$. We omit the details. \square

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