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A Class of Orthogonal Polynomials on the Radial Rays in the Complex Plane

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We introduce a class of polynomials orthogonal on some radial rays in the complex plane and investigate their existence and uniqueness. A recurrence relation for these polynomials, a representation, and the connection with standard polynomials orthogonal on $(0,1)$ are derived. It is shown that their zeros are simple and distributed symmetrically on the radial rays, with the possible exception of a multiple zero at the origin. An analogue of the Jacobi polynomials and the corresponding problem with the generalized Laguerre polynomials are also treated.

1. INTRODUCTION

Let $m \in \mathbb{N}$, $a_s > 0$, $s = 0, 1, \ldots, 2m-1$ and $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2m-1}$ be $(2m)$th roots of unity, i.e., $\varepsilon_s = \exp(i\pi s/m)$, $s = 0, 1, \ldots, 2m-1$. We study orthogonal polynomials relative to the inner product

$$(f, g) = \sum_{s=0}^{2m-1} \varepsilon_s^{-1} \int_{l_s} f(z)\overline{g(z)}\,|w(z)|\,dz, \tag{1.1}$$

where l_s are the radial rays in the complex plane which connect the origin $z = 0$ and the points $a_s \varepsilon_s$, $s = 0, 1, \ldots, 2m-1$, and $z \mapsto w(z)$ is a suitable complex (weight) function.

In this paper we consider the cases when $a_s = 1$ and $z \mapsto w(z)$ is a complex function such that

$$|w(x\varepsilon_s)| = w(x), \qquad s = 0, 1, \ldots, 2m-1,$$

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and $x \mapsto w(x)$ is a weight function on $(0,1)$ (nonnegative on $(0,1)$ and $\int_0^1 w(x)\,dx > 0$). Then, (1.1) can be written in the form

$$(f,g) = \int_0^1 \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\,dx. \tag{1.2}$$

In the case $m = 1$, (1.2) becomes

$$(f,g) = \int_{-1}^1 f(x)\overline{g(x)}\,w(x)\,dx,$$

so we have the standard case of polynomials orthogonal on $(-1,1)$ with respect to the even weight function $x \mapsto w(x)$.

Although some results hold regardless of whether the number of rays is even or odd, the case with an odd number of rays will not be considered here. The main reason for this decision is that the fundamental recurrence relation for orthogonal polynomials in that case is quite different from one in the even case.

The paper is organized as follows. In Section 2 we develop preliminary material on existence and uniqueness of the orthogonal polynomials, and in Section 3 we give the recurrence relation for these polynomials. Moment determinants and orthogonal polynomials for $m = 2$ and $w(x) = 1$ are discussed in Sections 4 and 5, respectively. A representation of the orthogonal polynomials and the connection with standard polynomials orthogonal on $(0,1)$ are discussed in Section 6. In Section 7 it is shown that their zeros are simple and distributed symmetrically on the radial rays, with the possible exception of a multiple zero at the origin. An analogue of the Jacobi polynomials is treated in Section 8, and a corresponding problem with the generalized Laguerre polynomials in Section 9, where we take $a_s = +\infty$, $s = 0,1,\ldots,2m-1$. Then, the inner product (1.2) reduces to

$$(f,g) = \int_0^{+\infty} \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\,dx. \tag{1.3}$$

The generalised Hermite polynomials on the radial rays were considered in [7], including a linear second-order differential equation for such polynomials. The type of connection between the general orthogonal polynomials and the orthogonal polynomials on the real ray also appeared in work by E. Hendriksen and H. van Rossum [5], where they considered an electrostatic interpretation of zeros. Also, we mention here the references by A. J. Duran [2, 3] and R. Smith [9, 10], which may have some connection to our results.

2. PRELIMINARIES, EXISTENCE, AND UNIQUENESS

First we see that

$$(f,f) = \int_0^1 \left(\sum_{s=0}^{2m-1} |f(x\varepsilon_s)|^2 \right) w(x)\,dx > 0,$$

except when $f(z) \equiv 0$. The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \left(\sum_{s=0}^{2m-1} \varepsilon_s^{p-q} \right) \int_0^1 x^{p+q} w(x)\,dx, \qquad p,q \geq 0. \quad (2.1)$$

The inner product (1.2) has the following property:

LEMMA 2.1. $(z^m f, g) = (f, z^m g)$.

Proof. Since $\varepsilon_s^m = \varepsilon_s^{-m} = (-1)^s$ we have

$$\begin{aligned}(z^m f, g) &= \int_0^1 \left(\sum_{s=0}^{2m-1} x^m \varepsilon_s^m f(x\varepsilon_s) \overline{g(x\varepsilon_s)} \right) w(x)\,dx \\ &= \int_0^1 \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s) \overline{x^m \varepsilon_s^m g(x\varepsilon_s)} \right) w(x)\,dx \\ &= (f, z^m g). \quad \blacksquare\end{aligned}$$

It is easy to verify the following

LEMMA 2.2. *Let* $p = 2mn + \nu$, $n = [p/(2m)]$, $0 \leq \nu \leq 2m-1$. *Then*

$$\sum_{s=0}^{2m-1} \varepsilon_s^p = \sum_{s=0}^{2m-1} \varepsilon_s^\nu = \begin{cases} 2m & \text{if } \nu = 0, \\ 0 & \text{if } 1 \leq \nu \leq 2m-1. \end{cases}$$

Thus, $\mu_{p,q}$ in (2.1) is different from zero only if $p \equiv q \pmod{2m}$; otherwise $\mu_{p,q} = 0$. Using the moment determinants

$$\Delta_0 = 1, \qquad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \; N \geq 1,$$

we can state the following existence result for the (monic) orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (1.2).

THEOREM 2.3. *If* $\Delta_N > 0$ *for all* $N \geq 1$ *the monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$, *orthogonal with respect to the inner product* (1.2), *exist uniquely.*

Proof. Write

$$\pi_N(z) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} z^\nu, \qquad \alpha_N^{(N)} = 1,$$

and consider the orthogonality conditions

$$(\pi_N, z^\varrho) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} (z^\nu, z^\varrho) = \sum_{\nu=0}^{N} \alpha_\nu^{(N)} \mu_{\nu,\varrho} = K_N \delta_{\varrho N}, \qquad \varrho \leq N,$$

where $K_N = \|\pi_N\|^2 \neq 0$ and $\delta_{\varrho N}$ is the Kronecker delta. These conditions are equivalent to the system of linear equations

$$\begin{bmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N1} \\ \vdots & & & \\ \mu_{0N} & \mu_{1N} & \cdots & \mu_{NN} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0^{(N)} \\ \alpha_1^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ K_N \end{bmatrix}. \tag{2.2}$$

Since $\Delta_{N+1} \neq 0$ the system (2.2) has a unique solution for the coefficients $\alpha_\nu^{(N)}$. For the monic polynomials we have $\alpha_N^{(N)} = 1$ and

$$\alpha_N^{(N)} = \frac{K_N \Delta_N}{\Delta_{N+1}} = \|\pi_N\|^2 \frac{\Delta_N}{\Delta_{N+1}} = 1,$$

i.e., $\|\pi_N\|^2 = \Delta_{N+1}/\Delta_N$. ∎

THEOREM 2.4. *For the polynomials* $\pi_N(z)$, *orthogonal with respect to the inner product* (1.2), *we have*

$$\pi_N(z\varepsilon_s) = \varepsilon_s^N \pi_N(z), \qquad s = 1, \ldots, 2m-1. \tag{2.3}$$

Proof. Let $\pi_N(z)$ be the (monic) polynomial of degree N orthogonal with respect to the inner product (1.2), i.e.,

$$(\pi_N, g) = \int_0^1 \left(\sum_{s=0}^{2m-1} \pi_N(x\varepsilon_s) \overline{g(x\varepsilon_s)} \right) w(x)\, dx = 0,$$

where $g(z)$ is an arbitrary polynomial of degree at most $N-1$ (i.e., $g \in \mathscr{P}_{N-1}$). For each j $(1 \leq j \leq 2m-1)$ we put $Q_{N,j}(z) = \varepsilon_j^{-N} \pi_N(z\varepsilon_j)$ and $H_j(z) = g(z\varepsilon_j)$. Evidently, the polynomials $Q_{N,j}(z)$ are monic.

We have

$$
\begin{aligned}
\left(Q_{N,j}, H_j\right) &= \int_0^1 \left(\sum_{s=0}^{2m-1} \varepsilon_j^{-N} \pi_N(x\varepsilon_j\varepsilon_s)\overline{H_j(x\varepsilon_s)} \right) w(x)\,dx \\
&= \varepsilon_j^{-N} \int_0^1 \left(\sum_{s=j}^{2m-1+j} \pi_N(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\,dx \\
&= \varepsilon_j^{-N}(\pi_N, g) = 0,
\end{aligned}
$$

because $\varepsilon_{2m-1+j} = \varepsilon_{j-1}$.

Since $H_j(z)$ can be every polynomial in $\mathscr{P}_{N-1}$, we conclude that $Q_{N,j}(z)$ is an orthogonal (monic) polynomial with respect to (1.2). Finally, from the uniqueness of $\pi_N(z)$ it follows that

$$
\varepsilon_s^{-N}\pi_N(z\varepsilon_s) = \pi_N(z), \qquad s = 1, \ldots, 2m-1,
$$

i.e., (2.3). ∎

3. RECURRENCE RELATION

We begin this section with the following auxiliary result:

LEMMA 3.1. *Let the inner product* $(\cdot,\cdot)$ *be given by* (1.2) *and let the corresponding system of monic orthogonal polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *exist. Then, for* $0 \le \nu < N \le 2m-1$, *we have* $(z^N, \pi_N) = 0$.

Proof. Let $0 \le \nu < N \le 2m-1$ and $\pi_\nu(z) = \sum_{j=0}^{\nu} \gamma_j^{(\nu)} z^j$, $\nu \ge 0$. Then

$$
\begin{aligned}
\left(z^N, \pi_\nu\right) &= \int_0^1 \left(\sum_{s=0}^{2m-1} x^N \varepsilon_s^N \overline{\pi_\nu(x\varepsilon_s)} \right) w(x)\,dx \\
&= \sum_{s=0}^{2m-1} \int_0^1 x^N \varepsilon_s^N \left(\sum_{j=0}^{\nu} \overline{\gamma}_j^{(\nu)} x^j \varepsilon_s^{-j} \right) w(x)\,dx \\
&= \sum_{j=0}^{\nu} \overline{\gamma}_j^{(\nu)} \left(\int_0^1 x^{N+j} w(x)\,dx \right) \left(\sum_{s=0}^{2m-1} \varepsilon_s^{N-j} \right).
\end{aligned}
$$

Since $0 \le j \le \nu \le N-1 \le 2m-2$, i.e., $1 \le N-\nu \le N-j \le 2m-1$, according to Lemma 2.2, we have that $\sum_{s=0}^{2m-1} \varepsilon_s^{N-j} = 0$, and therefore $(z^N, \pi_\nu) = 0$. ∎

Using the well-known Gram–Schmidt orthogonalizing process and Lemma 3.1, we get:

LEMMA 3.2. *The first* $2m$ *monic polynomials orthogonal with respect to the inner product* (1.2) *are given by* $\pi_N(z) = z^N$, $N = 0, 1, \dots, 2m-1$.

It is well known that an orthogonal sequence of polynomials satisfies a three-term recurrence relation if the inner product has the property $(zf, g) = (f, zg)$. In our case the corresponding property is given by $(z^m f, g) = (f, z^m g)$ (cf. Lemma 2.1) and the following result holds.

THEOREM 3.3. *Let the inner product* $(\cdot, \cdot)$ *be given by* (1.2) *and let the corresponding system of monic orthogonal polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *exist. They satisfy the recurrence relation*

$$\begin{gathered} \pi_{N+m}(z) = z^m \pi_N(z) - b_N \pi_{N-m}(z), \qquad N \geq m, \\ \pi_N(z) = z^N, \qquad N = 0, 1, \dots, 2m-1, \end{gathered} \tag{3.1}$$

where

$$b_N = \frac{(\pi_N, z^m \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = \frac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}. \tag{3.2}$$

Proof. Since $\pi_{N+m}(z) - z^m \pi_N(z)$ is a polynomial of degree at most $N + m - 1$, we can express it in terms of the orthogonal basis $\{\pi_\nu(z)\}_{\nu=0}^{N+m-1}$. Thus,

$$\pi_{N+m}(z) = z^m \pi_N(z) + \sum_{\nu=0}^{N+m-1} \beta_N^{(\nu)} \pi_\nu(z), \tag{3.3}$$

from which, for an arbitrary k, we have

$$(\pi_{N+m}, \pi_k) = (z^m \pi_N, \pi_k) + \sum_{\nu=0}^{N+m-1} \beta_N^{(\nu)} (\pi_\nu, \pi_k).$$

Because of orthogonality, we conclude that

$$\beta_N^{(\nu)} = -\frac{(z^m \pi_N, \pi_\nu)}{(\pi_\nu, \pi_\nu)}, \qquad 0 \leq \nu \leq N + m - 1.$$

Since $(z^m \pi_N, \pi_\nu) = (\pi_N, z^m \pi_\nu) = 0$ for $m + \nu < N$, i.e., $\nu < N - m$, we get

$$\beta_N^{(\nu)} = 0, \qquad \nu = 0, 1, \dots, N - m - 1.$$

Consider now the inner product $(z^m \pi_N, \pi_\nu)$ for $N - m \le \nu \le N + m - 1$. Using Theorem 2.4, we find that

$$\begin{aligned}
(z^m \pi_N, \pi_\nu) &= \int_0^1 \left(\sum_{s=0}^{2m-1} x^m \varepsilon_s^m \pi_N(x\varepsilon_s) \overline{\pi_\nu(x\varepsilon_s)} \right) w(x)\, dx \\
&= \int_0^1 \left(\sum_{s=0}^{2m-1} x^m \varepsilon_s^m \varepsilon_s^N \pi_N(x) \overline{\varepsilon_s^\nu \pi_\nu(x)} \right) w(x)\, dx \\
&= \left(\sum_{s=0}^{2m-1} \varepsilon_s^{m+N-\nu} \right) \int_0^1 x^m \pi_N(x) \overline{\pi_\nu(x)} w(x)\, dx.
\end{aligned}$$

In view of Lemma 2.2, the first factor on the right in the last equality is different from zero only for $m + N - \nu = 2m$, i.e, if $\nu = N - m$. Thus,

$$\beta_N^{(N-m)} = -\frac{(z^m \pi_N, \pi_{N-m})}{(\pi_{N-m}, \pi_{N-m})} = -\frac{(\pi_N, z^m \pi_{N-m})}{\|\pi_{N-m}\|^2}.$$

Denoting $\beta_N^{(N-m)}$ simply as $-b_N$, we see that (3.3) reduces to (3.1). ∎

4. MOMENT DETERMINANTS FOR $m = 2$ AND $w(x) = 1$

In this section we consider the inner product

$$\begin{aligned}
(f, g) = \int_0^1 \big[& f(x)\overline{g(x)} \\
& + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)} \big]\, dx. \qquad (4.1)
\end{aligned}$$

The moments are given by

$$\mu_{p,q} = (z^p, z^q) = \begin{cases} \dfrac{4}{p+q+1}, & p \equiv q \pmod 4, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, if $p = 4i + \nu$ and $q = 4j + \nu$, $\nu \in \{0, 1, 2, 3\}$, we have

$$\mu_{4i+\nu, 4j+\nu} = \frac{4}{4(i+j) + 2\nu + 1}, \qquad i, j \ge 0. \qquad (4.2)$$

Our purpose is now to evaluate the moment determinants

$$\Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \end{vmatrix}, \qquad N \geq 1. \tag{4.3}$$

For every $k \in \mathbb{N}$ we define the determinants

$$C_k = \begin{vmatrix} \mu_{00} & 0 & \mu_{40} & 0 & \cdots & \\ 0 & \mu_{22} & 0 & \mu_{62} & & \\ \mu_{04} & 0 & \mu_{44} & 0 & & \\ 0 & \mu_{26} & 0 & \mu_{66} & & \\ \vdots & & & & \ddots & \\ & & & & & \mu_{2k-2,2k-2} \end{vmatrix}, \tag{4.4}$$

$$D_k = \begin{vmatrix} \mu_{11} & 0 & \mu_{51} & 0 & \cdots & \\ 0 & \mu_{33} & 0 & \mu_{73} & & \\ \mu_{15} & 0 & \mu_{55} & 0 & & \\ 0 & \mu_{37} & 0 & \mu_{77} & & \\ \vdots & & & & \ddots & \\ & & & & & \mu_{2k-1,2k-1} \end{vmatrix}, \tag{4.5}$$

which can be expressed in terms of the determinants

$$E_0^{(\nu)} = 1,$$

$$E_n^{(\nu)} = \begin{vmatrix} \mu_{\nu,\nu} & \mu_{4+\nu,\nu} & \cdots & \mu_{4(n-1)+\nu,\nu} \\ \mu_{\nu,4+\nu} & \mu_{4+\nu,4+\nu} & \cdots & \mu_{4(n-1)+\nu,4+\nu} \\ \vdots & & & \\ \mu_{\nu,4(n-1)+\nu} & \mu_{4+\nu,4(n-1)+\nu} & \cdots & \mu_{4(n-1)+\nu,4(n-1)+\nu} \end{vmatrix},$$

where $\nu = 0, 1, 2, 3$. We first interpret these determinants in terms of Hilbert-type determinants. Namely, because of (4.2), we have

$$E_n^{(\nu)} = 4^n \det\left[\frac{1}{4(i+j) + 2\nu - 7} \right]_{i,j=1}^{n}. \tag{4.6}$$

In order to evaluate the determinants in (4.6), we use Cauchy's formula (see Muir [8, p. 345])

$$\det\left[\frac{1}{a_i+b_j}\right]_{i,j=1}^{n} = \frac{\prod_{i>j=1}^{n}(a_i-a_j)(b_i-b_j)}{\prod_{i,j=1}^{n}(a_i+b_j)}$$

with $a_i = 4i$ and $b_j = 4j + 2\nu - 7$. Thus, we obtain:

LEMMA 4.1. *We have*

$$E_n^{(\nu)} = 4^{n^2}\frac{(0!1!\cdots(n-1)!)^2}{\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}(4i+4j+2\nu+1)}, \qquad n \geq 1.$$

In our further investigations, we need a quotient of the determinants $E_n^{(\nu)}$.

LEMMA 4.2. *For* $\nu = 0, 1, 2, 3$ *we have*

$$\frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} = \frac{4}{8n+2\nu+1}\left(\prod_{k=n}^{2n-1}\frac{4(k-n+1)}{4k+2\nu+1}\right)^2, \qquad n \geq 1,$$

and $E_1^{(\nu)}/E_0^{(\nu)} = 4/(2\nu+1)$.

This follows immediately from Lemma 4.1.

LEMMA 4.3. *For the determinants* (4.4) *we have*

$$C_k = E_{k/2}^{(0)}E_{k/2}^{(2)},\ k(\mathit{even}) \geq 2; \qquad C_k = E_{(k+1)/2}^{(0)}E_{(k-1)/2}^{(2)},\ k(\mathit{odd}) \geq 1. \tag{4.7}$$

Proof. Similarly as in [4], we use Laplace expansion for determinants. Let first k be even. Expanding by columns numbered $1, 3, \ldots, k-1$, one finds that only one nonzero contribution results, namely from the minor and cominor pair

$$\begin{pmatrix} 1 & 3 & 5 & \cdots & k-1 \\ 1 & 3 & 5 & \cdots & k-1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 4 & 6 & \cdots & k \\ 2 & 4 & 6 & \cdots & k \end{pmatrix}. \tag{4.8}$$

Since the matrix C_k is symmetric, and the sign associated with the pair (4.8) is $(-1)^{k^2/2}$, one immediately obtains the first relation in (4.7).

Similarly, Laplace expansion by columns $1, 3, \ldots, k$ gives the result for odd k. ∎

Also, we can prove:

LEMMA 4.4. *For the determinants* (4.5) *we have*

$$D_k = E_{k/2}^{(1)} E_{k/2}^{(3)}, \, k(even) \geq 2; \qquad D_k = E_{(k+1)/2}^{(1)} E_{(k-1)/2}^{(3)}, \, k(odd) \geq 1.$$

Using the same techniques we find:

LEMMA 4.5. *For the moment determinants* (4.3) *we have*

$$\Delta_{2k} = C_k D_k \qquad and \qquad \Delta_{2k+1} = C_{k+1} D_k,$$

where C_k and D_k are given by (4.4) *and* (4.5), *respectively*.

Combining Lemmas 4.1, 4.3, 4.4, and 4.5, we obtain:

LEMMA 4.6. *We have*

$$\begin{aligned} \Delta_{4n} &= E_n^{(0)} E_n^{(1)} E_n^{(2)} E_n^{(3)}, \\ \Delta_{4n+1} &= E_{n+1}^{(0)} E_n^{(1)} E_n^{(2)} E_n^{(3)}, \\ \Delta_{4n+2} &= E_{n+1}^{(0)} E_{n+1}^{(1)} E_n^{(2)} E_n^{(3)}, \\ \Delta_{4n+3} &= E_{n+1}^{(0)} E_{n+1}^{(1)} E_{n+1}^{(2)} E_n^{(3)}. \end{aligned}$$

5. ORTHOGONAL POLYNOMIALS FOR $m = 2$ AND $w(x) = 1$

We note, first of all, from Lemmas 4.1 and 4.6, that $\Delta_N > 0$ for all $N \geq 1$, and therefore, the orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ with respect to the inner product (4.1) exist uniquely, and

$$(\pi_N, \pi_N) = \|\pi_N\|^2 = \frac{\Delta_{N+1}}{\Delta_N} > 0. \tag{5.1}$$

THEOREM 5.1. *The* (*monic*) *polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$, orthogonal with respect to the inner product* (4.1), *satisfy the recurrence relation*

$$\begin{aligned} \pi_{N+2}(z) &= z^2 \pi_N(z) - b_N \pi_{N-2}(z), \qquad N \geq 2, \\ \pi_N(z) &= z^N, \qquad N = 0, 1, 2, 3, \end{aligned} \tag{5.2}$$

where

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 0,1, \\ \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 2,3. \end{cases} \tag{5.3}$$

Proof. Since (see (3.2))

$$b_N = \frac{\|\pi_N\|^2}{\|\pi_{N-2}\|^2} = \frac{\Delta_{N+1}}{\Delta_N} \cdot \frac{\Delta_{N-2}}{\Delta_{N-1}}, \qquad N \geq 2,$$

using Lemmas 4.6 and 4.2, we find for $\nu = 0, 1$,

$$b_{4n+\nu} = \frac{\Delta_{4n+\nu+1}/\Delta_{4n+\nu}}{\Delta_{4(n-1)+\nu+3}/\Delta_{4(n-1)+\nu+2}} = \frac{E_{n+1}^{(\nu)}/E_n^{(\nu)}}{E_n^{(\nu+2)}/E_{n-1}^{(\nu+2)}},$$

i.e.,

$$b_{4n+\nu} = \frac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)}.$$

Similarly, for $\nu = 2, 3$, we have

$$b_{4n+\nu} = \frac{\Delta_{4n+\nu+1}/\Delta_{4n+\nu}}{\Delta_{4n+\nu-1}/\Delta_{4n+\nu-2}} = \frac{E_{n+1}^{(\nu)}/E_n^{(\nu)}}{E_{n+1}^{(\nu+2)}/E_n^{(\nu+2)}},$$

i.e.,

$$b_{4n+\nu} = \frac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)}. \qquad \blacksquare$$

Remark 5.1. From (5.3) we conclude that

$$b_N \to \frac{1}{4} \qquad \text{as } N \to +\infty,$$

just like in Szegő's theory for orthogonal polynomials on the interval $(-1, 1)$.

Remark 5.2. Taking

$$\pi_{-2}(z) = \pi_{-1}(z) = 0, \qquad \pi_0(z) = 1, \qquad \pi_1(z) = z,$$

the recurrence relation (5.2) holds for every $N \geq 0$.

Remark 5.3. Since

$$\|\pi_N\|^2 = \begin{cases} b_N b_{N-2} \cdots b_2 \|\pi_0\|^2, & N \text{ even}, \\ b_N b_{N-2} \cdots b_3 \|\pi_1\|^2, & N \text{ odd}, \end{cases}$$

and $\|\pi_0\|^2 = \Delta_1/\Delta_0 = \mu_{00} = 4$ ($\Delta_0 \equiv 1$),

$$\|\pi_1\|^2 = \Delta_2/\Delta_1 = \mu_{00}\,\mu_{11}/\mu_{00} = \mu_{11} = 4/3,$$

we can define $b_0 = 4$, $b_1 = 4/3$, so that for every $N \geq 0$ we have

$$\|\pi_N\|^2 = \begin{cases} b_N b_{N-2} \cdots b_0, & N \text{ even}, \\ b_N b_{N-2} \cdots b_1, & N \text{ odd}. \end{cases}$$

Remark 5.4. Let $N = 4n + \nu$, $n = [N/4]$, $0 \leq \nu \leq 3$. Since (cf. (5.1))

$$\|\pi_N\|^2 = \frac{\Delta_{N+1}}{\Delta_N} = \frac{\Delta_{4n+\nu+1}}{\Delta_{4n+\nu}} = \frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}},$$

Lemma 4.2 gives the norms of the polynomials $\{\pi_N(z)\}$. Namely,

$$\|\pi_N\|^2 = \frac{4}{2N+1}, \qquad 0 \leq N \leq 3,$$

$$\|\pi_N\|^2 = \|\pi_{4n+\nu}\|^2 = \frac{4}{8n+2\nu+1} \left(\prod_{k=n}^{2n-1} \frac{4(k-n+1)}{4k+2\nu+1} \right)^2, \qquad N \geq 4.$$

6. A REPRESENTATION OF THE POLYNOMIALS $\pi_N(z)$

In this section we again consider the general case of the inner product (1.2) for which the corresponding system of the monic orthogonal polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ exists and satisfies the recurrence relation (3.1). Based on this recurrence relation or on formula (2.3) from Theorem 2.4, we can conclude and easily prove that $\pi_N(z)$ are incomplete polynomials with the following representation:

LEMMA 6.1. *The polynomials $\pi_N(z)$ can be expressed in the form*

$$\pi_N(z) = \sum_{i=0}^{[N/2m]} \gamma_i^{(N)} z^{N-2mi}, \tag{6.1}$$

where $\gamma_i^{(N)}$ are real coefficients and $\gamma_0^{(N)} = 1$.

Indeed, from Theorem 2.4 it follows that every zero $z_0 \neq 0$ of $\pi_N(z)$ leads to a set of $2m$ zeros of the form $z_0 \varepsilon_s$ $(0 \leq s \leq 2m-1)$ and therefore $\pi_N(z)$ has a factor $z^{2m} - z_0^{2m}$; apart from these factors there is a factor z^ν with $\nu = N - 2m[N/2m]$.

Taking $N = 2mn + \nu$, $n = [N/2m]$, $\nu \in \{0, 1, \ldots, 2m-1\}$, we see that (6.1) becomes

$$\pi_{2mn+\nu}(z) = \sum_{i=0}^{n} \gamma_i^{(N)} z^{2m(n-i)+\nu}, \tag{6.2}$$

from which there follows immediately:

LEMMA 6.2. *The polynomials from Theorem* 3.3 *can be expressed in the form*

$$\pi_{2mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^{2m}), \qquad \nu = 0, 1, \ldots, 2m-1;\ n = 0, 1, \ldots, \tag{6.3}$$

where $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m-1$, *are monic polynomials of exact degree* n.

THEOREM 6.3. *The monic polynomials* $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m-1$, *defined in* (6.3), *satisfy the two relations*

$$q_n^{(\nu+m)}(t) = q_n^{(\nu)}(t) - b_N q_{n-1}^{(\nu+m)}(t), \qquad 0 \leq \nu \leq m-1, \tag{6.4}$$

and

$$q_{n+1}^{(\nu-m)}(t) = t q_n^{(\nu)}(t) - b_N q_n^{(\nu-m)}(t), \qquad m \leq \nu \leq 2m-1, \tag{6.5}$$

where $N = 2mn + \nu$.

Proof. Let $N = 2mn + \nu$, $n = [N/2m]$. Then, for $0 \leq \nu \leq m-1$, we have that $N + m = 2mn + \nu + m$ and $N - m = 2m(n-1) + \nu + m$. Using the recurrence relation (3.1) and the representation (6.3), we obtain (6.4). Similarly, for $m \leq \nu \leq 2m-1$, we have that $N + m = 2m(n+1) + \nu - m$ and $N - m = 2mn + \nu - m$, and then (3.1) reduces to (6.5). ∎

THEOREM 6.4. *The monic polynomials* $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m-1$, *defined in* (6.3), *satisfy the three-term recurrence relation*

$$\begin{aligned} q_{n+1}^{(\nu)}(t) &= \left(t - a_n^{(\nu)}\right) q_n^{(\nu)}(t) - b_n^\nu q_{n-1}^{(\nu)}(t), \qquad n = 0, 1, \ldots, \\ q_0^{(\nu)}(t) &= 1, \qquad q_{-1}^{(\nu)}(t) = 0, \end{aligned} \tag{6.6}$$

where the recursion coefficients $a_n^{(\nu)}$ *and* $b_n^{(\nu)}$ *are given in terms of the b-coefficients as*

$$a_n^{(\nu)} = b_N + b_{N+m}, \qquad b_n^{(\nu)} = b_{N-m} b_N, \qquad N = 2mn + \nu.$$

Proof. Suppose that $N = 2mn + \nu$, $n = [N/2m]$, and $0 \le \nu \le m - 1$. Then, combining (6.5) in the form

$$q_{n+1}^{(\nu)}(t) = tq_n^{(\nu+m)}(t) - b_{N+m}q_n^{(\nu)}(t) \tag{6.7}$$

and (6.4) we obtain

$$q_{n+1}^{(\nu)}(t) = (t - b_{N+m})q_n^{(\nu)}(t) - b_N tq_{n-1}^{(\nu+m)}(t).$$

Replacing now n by $n - 1$ in (6.7) and using the last equality we get (6.6). In a similar way we prove the case when $m \le \nu \le 2m - 1$. ∎

The three-term recurrence relation (6.6) shows that the monic polynomial systems $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$, $\nu = 0, 1, \ldots, 2m - 1$, are orthogonal. In the following theorem we investigate this orthogonality.

THEOREM 6.5. *Let $x \mapsto w(x)$ be a weight function in the inner product* (1.2) *which guarantees the existence of the polynomials $\pi_N(z)$, i.e., $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, 2m - 1$, determined by* (6.3). *For any $\nu \in \{0, 1, \ldots, 2m - 1\}$, the sequence of polynomials $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$ is orthogonal on* $(0, 1)$ *with respect to the weight function*

$$t \mapsto w_\nu(t) = t^{(2\nu+1-2m)/2m} w(t^{1/2m}). \tag{6.8}$$

Proof. Let $N = 2mn + \nu$, $n = [N/2m]$, and $K = 2mk + \nu$, $k = [K/2m]$. Consider the inner product

$$(\pi_N, \pi_K) = \int_0^1 \left(\sum_{s=0}^{2m-1} \pi_N(x\varepsilon_s)\overline{\pi_K(x\varepsilon_s)} \right) w(x)\, dx,$$

which can be reduced to

$$\begin{aligned}(\pi_N, \pi_K) &= \int_0^1 \left(\sum_{s=0}^{2m-1} \varepsilon_s^{\nu}\pi_N(x)\,\overline{\varepsilon}_s^{\nu}\overline{\pi_K(x)} \right) w(x)\, dx \\ &= 2m\int_0^1 \pi_N(x)\overline{\pi_K(x)}w(x)\, dx \\ &= 2m\int_0^1 x^{2\nu}q_n^{(\nu)}(x^{2m})q_k^{(\nu)}(x^{2m})w(x)\, dx,\end{aligned}$$

using the property $\pi_N(x\varepsilon_s) = \varepsilon_s^N\pi_N(x)$, $s = 0, 1, \ldots, 2m - 1$, from Theorem 2.4, and the representation (6.3). Changing variable $x^{2m} = t$ in the last integral, we conclude that

$$(\pi_N, \pi_K) = (\pi_{2mn+\nu}, \pi_{2mk+\nu}) = \int_0^1 q_n^{(\nu)}(t)q_k^{(\nu)}(t)w_\nu(t)\, dt = 0, \ n \ne k,$$

where $w_\nu(t)$ is given by (6.8). ∎

Remark 6.1. The question of the existence of the polynomials $\pi_N(z)$ is reduced to the existence of polynomials $q_n^{(\nu)}(t)$, orthogonal on $(0,1)$ with respect to the weight function $w_\nu(t)$, for every $\nu = 0, 1, \ldots, 2m-1$.

7. ZEROS OF $\pi_N(z)$

THEOREM 7.1. *Let* $N = 2mn + \nu$, $n = [N/2m]$, $\nu \in \{0, 1, \ldots, 2m-1\}$. *All zeros of the polynomial* $\pi_N(z)$ *are simple and located symmetrically on the radial rays* l_s, $s = 0, 1, \ldots, 2m-1$, *with the possible exception of a multiple zero of order* ν *at the origin* $z = 0$.

Proof. In view of (6.3), the polynomial $\pi_N(z)$ can be expressed in the form $\pi_N(z) = z^\nu q_n^{(\nu)}(z^{2m})$, $\nu \in \{0, 1, \ldots, 2m-1\}$, where $q_n^{(\nu)}(t)$ is orthogonal on $(0,1)$ with respect to the weight (6.8). It is well known that the zeros of $q_n^{(\nu)}(t)$ are real and distinct and are located in $(0,1)$. Let $\tau_k^{(n,\nu)}$, $k = 1, \ldots, n$, denote these zeros in increasing order,

$$\tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)}.$$

Each zero $\tau_k^{(n,\nu)}$ generates $2m$ zeros $z_{k,s}^{(n,\nu)}$, $s = 0, 1, \ldots, 2m-1$, on the radial rays l_s,

$$z_{k,s}^{(n,\nu)} = \sqrt[2m]{\tau_k^{(n,\nu)}}\, e^{is\pi/m}, \qquad s = 0, 1, \ldots, 2m-1,$$

where $i = \sqrt{-1}$. If $\nu > 0$, there exists a zero of order ν at the origin $z = 0$. ∎

8. AN ANALOGUE OF THE JACOBI POLYNOMIALS

Let $\hat{P}_n^{(\alpha,\beta)}(x)$ be the monic Jacobi polynomials orthogonal with respect to the weight $x \mapsto (1-x)^\alpha(1+x)^\beta$ on $(-1,1)$ and let $\overline{P}_n^{(\alpha,\beta)}(t)$ be their transformed form (again monic) on $(0,1)$. Then we have

$$\hat{P}_{n+1}^{(\alpha,\beta)}(x) = (x - \hat{\alpha}_n)\hat{P}_n^{(\alpha,\beta)}(x) - \hat{\beta}_n \hat{P}_{n-1}^{(\alpha,\beta)}(x),$$

where (cf. [6, p. 45])

$$\hat{\alpha}_k = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

$$\hat{\beta}_k = \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2\big((2k+\alpha+\beta)^2 - 1\big)},$$

and

$$\overline{P}_{n+1}^{(\alpha,\beta)}(t) = (t-\overline{\alpha}_n)\overline{P}_n^{(\alpha,\beta)}(t) - \overline{\beta}_n\overline{P}_{n-1}^{(\alpha,\beta)}(t),$$

where

$$\hat{P}_n^{(\alpha,\beta)}(x) = \hat{P}_n^{(\alpha,\beta)}(2t-1) = 2^n\overline{P}_n^{(\alpha,\beta)}(t),$$
$$\overline{\alpha}_n = \frac{1}{2}(1+\hat{\alpha}_n), \qquad \overline{\beta}_n = \frac{1}{4}\hat{\beta}_n. \tag{8.1}$$

In the sequel we need the following lemma which follows immediately from Bateman and Erdélyi [1, Sect. 10.8; formulas (33) and (36)].

LEMMA 8.1. *For the monic Jacobi polynomials* $\hat{P}_n^{(\alpha,\beta)}(x)$ *the following relations hold*,

$$\hat{P}_{n+1}^{(\alpha,\beta)}(x) = (1+x)\hat{P}_n^{(\alpha,\beta+1)}(x) - c_n\hat{P}_n^{(\alpha,\beta)}(x), \tag{8.2}$$
$$\hat{P}_n^{(\alpha,\beta)}(x) = \hat{P}_n^{(\alpha,\beta-1)}(x) - d_n\hat{P}_{n-1}^{(\alpha,\beta)}(x), \tag{8.3}$$

where

$$c_n = \frac{2(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)},$$
$$d_n = \frac{2n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}.$$

Consider now the polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product (1.2), where the weight function is given by

$$w(x) = (1-x^{2m})^{\alpha}x^{2m\gamma}, \qquad \alpha > -1,\ \gamma > -\frac{1}{2m}. \tag{8.4}$$

THEOREM 8.2. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (1.2), *where the weight function is given by* (8.4), *can be expressed in the form*

$$\pi_N(z) = 2^{-n}z^{\nu}\hat{P}_n^{(\alpha,\beta_\nu)}(2z^{2m}-1), \qquad N = 2mn+\nu,\ n = [N/2m], \tag{8.5}$$

where $\nu \in \{0,1,\dots,2m-1\}$, $\beta_\nu = \gamma + (2\nu+1-2m)/(2m)$, *and* $\hat{P}_n^{(\alpha,\beta)}(x)$ *denotes the monic Jacobi polynomial orthogonal with respect to the weight* $x \mapsto (1-x)^{\alpha}(1+x)^{\beta}$ *on* $(-1,1)$. *The polynomials* $\pi_N(z)$ *satisfy the* *recurrence relation* (3.1), *where*

$$b_{2mn+\nu} = \begin{cases} \dfrac{n(n+\alpha)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } 0 \le \nu \le m-1, \\ \dfrac{(n+\beta_\nu)(n+\alpha+\beta_\nu)}{(2n+\alpha+\beta_\nu)(2n+\alpha+\beta_\nu+1)} & \text{if } m \le \nu \le 2m-1. \end{cases} \tag{8.6}$$

Proof. The weights $t \mapsto w_\nu(t)$, given by (6.8), reduce in this case to

$$w_\nu(t) = (1-t)^\alpha t^{\gamma+(2\nu+1-2m)/2m}, \qquad \nu = 0, 1, \ldots, 2m-1.$$

Taking $t = z^{2m}$, i.e,. $x = 2z^{2m} - 1$, in (8.1), yields immediately the representation (8.5).

In order to determine b_N in the recurrence relation (3.1) we combine (8.5) and (8.2) or (8.3), taking $x = 2z^{2m} - 1$ and $\beta = \beta_\nu = \gamma + (2\nu + 1 - 2m)/2m$.

Let $N = 2mn + \nu$, $n = [N/2m]$. Since

$$\beta_\nu + 1 = \gamma + \frac{2\nu + 1 - 2m}{2m} + 1 = \gamma + \frac{2(\nu+m) + 1 - 2m}{2m} = \beta_{\nu+m}, \tag{8.7}$$

for $\nu = 0, 1, \ldots, m-1$, (8.2) reduces to

$$\begin{aligned} 2^{n+1} z^{-\nu} \pi_{2m(n+1)+\nu}(z) &= 2z^{2m} 2^n z^{-(\nu+m)} \pi_{2mn+\nu+m}(z) \\ &\quad - c_n 2^n z^{-\nu} \pi_{2mn+\nu}(z), \end{aligned}$$

i.e., $\pi_{N+2m}(z) = z^m \pi_{N+m}(z) - (c_n/2)\pi_N(z)$. Thus,

$$b_{N+m} = \frac{c_n}{2} = \frac{(n+\beta_\nu+1)(n+\alpha+\beta_\nu+1)}{(2n+\alpha+\beta_\nu+2)(2n+\alpha+\beta_\nu+1)}.$$

According to (8.7), the last equality gives $b_{2mn+\nu}$, for $\nu = m, \ldots, 2m-1$.

In a similar way, using (8.5) and (8.3), for $\nu = m, \ldots, 2m-1$, we obtain

$$b_{N-m} = \frac{d_n}{2} = \frac{n(n+\alpha)}{(2n+\alpha+\beta_{\nu-m})(2n+\alpha+\beta_{\nu-m}+1)},$$

from which we determine $b_{2mn+\nu}$, for $\nu = 0, 1, \ldots, m-1$. ∎

Remark 8.1. For $\alpha = \gamma = 0$ and $m = 2$, (8.6) reduces to (5.3).

9. AN ANALOGUE OF THE GENERALIZED LAGUERRE POLYNOMIALS

Using the same method as in the previous section, we investigate the corresponding problem on $(0, +\infty)$ with the inner product (1.3), i.e.,

$$(f, g) = \int_0^{+\infty} \left(\sum_{s=0}^{2m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\, dx, \tag{9.1}$$

where

$$w(x) = x^{2m\gamma}\exp(-x^{2m}), \qquad \gamma > -\frac{1}{2m}. \tag{9.2}$$

Let $\hat{L}_n^{(s)}(t)$ be the monic generalized Laguerre polynomials orthogonal with respect to the weight $t \mapsto t^s e^{-t}$ on $(0, +\infty)$. They satisfy the three-term recurrence relation (cf. [6, p. 46])

$$\hat{L}_{n+1}^{(s)}(t) = (t - (2n + s + 1))\hat{L}_n^{(s)}(t) - n(n + s)\hat{L}_{n-1}^{(s)}(t),$$

as well as the following relations (see [11])

$$t\hat{L}_{n-1}^{(s+1)}(t) = \hat{L}_n^{(s)}(t) + (n + s)\hat{L}_{n-1}^{(s)}(t),$$
$$\hat{L}_n^{(s)}(t) = \hat{L}_n^{(s-1)}(t) - n\hat{L}_{n-1}^{(s)}(t).$$

THEOREM 9.1. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to the inner product* (9.1), *where the weight function is given by* (9.2), *can be expressed in the form*

$$\pi_N(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^{2m}), \qquad N = 2mn + \nu,\, n = [N/2m],$$

where $\nu \in \{0, 1, \ldots, 2m - 1\}$, $\alpha_\nu = \gamma + (2\nu + 1 - 2m)/(2m)$, *and* $\hat{L}_n^{(s)}(t)$ *denotes the monic generalized Laguerre polynomial orthogonal with respect to the weight* $t \mapsto t^s e^{-t}$ *on* $(0, +\infty)$. *The polynomials* $\pi_N(z)$ *satisfy the recurrence relation* (3.1), *where*

$$b_{2mn+\nu} = \begin{cases} n + 1 + \alpha_\nu & \text{if } 0 \le \nu \le m - 1, \\ n & \text{if } m \le \nu \le 2m - 1. \end{cases}$$

The proof of this theorem is quite similar to the proof of Theorem 8.2.

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