ELSEVIER Applied Mathematics and Computation 128 (2002) 365–378

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Weighted integration of periodic functions on the real line ☆

Giuseppe Mastroianni [a], Gradimir V. Milovanović [b,*]

[a] *Dipartimento di Matematica, Universitá degli Studi della Basilicata, 85100 Potenza, Italy*

[b] *Faculty of Electronic Engineering, Department of Mathematics, University of Niš, P.O. Box 73, 18000 Niš, Serbia, Yugoslavia*

Abstract

Integration of periodic functions on the real line with an even rational weight function is considered. A transformation method of such integrals to the integrals on $(-1,1)$ with respect to the Szegő–Bernstein weights and a construction of the corresponding Gaussian quadrature formulas are given. The recursion coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials were obtained in an analytic form. Numerical examples are also included. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Gauss-type quadratures; Error term; Convergence; Orthogonal polynomials; Nonnegative measure; Weights; Chebyshev weight; Szegő–Bernstein weights; Nodes; Modified moments; Chebyshev polynomials

1. Introduction

We consider integrals of (2π)-periodic functions over the real line $\mathbb{R}$,

$$I(f) = \int_{\mathbb{R}} f(t)w(t)\,\mathrm{d}t, \tag{1.1}$$

☆ Research of the authors supported by GNIM Project 2000: Teoria costruttiva delle funzioni e problemi connessi.

* Corresponding author.

E-mail address: grade@gauss.elfak.ni.ac.yu (G.V. Milovanović).

PII: S0096-3003(01)00080-7

with a given even rational weight function of the form

$$w(t) = \frac{P(t^2)}{Q(t^2)}, \tag{1.2}$$

where

$$Q(t) = \prod_{k=1}^{n} \left(t + b_k^2\right), \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n,$$

and $P(t)$ is a polynomial of degree at most $m < n$, which is nonnegative on the half line $[0, +\infty)$.

Problem (1.1) can be simplified by first obtaining the partial fraction decomposition of (1.2) in the form

$$w(t) = \sum_{j=1}^{m} \sum_{v=1}^{r_j} \frac{C_{jv}}{(t^2 + b_j^2)^v},$$

where the sum is over all pairs of conjugate complex poles $\pm \mathrm{i} b_j$ of $Q(t^2)$, with corresponding multiplicities r_j $(j = 1, \ldots, m)$. Here, $\sum_{j=1}^{m} r_j = n$.

Thus, without loss of generality we can consider only weights of the form

$$w_v(t) = w_v(t; b) = \frac{1}{(t^2 + b^2)^v} \quad (v \geqslant 1), \tag{1.3}$$

i.e., integrals

$$I_v(f) = I_v(f; b) = \int_{\mathbb{R}} f(t) \frac{\mathrm{d}t}{(t^2 + b^2)^v} \quad (b > 0, v \geqslant 1). \tag{1.4}$$

The paper is organized as follows. In Section 2 we develop a transformation method for reducing the previous integrals $I_v(f; b)$ to the integrals on $(-1, 1)$ with respect to the Szegő–Bernstein weights (SBWs). Section 3 is devoted to the corresponding Gaussian formulas. For appropriate values of v we obtain the explicit expressions for the recursion coefficients in the three-term-recurrence relation for the corresponding (monic) orthogonal polynomials. Finally, some numerical examples are considered in Section 4.

2. Reduction of integrals to a finite interval

In this section we will show how to reduce the integral (1.4) to an integral on the finite interval. For this purpose we need the sum of the following series

$$W_v(\tau) = W_v(\tau; b) = \sum_{k=-\infty}^{+\infty} w_v(2k\pi + \tau) = \sum_{k=-\infty}^{+\infty} \frac{1}{\left[(2k\pi + \tau)^2 + b^2\right]^v}. \tag{2.1}$$

Since (cf. [6, p. 685])

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(k+\alpha)^2+\beta^2} = \frac{\pi}{\beta}\cdot\frac{\sinh 2\pi\beta}{\cosh 2\pi\beta - \cos 2\pi\alpha},$$

in the simplest, but the most important case $v=1$, for $2\pi\alpha=\tau$ and $2\pi\beta=b$, we obtain

$$W_1(\tau) = W_1(\tau;b) = \frac{\sinh b}{2b}\cdot\frac{1}{\cosh b - \cos\tau}. \tag{2.2}$$

In a general case we can prove:

Lemma 2.1. *Let* w_v *be given by* (1.3), $\xi^{\pm} = -(\tau \pm \mathrm{i}b)/(2\pi)$, *and* $\zeta = -\xi^{+}$. *Then*

$$W_v(\tau) = -\frac{(2\pi)^{1-2v}}{2(v-1)!}\left\{\lim_{z\to\xi^+}\frac{\mathrm{d}^{v-1}}{\mathrm{d}z^{v-1}}\left[\frac{\cot\pi z}{(z+\bar{\zeta})^v}\right] + \lim_{z\to\xi^-}\frac{\mathrm{d}^{v-1}}{\mathrm{d}z^{v-1}}\left[\frac{\cot\pi z}{(z+\zeta)^v}\right]\right\}. \tag{2.3}$$

The proof of this result can be done by an integration of the function $z\mapsto g(z)=\pi\cot(\pi z)w_v(2\pi z+\tau)$ over the rectangular contour C_N with vertices at the points $(N+(1/2))(\pm 1\pm \mathrm{i})$, where $N\in\mathbb{N}$ is such that the poles $\xi^{\pm}$ of the function g are inside of C_N. Then, taking $N\to+\infty$, the corresponding integral over C_N tends to zero, because $w_v(z)=\mathrm{O}(1/z^{2v})$ when $z\to\infty$. Then, by Cauchy's residue theorem, we get

$$W_v(\tau) = \sum_{k=-\infty}^{+\infty} w_v(2k\pi+\tau) = -\left(\operatorname*{Res}_{z=\xi^+} g(z) + \operatorname*{Res}_{z=\xi^-} g(z)\right),$$

i.e., (2.3).

For $v=1$, (2.3) reduces to (2.2). When $v=2$ we have

$$W_2(\tau) = \frac{b\cosh b - \sinh b}{4b^3}\cdot\frac{\cos\tau + a}{(\cosh b - \cos\tau)^2},$$

where

$$a = \frac{\sinh 2b - 2b}{2b\cosh b - 2\sinh b}.$$

Using MATHEMATICA package:

```
In[1]:= g[z_,t_,b_,n_]:= Pi Cot[Pi z]/((2Pi z + t)^2 + b^2)^n
In[2]:= suma[t_,b_,n_]:= -(Residue[g[z,t,b,n],z,-(t + I b)/(2Pi)]
          + Residue[g[z,t,b,n],z,-(t - Ib)/(2Pi)])
In[3]:= pol[t_,b_,n_]:= ComplexExpand[
          suma[t,b,n]] * (Cosh[b] - Cos[t])^n//Simplify
```

we can suspect the following form of our sum

$$W_v(\tau) = \frac{p_v(\cos\tau)}{(\cosh b - \cos\tau)^v},$$

where $p_v(x)$ is an algebraic polynomial. Indeed, we can prove the following result:

Theorem 2.2. *Let $x = \cos\tau$ and $c = \cosh b$. Then*

$$W_v(\tau) = W_v(\tau; b) = \frac{p_v(x)}{(c-x)^v} \qquad (v = 1, 2, \ldots), \tag{2.4}$$

where $p_v(x) = p_v(x; b)$ is a nonnegative polynomial on $[-1, 1]$ of degree $v - 1$. These polynomials satisfy the recurrence relation

$$p_{v+1}(x) = \frac{1}{2bv}\left\{v\sqrt{c^2-1}\,p_v(x) - (c-x)\frac{\partial p_v(x)}{\partial b}\right\}, \tag{2.5}$$

where $p_1(x) = \sqrt{c^2-1}/(2b)$.

Proof. We start with (2.2) written in the form $(c-x)W_1(\tau) = p_1(x)$, where

$$p_1(x) = \frac{\sinh b}{2b} = \frac{\sqrt{c^2-1}}{2b}, \quad x = \cos\tau, \ \ c = \cosh b.$$

Thus, the formula (2.4) is true for $v = 1$.

Suppose that (2.4) holds for some $v(\geqslant 1)$. Then, differentiating

$$(c-x)^v W_v(\tau) = p_v(x)$$

with respect to b, we get

$$v(c-x)^{v-1}\frac{\mathrm{d}c}{\mathrm{d}b}W_v(\tau) + (c-x)^v\frac{\partial W_v(\tau)}{\partial b} = \frac{\partial p_v(x)}{\partial b},$$

from which it follows

$$v\sqrt{c^2-1}(c-x)^v W_v(\tau) - 2bv(c-x)^{v+1}W_{v+1}(\tau) = (c-x)\frac{\partial p_v(x)}{\partial b},$$

i.e.,

$$(c-x)^{\nu+1}W_{\nu+1}(\tau)=\frac{1}{2b\nu}\left\{\nu\sqrt{c^2-1}p_\nu(x)-(c-x)\frac{\partial p_\nu(x)}{\partial b}\right\}=:p_{\nu+1}(x).$$

Thus, the result is proved. □

We are ready now to give a transformation of the integral (1.3) to one on a finite interval. Putting $t=2k\pi+\tau$ and using the periodicity of the function f,

$$f(t)=f(2k\pi+\tau)=f(\tau),$$

we have

$$\begin{aligned}I_\nu(f)=I_\nu(f;b)&=\sum_{k=-\infty}^{+\infty}\int_{(2k-1)\pi}^{(2k+1)\pi}f(t)w_\nu(t)\,\mathrm{d}t\\&=\sum_{k=-\infty}^{+\infty}\int_{-\pi}^{\pi}f(\tau)w_\nu(2k\pi+\tau)\,\mathrm{d}\tau\\&=\int_{-\pi}^{\pi}f(\tau)\left(\sum_{k=-\infty}^{+\infty}w_\nu(2k\pi+\tau)\right)\mathrm{d}\tau,\end{aligned}$$

because of the uniform convergence of the series (2.1). Thus,

$$I_\nu(f)=\int_{-\pi}^{\pi}f(\tau)W_\nu(\tau)\,\mathrm{d}\tau,$$

where $W_\nu(\tau)$ is defined by (2.1) and given by (2.4). We see that $W_\nu(-\tau)=W_\nu(\tau)$, i.e., W_ν is an even weight function.

Because of the last property of the weight function, we have

$$\begin{aligned}I_\nu(f)=I_\nu(f;b)&=\int_{-\pi}^{0}f(\tau)W_\nu(\tau)\,\mathrm{d}\tau+\int_{0}^{\pi}f(\tau)W_\nu(\tau)\,\mathrm{d}\tau\\&=\int_{0}^{\pi}(f(\tau)+f(-\tau))W_\nu(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Changing the variables $\cos\tau=x$ and putting

$$f(\tau)+f(-\tau)=F(\cos\tau),\tag{2.6}$$

we get the following result:

Theorem 2.3. *The integral* (1.3) *can be transformed to the form*

$$I_\nu(f)=I_\nu(f;b)=\int_{-1}^{1}F(x)\frac{p_\nu(x)}{(c-x)^\nu}\cdot\frac{\mathrm{d}x}{\sqrt{1-x^2}},\tag{2.7}$$

where $c = \cosh b$, $p_\nu(x)$ *is a polynomial determined by the recurrence relation* (2.5), *and F is defined by* (2.6).

3. Gaussian type formulae for Szegő–Bernstein weights

In order to evaluate the integral (2.7) it would seem more natural and simpler to apply the Gauss–Chebyshev quadrature formula, i.e., taking $x \mapsto \Phi(x) = F(x)p_\nu(x)/(c-x)^\nu$ $(c > 1)$ as an integrating function with respect to the Chebyshev weight (ChW) $v_0(x) = (1-x^2)^{-1/2}$.

In this case, when for some $r \geqslant 1$ the function F satisfies the condition $\int_{-1}^{1} F^{(r)}(x)(\sqrt{1-x^2})^{(r-1)}\,\mathrm{d}x < +\infty$, the error $R_n(\Phi)_{v_0}$ of the n-point Gauss–Chebyshev quadrature can be estimated as follows (see [5])

$$|R_n(\Phi)_{v_0}| \leqslant \frac{A}{n^r}\int_{-1}^{1}\left|\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[\frac{F(x)p_\nu(x)}{(c-x)^\nu}\right]\right|(1-x^2)^{(r-1)/2}\,\mathrm{d}x,$$

where $A > 0$ is a constant independent on Φ and n. Hence, when $c > 1$ is very close to 1, even if the integrand is bounded, it gives a very large bound.

On the contrary, if we take $v_\nu(x) = (1-x^2)^{-1/2}/(c-x)^\nu$ as a weight function (Szegő–Bernstein weight), then the error of the corresponding Gaussian formula is bounded as follows

$$|R_n(\Psi)| \leqslant \frac{B}{n^r}\int_{-1}^{1}\left|\frac{\mathrm{d}^r}{\mathrm{d}x^r}[F(x)p_\nu(x)]\right|(1-x^2)^{(r-1)/2}\frac{\mathrm{d}x}{(c-x)^\nu},$$

where $B > 0$ is a constant independent on Ψ $(\Psi(x) = F(x)p_\nu(x))$ and n. It is clear that the last integral is much smaller then the previous one. Also, some numerical evidences confirm this argument (see Section 4).

Thus, for evaluating the integral (2.7) it is more convenient to construct the Gaussian quadratures

$$\int_{-1}^{1}\Psi(x)\,\mathrm{d}\lambda_\nu(x) = \sum_{k=1}^{n}A_k^{(n)}\Psi(x_k^{(n)}) + R_n(\Psi)_{v_\nu}, \quad R_n(\mathscr{P}_{2n-1})_{v_\nu} \equiv 0, \tag{3.1}$$

for the measure

$$\mathrm{d}\lambda_\nu(x) = v_\nu(x)\,\mathrm{d}x = \frac{\mathrm{d}x}{(c-x)^\nu\sqrt{1-x^2}} \quad (\nu \geqslant 1), \tag{3.2}$$

where the function Ψ includes the algebraic polynomial $p_\nu(x)$, i.e., $\Psi(x) = F(x)p_\nu(x)$. Here, $\mathscr{P}_{2n-1}$ denotes the set of all polynomials of degree at most $2n-1$.

It is well known that the corresponding orthogonal polynomials $\pi_{n,\nu}(x)$ for the measure (3.2) can be calculated explicitly provided $\nu < 2n$ (cf. [7, p. 31]). On the other side, there is a nonlinear algorithm to produce the recursion coefficients in the three-term recurrence relation for the monic polynomials $\pi_{n,v}(x)$,

$$\begin{aligned} &\pi_{n+1,v}(x) = (x - \alpha_n^{(v)})\pi_{n,v}(x) - \beta_n^{(v)}\pi_{n-1,v}(x), \quad n \geqslant 0, \\ &\pi_{0,v}(x) = 1, \pi_{-1,v}(x) \quad \left(\beta_0^{(v)} \triangleq m_0^{(v)} = \int_{-1}^{1} \mathrm{d}\lambda_v(x)\right) \end{aligned} \tag{3.3}$$

in terms of ones for the polynomials $\pi_{n,v-1}(x)$ orthogonal with respect to the measure $\mathrm{d}\lambda_{v-1}(x) = \mathrm{d}\lambda_v(x)/(c-x)$. However, such an algorithm is quite numerically unstable unless c is very close to the support interval of the measure (see [3, p. 102]). Two numerical algorithms for this purpose were also discussed in [1]. Our goal in this paper is to find analytic expressions for the recursion coefficients for some appropriate values of v.

Knowing these coefficients, $\alpha_k^{(v)}$, $\beta_k^{(v)}$, $k \geqslant 0$, one can easily obtain the n-point Gaussian quadrature formula (3.1) for any n. The nodes $x_k^{(n)}$, indeed, are the eigenvalues of the symmetric (tridiagonal) Jacobi matrix

$$J_n(\mathrm{d}\lambda_v) = \begin{bmatrix} \alpha_0^v & \sqrt{\beta_1^{(v)}} & & & \mathrm{O} \\ \sqrt{\beta_1^{(v)}} & \alpha_1^{(v)} & \sqrt{\beta_2^{(v)}} & & \\ & \sqrt{\beta_2^{(v)}} & \alpha_2^{(v)} & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}^{(v)}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}^{(v)}} & \alpha_{n-1}^{(v)} \end{bmatrix}$$

while the weights (Christoffel numbers) $A_k^{(n)}$ are given by $A_k^{(n)} = \beta_0^{(v)} v_{k,1}^2$ in terms of the first components $v_{k,1}$ of the corresponding normalized eigenvectors (cf. [2, Section 5.1; 4]).

Firstly, we introduce the modified moments for $\mathrm{d}\lambda_v(x)$ by the orthogonal polynomials $\pi_{n,v-1}(x)$,

$$m_n^{(v)} = \int_{-1}^{1} \pi_{n,v-1}(x)\,\mathrm{d}\lambda_v(x) = \int_{-1}^{1} \frac{\pi_{n,v-1}(x)\,\mathrm{d}x}{(c-x)^v\sqrt{1-x^2}} \quad (n \geqslant 0). \tag{3.4}$$

Notice that $\pi_{n,0}(x)$ are the monic Chebyshev polynomials of the first kind $\hat{T}_n(x)$ ($\hat{T}_0(x) = 1$, $\hat{T}_n(x) = 2^{1-n}\cos(n \arccos x)$, $n \geqslant 1$).

It is easy to prove the following auxiliary result:

Lemma 3.1. *For the first moment we have*

$$m_0^{(v)} = \int_{-1}^{1} \frac{\mathrm{d}x}{(c-x)^v\sqrt{1-x^2}} = \frac{\pi Q_{v-1}(c)}{(c^2-1)^{v-1/2}},$$

where

$$Q_\nu(c) = \frac{1}{\nu}\left[(2\nu-1)cQ_{\nu-1}(c) - (c^2-1)Q'_{\nu-1}(c)\right], \quad Q_0(c) = 1.$$

Thus, we find

$$Q_1(c) = c, \quad Q_2(c) = c^2 + \frac{1}{2}, \quad Q_3(c) = c^3 + \frac{3}{2}c,$$

$$Q_4(c) = c^4 + 3c^2 + \frac{3}{8}, \text{ etc.}$$

According to [6, p. 415] we have

$$Q_\nu(c) = (c^2-1)^{\nu/2} P_\nu\left(\frac{c}{\sqrt{c^2-1}}\right),$$

where P_ν is the Legendre polynomial of the order ν.

In order to get connection with the Chebyshev measure $d\lambda_0(x)$ it is convenient to put $Q_{-1}(c) = (c^2-1)^{-1/2}$. Then it gives $m_0^{(0)} = \pi$.

Now, we can prove:

Theorem 3.2. *The polynomials $\pi_{n,\nu}(x)$ can be expressed in terms of polynomials $\{\pi_{k,\nu-1}(x)\}$ in the form*

$$\pi_{n,\nu}(x) = \pi_{n,\nu-1}(x) - q_n^{(\nu)}\pi_{n-1,\nu-1}(x), \tag{3.5}$$

where $q_n^{(\nu)} = m_n^{(\nu)}/m_{n-1}^{(\nu)}$ and the moments $m_n^{(\nu)}$ are given by (3.4). *If $\alpha_n^{(\nu-1)}$ and $\beta_n^{(\nu-1)}$ are the recursion coefficients in* (3.3) *for polynomials $\{\pi_{n,\nu-1}(x)\}$, and*

$$r_\nu = \frac{m_0^{(\nu-1)}}{m_0^{(\nu)}} = (c^2-1)\frac{Q_{\nu-2}(c)}{Q_{\nu-1}(c)}, \tag{3.6}$$

where the polynomials $Q_\nu(c)$ are defined in Lemma 3.1, *then*

$$q_1^{(\nu)} = c - \alpha_0^{(\nu-1)} - r_\nu, \quad q_{n+1}^{(\nu)} = c - \alpha_n^{(\nu-1)} - \frac{\beta_n^{(\nu-1)}}{q_n^{(\nu)}} \quad (n \geqslant 1). \tag{3.7}$$

The coefficients in (3.3) *are given by*

$$\alpha_0^{(\nu)} = \alpha_0^{(\nu-1)} + q_1^{(\nu)}, \quad \alpha_n^{(\nu)} = \alpha_n^{(\nu-1)} + q_{n+1}^{(\nu)} - q_n^{(\nu)} \quad (n \geqslant 1)$$

and $\beta_0^{(\nu)} \triangleq m_0^{(\nu)} = \pi Q_{\nu-1}(c)/(c^2-1)^{\nu-1/2}$,

$$\beta_n^{(\nu)} = \beta_n^{(\nu-1)} + q_n^{(\nu)}\left[\alpha_n^{(\nu-1)} - \alpha_{n-1}^{(\nu-1)} + q_{n+1}^{(\nu)} - q_n^{(\nu)}\right] \quad (n \geqslant 1).$$

Alternatively,

$$\beta_n^{(\nu)} = \beta_{n-1}^{(\nu-1)}\frac{q_n^{(\nu)}}{q_{n-1}^{(\nu)}} \quad (n \geqslant 2).$$

Proof. Putting

$$\pi_{n,v}(x) = \pi_{n,v-1}(x) - \sum_{k=0}^{n-1} q_{n,k}^{(v)} \pi_{k,v-1}(x)$$

and using the inner product with respect to the measure $d\lambda_{v-1}$,

$$(f,g)_{v-1} = \int_{-1}^{1} f(x)g(x)\, d\lambda_{v-1},$$

because of orthogonality, we obtain that for each $0 \leqslant i \leqslant n-2$,

$$(\pi_{n,v}, \pi_{i,v-1})_{v-1} = -q_{n,i}^{(v)} (\pi_{i,v-1}, \pi_{i,v-1})_{v-1}$$

and

$$(\pi_{n,v}, \pi_{i,v-1})_{v-1} = \int_{-1}^{1} (c-x)\pi_{n,v}(x)\pi_{i,v-1}(x)\, d\lambda_v(x)$$
$$= -\int_{-1}^{1} \pi_{n,v}(x)(x\pi_{i,v-1}(x))\, d\lambda_v(x) = 0.$$

Thus, we conclude that $q_{n,i}^{(v)} = 0$ for such values of i and the formula (3.5) is true, where we put $q_{n,n-1}^{(v)} \equiv q_n^{(v)}$.

From (3.5), because of orthogonality

$$0 = (\pi_{n,v}, 1)_v = (\pi_{n,v-1}, 1)_v - q_n^{(v)} (\pi_{n-1,v-1}, 1)_v,$$

we get $q_n^{(v)} = m_n^{(v)} / m_{n-1}^{(v)}$, where the modified moments are defined by (3.4).

Using the recurrence relation for polynomials $\{\pi_{n,v-1}(x)\}$ we find that

$$m_{n+1}^{(v)} = (c - \alpha_n^{(v-1)}) m_n^{(v)} - \beta_n^{(v-1)} m_{n-1}^{(v)} - \int_{-1}^{1} \pi_{n,v-1}(x)\, d\lambda_{v-1},$$

which gives

$$m_1^{(v)} = (c - \alpha_0^{(v-1)}) m_0^{(v)} - m_0^{(v-1)}$$

and

$$m_{n+1}^{(v)} = (c - \alpha_n^{(v-1)}) m_n^{(v)} - \beta_n^{(v-1)} m_{n-1}^{(v)} \quad (n \geqslant 1).$$

These equalities give (3.7).

Finally, changing $\pi_{k,v}(x)$ $(k = n-1, n, n+1)$ in the recurrence relation (3.3) by (3.5) and using the corresponding relation for polynomials $\{\pi_{k,v-1}(x)\}$ we get for $n \geqslant 2$

$$\pi_{n+1,v-1}(x) = \left(x - \alpha_n^{(v)} + q_{n+1}^{(v)} - q_n^{(v)}\right)\pi_{n,v-1}(x) - \left(\beta_n^{(v)} - q_n^{(v)}\alpha_n^{(v)} + q_n^{(v)}\alpha_{n-1}^{(v-1)}\right)\pi_{n-1,v-1}(x) + \left(\beta_n^{(v)} q_{n-1}^{(v)} - \beta_{n-1}^{(v-1)} q_n^{(v)}\right)\pi_{n-2,v-1}(x).$$

Comparing with the recurrence relation for $\{\pi_{n,\nu-1}(x)\}$ we obtain formulas for the recursion coefficients. The case $n=1$ should be considered separately. $\square$

Notice that in Chebyshev case ($\nu=0$) we have

$$\alpha_n^{(0)}=0\quad (n\geqslant 0),\quad \beta_0^{(0)}\triangleq\pi,\quad \beta_1^{(0)}=\frac{1}{2},\quad \beta_n^{(0)}=\frac{1}{4}\quad (n\geqslant 2).$$

Also, from (3.6) it follows

$$r_1=\sqrt{c^2-1},\quad r_2=\frac{c^2-1}{c},\quad r_3=\frac{c(c^2-1)}{c^2+(1/2)},$$
$$r_4=\frac{(c^2+(1/2))(c^2-1)}{c^3+(3/2)c},\ \text{etc.}$$

Now, using the previous theorem we give explicit expressions for recursion coefficients for some important special cases, where $c=\cosh b$.

Case $\nu=1$. Here we have $q_1^{(1)}=\mathrm{e}^{-b}$, $q_n^{(1)}=(1/2)\mathrm{e}^{-b}$ $(n\geqslant 2)$, and the recursion coefficients

$$\alpha_0^{(1)}=\mathrm{e}^{-b},\quad \alpha_1^{(1)}=-\frac{1}{2}\mathrm{e}^{-b},\quad \alpha_n^{(1)}=0\quad (n\geqslant 2),$$
$$\beta_0^{(1)}\triangleq\frac{\pi}{\sinh b},\quad \beta_1^{(1)}=\frac{1}{2}\left(1-\mathrm{e}^{-2b}\right),\quad \beta_n^{(1)}=\frac{1}{4}\quad (n\geqslant 2).$$

Case $\nu=2$. Here, $q_1^{(2)}=\mathrm{e}^{-b}\tanh b$, $q_n^{(2)}=(1/2)\mathrm{e}^{-b}$ $(n\geqslant 2)$, and

$$\alpha_0^{(2)}=\frac{1}{\cosh b},\quad \alpha_1^{(2)}=-\mathrm{e}^{-b}\tanh b,\quad \alpha_n^{(2)}=0\quad (n\geqslant 2),$$
$$\beta_0^{(2)}\triangleq\frac{\pi\cosh b}{\sinh^3 b},\quad \beta_1^{(2)}=\frac{1}{2}\left(1-\mathrm{e}^{-2b}\right)\tanh^2 b,\quad \beta_2^{(2)}=\frac{1}{4}\left(1+\mathrm{e}^{-2b}\right),$$

and $\beta_n^{(2)}=(1/4)$ $(n\geqslant 3)$.

Case $\nu=3$. Here, $q_1^{(3)}=\sinh b\tanh b/(2+\cosh(2b))$, $q_2^{(3)}=\mathrm{e}^{-2b}\cosh b$, $q_n^{(3)}=\frac{1}{2}\mathrm{e}^{-b}$ $(n\geqslant 3)$, and

$$\alpha_0^{(3)}=\frac{3\cosh b}{2+\cosh(2b)},\quad \alpha_1^{(3)}=\mathrm{e}^{-2b}\cosh b+\left(\mathrm{e}^{-b}+\frac{\sinh b}{2+\cosh(2b)}\right)\tanh b,$$
$$\alpha_2^{(3)}=\frac{1}{2}\mathrm{e}^{-3b},\quad \alpha_n^{(3)}=0\quad (n\geqslant 3);$$
$$\beta_0^{(3)}\triangleq\frac{\pi\left(\cosh^2 b+(1/2)\right)}{\sinh^5 b},\quad \beta_1^{(3)}=\frac{(1-\mathrm{e}^{-2b})^4}{2(1-4\mathrm{e}^{-2b}+\mathrm{e}^{-4b})^2},$$

$$\beta_2^{(3)} = \frac{1}{4}\left(1 + 3e^{-2b} - 3e^{-4b} - e^{-6b}\right), \quad \beta_n^{(3)} = \frac{1}{4} \quad (n \geqslant 3).$$

As we can see the recurrence coefficients for polynomials $\pi_{n,v}(x)$ reduce to the corresponding coefficients for Chebyshev polynomials for $n \geqslant n_0$ $(n \in \mathbb{N})$. Precisely, calculations show that

$$\alpha_n^{(v)} = \alpha_n^{(0)} = 0, \quad n \geqslant \left[\frac{v+1}{2}\right] + 1$$

and

$$\beta_n^{(v)} = \beta_n^{(0)} = \frac{1}{4}, \quad n \geqslant \left[\frac{v}{2}\right] + 1.$$

4. Numerical examples

In order to illustrate the presented transformation method, we consider in this section a few numerical examples. All computations were done in D-arithmetic on the Workstation Digital Ultimate Alpha 533au2 (with machine precision $\approx 2.22 \times 10^{-16}$).

Example 4.1. Consider integrals of the form

$$I_v(f;b) = \int_{-\infty}^{+\infty} \frac{2\sin 2t - 1}{3 + 2\cos 3t} \cdot \frac{e^{-\cos 2t}}{(t^2 + b^2)^v}\, dt \quad (v \geqslant 1).$$

The function

$$f(t) = \frac{2\sin 2t - 1}{3 + 2\cos 3t} e^{-\cos 2t}$$

is (2π)-periodic and its graph on the interval $[-\pi, \pi]$ is displayed in Fig. 1.

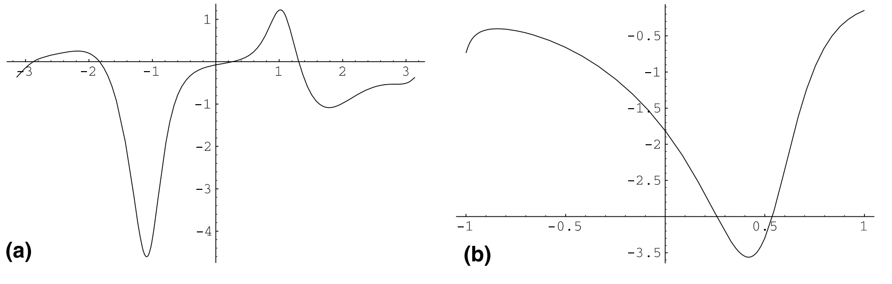

Fig. 1. The periodic function $f(t)$ (a). The function $F(x)$ obtained by transformation (2.6) (b).

Since

$$f(\tau) + f(-\tau) = -\frac{2e^{-\cos 2\tau}}{3 + 2\cos 3\tau},$$

putting $x = \cos\tau$ and using (2.6) we find

$$F(x) = F(\cos\tau) = f(\tau) + f(-\tau) = \frac{-2e^{1-2x^2}}{3 - 6x + 8x^3}$$

and according to (2.7),

$$I_\nu(f;b) = \int_{-1}^{1} F(x)\frac{\pi_\nu(x)}{(c-x)^\nu}\frac{dx}{\sqrt{1-x^2}},$$

where $c = \cosh b$.

Let $\nu = 1$. Applying Gaussian quadratures with the ChW for $n = 5(5)50$ and taking $b = 10^m$ $(m = -2, -1, 0)$ we get approximations of $I_1(f;b)$ with relative errors given in Table 1. Numbers in parentheses indicate decimal exponents.

Taking Gaussian quadratures for $n = 5(5)50$, with SBW, $v_1(x) = (1-x^2)^{-1/2}/(c-x)$, the corresponding errors are also presented in the same table. The corresponding exact values of $I_1(f;b)$ are obtained using Gaussian quadratures with SBW in Q-arithmetic (machine precision $\approx 1.93 \times 10^{-34}$):

$$I_1(f;0.01) = -0.25865882162418231278822\ldots \times 10^2 \quad (c = 1.0000500\ldots),$$
$$I_1(f;0.10) = -0.49680128779962862283355\ldots \times 10^1 \quad (c = 1.0050041\ldots),$$
$$I_1(f;1.00) = -0.16732154097453311127226\ldots \times 10^1 \quad (c = 1.5430806\ldots).$$

Table 1
Relative errors in Gaussian approximations of the integral $I_1(f;b)$ with respect to the Chebyshev weight (ChW) and the Szegő–Bernstein weight (SBW)

b	$b = 0.01$		$b = 0.1$		$b = 1.0$	
n	ChW	SBW	ChW	SBW	ChW	SBW
5	8.4 (−1)	1.3 (−2)	2.2 (−1)	6.3 (−2)	1.2 (−2)	5.5 (−2)
10	8.0 (−1)	2.4 (−4)	1.1 (−1)	1.5 (−3)	2.7 (−3)	3.5 (−3)
15	7.6 (−1)	1.1 (−5)	4.3 (−2)	4.2 (−5)	1.5 (−4)	7.0 (−5)
20	7.2 (−1)	9.0 (−7)	1.6 (−2)	4.4 (−6)	1.0 (−6)	3.5 (−6)
25	6.7 (−1)	1.6 (−8)	6.0 (−3)	9.7 (−8)	1.8 (−7)	2.3 (−7)
30	6.3 (−1)	7.4 (−10)	2.2 (−3)	2.8 (−9)	9.8 (−9)	4.6 (−9)
35	5.9 (−1)	5.9 (−11)	8.2 (−4)	2.9 (−10)	6.7 (−11)	2.3 (−10)
40	5.5 (−1)	1.0 (−12)	3.0 (−4)	6.4 (−12)	1.2 (−11)	1.5 (−11)
45	5.2 (−1)	4.8 (−14)	1.1 (−4)	1.8 (−13)	6.5 (−13)	3.1 (−13)
50	4.8 (−1)	4.7 (−15)	4.1 (−5)	1.9 (−14)	5.2 (−15)	1.6 (−14)

As can be seen, for smaller values of b (c is close to 1) the Gauss–Chebyshev quadratures ChW cannot be used directly. When b increases the both quadratures become comparable. However, by writing $I_1(f;b)$ in the form

$$I_1(f;b) = \frac{\pi}{2b}F(c) - \frac{\sinh b}{2b}\int_{-1}^{1}\frac{F(c)-F(x)}{c-x}\frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

the Gauss–Chebyshev quadratures can be applied directly.

Consider now the case $v = 2$, with the functions ϕ_k and the corresponding weights v_k ($k = 0, 1, 2$), where

$$\phi_k(x) = \frac{F(x)p_2(x)}{(c-x)^{2-k}}, \qquad v_k(x) = \frac{1}{(c-x)^k\sqrt{1-x^2}}.$$

Applying the Gaussian quadratures with the Chebyshev weight ChW ($k = 0$) and the Szegő–Bernstein weights SBW_1 ($k = 1$) and SBW_2 ($k = 2$) we get approximations of the integral $I_2(f;b)$. The exact values of this integral for some selected b are:

$$I_2(f;0.01) = -0.11561838211404870282\ldots \times 10^6,$$
$$I_2(f;0.10) = -0.12147069135884123005\,93\ldots \times 10^3.$$

The relative errors in Gaussian approximations for $n = 5(5)50$ are presented in Table 2.

The advantage of quadrature formulas for $k = 2$ (in this case $v = 2$) is evident. When b increases all quadratures give similar results.

Example 4.2. Consider now the integral (1.4), with a nonanalytic function $f(t) = |\cos(t/2)|^{2\alpha}$ ($\alpha > 0$). After the transformation we obtain the integral

Table 2
Relative errors in Gaussian approximations of the integral $I_2(f;b)$ with respect to the Chebyshev weight (ChW) and to the Szegő–Bernstein weights (SBW_1 and SBW_2)

b	$b = 0.01$			$b = 0.1$		
n	ChW	SBW_1	SBW_2	ChW	SBW_1	SBW_2
5	1.0 (0)	9.1 (−1)	5.5 (−7)	8.9 (−1)	3.7 (−1)	1.1 (−3)
10	1.0 (0)	8.3 (−1)	1.0 (−7)	6.2 (−1)	1.4 (−1)	6.7 (−5)
15	1.0 (0)	7.5 (−1)	4.7 (−9)	3.4 (−1)	5.0 (−2)	4.3 (−6)
20	9.9 (−1)	6.8 (−1)	1.9 (−11)	1.6 (−1)	1.9 (−2)	6.4 (−8)
25	9.9 (−1)	6.1 (−1)	4.6 (−12)	7.4 (−2)	6.8 (−3)	4.4 (−9)
30	9.8 (−1)	5.5 (−1)	2.6 (−12)	3.2 (−2)	2.5 (−3)	2.8 (−10)
35	9.7 (−1)	5.0 (−1)	2.3 (−12)	1.3 (−2)	9.2 (−4)	4.3 (−12)
40	9.6 (−1)	4.5 (−1)	2.3 (−12)	5.6 (−3)	3.4 (−4)	3.1 (−13)
45	9.5 (−1)	4.1 (−1)	2.3 (−12)	2.3 (−3)	1.2 (−4)	1.4 (−15)
50	9.3 (−1)	3.7 (−1)	2.3 (−12)	9.2 (−4)	4.6 (−5)	2.0 (−14)

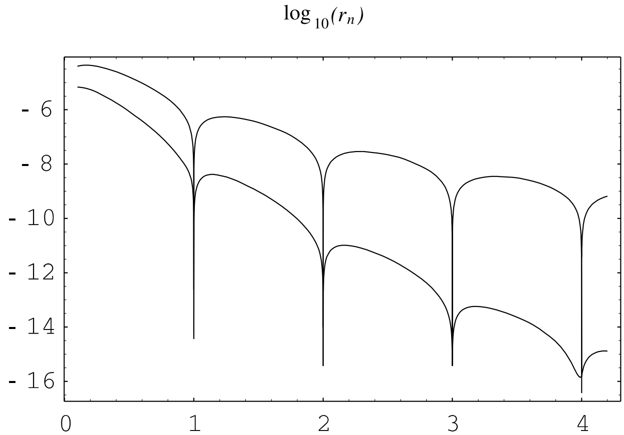

Fig. 2. Relative errors in Gaussian approximations with $n = 5$ (upper curve) and $n = 20$ nodes (lower curve) for $0 \leqslant \alpha < 4.5$.

$$J(\alpha) = A \int_{-1}^{1} \left(\frac{1+x}{2} \right)^{\alpha} \frac{\mathrm{d}x}{(c-x)\sqrt{1-x^2}},$$

where $A = 2p_1(x) = \sinh b/b$.

In order to evaluate this integral, we apply Gaussian rule in n points with SBW ($v = 1$). A typical behavior of the relative error r_n of Gaussian approximations with respect to the parameter α ($0 \leqslant \alpha < 4.5$) is displayed in Fig. 2 in the log-scale. Two cases for $n = 5$ and $n = 20$ are given, whereas $b = 0.01$. It is clear that the rapidly increasing of accuracy achieves when the parameter α tends to an integer (i.e., when f becomes an analytic function).

References

[1] B. Fischer, G. Golub, How to generate unknown orthogonal polynomials out of known orthogonal polynomials, J. Comput. Appl. Math. 43 (1992) 99–115.

[2] W. Gautschi, A survey of Gauss–Christoffel quadrature formulae, in: P.L. Butzer, F. Fehér (Eds.), E.B. Christoffel, Birkhäuser, Basel, 1981, pp. 72–147.

[3] W. Gautschi, Orthogonal polynomials: applications and computation, Acta Numerica (1996) 45–119.

[4] G. Golub, J.H. Welsch, Calculation of Gauss quadrature rules, Math. Comp. 44 (1969) 221–230.

[5] G. Mastroianni, Generalized Christoffel functions and error of positive quadrature, Numer. Algorithms 10 (1995) 113–126.

[6] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series. Elementary Functions, Nauka, Moscow, 1981 (in Russian).

[7] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., vol. 23, fourth ed., Amer. Math. Soc., Providence, RI, 1975.