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# A Tribute to S. B. Prešić

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# CONTRIBUTION AND INFLUENCE OF S.B. PREŠIĆ TO NUMERICAL FACTORIZATION OF POLYNOMIALS

**Gradimir V. Milovanović**

ABSTRACT. This paper is devoted to contributions of S.B. Prešić in numerical factorization of algebraic polynomials, as well as to influence of his work in this subject. Beside a general factorization of polynomials, we consider some important special cases and point out some accelerated iterative formulas.

## 1. Introduction

The numerical factorization of algebraic polynomials is a very important mathematical subject. There are several methods for it in the literature, beginning with the well-known methods of Bairstow [2] and of Lin [19–20]. Many of them are quadratically convergent, but most require a sufficiently close starting values for factorization. In their survey paper, Householder and Stewart [14] mentioned also the method of Graeffe and the *qd* algorithm, though they are not primarily for this assignment. A number of these methods can be related to an algorithm proposed by Sebastião e Silva [38]. Some generalizations of this algorithm were given by Householder [11] in 1971 (see also [12], [41], [6]). In addition we mention also a method of Samelson [36] from 1959, which generalizes the Bauer-Samelson iteration [3]. In his paper Samelson noted that his method is related to Bairstow's method. Taking a monic algebraic polynomial over the field of complex numbers, with zeros $z_1, z_2, \dots, z_n$, i.e.,

$$P(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n = \prod_{k=1}^{n} (z - z_k), \tag{1.1}$$

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Samelson [36] seeks its factorization by two factors

$$u(z) = (z - z_1)(z - z_2) \cdots (z - z_m)$$

and

$$v(z) = (z - z_{m+1})(z - z_{m+2}) \cdots (z - z_n).$$

Let $p$ and $q$ be monic polynomials of degree $m$ and $n - m$ approximating $u$ and $v$ repectively. Then his quadratically convergent iterative procedure defines improved approximations $p^*$ and $q^*$ by the formula

$$p^*q + q^*p = P + pq. \tag{1.2}$$

If $p$ and $q$ are relatively prim, then $p^*$ and $q^*$ are uniquely defined by (1.2) Samelson's iteration was discovered independently by Stewart [39], who characterized $pq^*$ as the linear combination of $P, q, zq, \ldots, z^{m-1}q$ that is divisible by $p$. Householder and Stewart [13] gave the exact connection between these characterizations (see also [14] and [40] for another derivation of Samelson's method and the corresponding error bounds for the iteration, as well as the paper of Schröder [37] for a connection with Newton's method).

In 1966 and 1968 S.B. Prešić [34–35], inspired only by some results of D. Marković [21], gave an iterative method for numerical factorization of algebraic polynomials by $s$ $(2 \le s \le n)$ factors. The purpose of the present paper is to show contributions of S. Prešić, as well as to point out an influence of S. Prešić's work to this subject. The paper is organized as follows. In Section 2 we explain S. Prešić's approach to numerical factorization of polynomials and give an example on $2-2$ factorization of a polynomial of fourth degree. Sections 3 and 4 are dedicated to an $1 - 1 - \cdots - 1$ factorization and some accelerated iterative formulas, respectively.

Later, in 1969 Dvorčuk [9] considered a factorization into quadratic factors, and in 1971 Grau [10] used a Newton-type of approximation for simultaneously improving a complete set of approximate factors for a given polynomial. Recently, Carstensen [4] and Carstensen and Sakurai [5] gave some generalizations of this method.

Here, we mention also that in the last period many papers have been published on factorization of polynomials over finite fields, on factorization methods for multivariable polynomials, as well as on factorization of matrix polynomials.

## 2. S. Prešić's approach to numerical factorization

Let $P$ be a monic algebraic polynomial over the field of complex numbers given by (1.1) and let it be expressed in a factorized form

$$P(z) = A_1(z)A_2(z) \cdots A_s(z) \qquad (2 \le s \le n), \tag{2.1}$$

where $A_\nu(z)$ are monic polynomials of degree $n_\nu$, i.e.,

$$A_\nu(z) = \sum_{i=0}^{n_\nu} a_{\nu i} z^{n_\nu - i}, \qquad a_{\nu 0} = 1 \quad (\nu = 1, 2, \ldots, s), \tag{2.2}$$

and $\sum_{\nu=1}^{s} n_\nu = n$. The case $s = 2$ is mentioned in the previous section.

Assuming that zeros of (1.1) are simple, S.B. Prešić gave an iterative method for numerical determination such a factorization, so-called $n_1 - n_2 - \cdots - n_s$ factorization, in which successive iterated monic factors

$$A_\nu^{(k)}(z) = \sum_{i=0}^{n_\nu} a_{\nu i}^{(k)} z^{n_\nu - i}, \quad a_{\nu 0}^{(k)} = 1 \qquad (\nu = 1, 2, \ldots, s) \tag{2.3}$$

are determined from the relation

$$A_1^{(k+1)} A_2^{(k)} \cdots A_s^{(k)} + A_1^{(k)} A_2^{(k+1)} \cdots A_s^{(k)} + \cdots + A_1^{(k)} A_2^{(k)} \cdots A_s^{(k+1)} - (s-1) A_1^{(k)} A_2^{(k)} \cdots A_s^{(k)} = P,$$

i.e.,

$$A_1^{(k)}(z) A_2^{(k)}(z) \cdots A_s^{(k)}(z) \left( \sum_{\nu=1}^{s} \frac{A_\nu^{(k+1)}(z)}{A_\nu^{(k)}(z)} - s + 1 \right) \equiv P(z). \tag{2.4}$$

Taking the coefficients $a_{\nu i}$ of polynomials (2.2) as coordinates of an $n$-dimensional vector

$$a = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n_1} \quad a_{21} \quad a_{2n_2} \quad \cdots \quad a_{22} \quad \cdots \quad a_{s1} \quad a_{22} \quad \cdots \quad a_{sn_s}]^T$$

and $a_{\nu i}^{(k)}$ (coefficients of iterated factors (2.3)) as coordinates of the corresponding also $n$-dimensional vector $a^{(k)}$, S. Prešić observed that (2.4) implies a system of linear equations of the form

$$A_n(a^{(k)}) a^{(k+1)} = b_n(a^{(k)}, p) \tag{2.5}$$

where $A_n$ is an $n \times n$ matrix depending only on $a^{(k)}$, and $b_n$ is an $n$-dimensional vector depending also on $a^{(k)}$ and on coefficients of the polynomial (1.1), $p = [p_1 \quad p_2 \quad \cdots \quad p_n]^T$. Further, he concluded that there exists a neighbourhood $V$ of $a \in \mathbb{C}^n$, such that (2.5) can be expressed in the following form

$$a^{(k+1)} = F(a^{(k)}) \qquad (k = 0, 1, \ldots;\ a^{(k)} \in V), \tag{2.6}$$

where $F: V \to V$ is an enough times differentiable operator (in Fréchet sense). Practically, S. Prešić proved that $F(a) = a$ and $F'_{(a)}$ is a zero operator, so that

$$\|a^{(k+1)} - a\| = O(\|a^{(k)} - a\|^2) \qquad \left(a = \lim_{k \to +\infty} a^{(k)}\right).$$

Thus, S. Prešić's result can be sumarized as:

THEOREM 1.1. *There is an neighbourhood $V$ of $\boldsymbol{a} \in \mathbb{C}^n$ so that for an arbitrary $\boldsymbol{a}^{(0)} \in V$, the iterative process (2.6) quadratically converges to $\boldsymbol{a}$.*

Thus,

$$\lim_{k\to+\infty} A_\nu^{(k)}(z) = A_\nu(z) \qquad (\nu = 1, 2, \dots, s),$$

give the factorization (2.1).

In his paper [35], S. Prešić derived formulas for a $2-2-2$ factorization of a polynomial of degree 6. Here, as an illustration, we give a simpler case when $P(z) = z^4 + p_1 z^3 + p_2 z^2 + p_3 z + p_4$ and when we seek its $2-2$ factorization, with

$$A_1(z) = z^2 + a_{11} z + a_{12}, \quad A_2(z) = z^2 + a_{21} z + a_{22}.$$

In that case the system (2.5) becomes

$$\begin{aligned}
a_{11}^{(k+1)} \qquad\qquad\qquad + \quad a_{21}^{(k+1)} \qquad\qquad\qquad &= b_1^{(k)},\\
a_{21}^{(k)} a_{11}^{(k+1)} + \quad a_{12}^{(k+1)} + a_{11}^{(k)} a_{21}^{(k+1)} + \quad a_{22}^{(k+1)} &= b_2^{(k)},\\
a_{22}^{(k)} a_{11}^{(k+1)} + a_{21}^{(k)} a_{12}^{(k+1)} + a_{12}^{(k)} a_{21}^{(k+1)} + a_{11}^{(k)} a_{22}^{(k+1)} &= b_3^{(k)},\\
a_{22}^{(k)} a_{12}^{(k+1)} \qquad\qquad\qquad + a_{12}^{(k)} a_{22}^{(k+1)} &= b_4^{(k)},
\end{aligned}$$

where

$$\begin{aligned}
b_1^{(k)} &= p_1, & b_2^{(k)} &= p_2 + a_{11}^{(k)} a_{21}^{(k)},\\
b_3^{(k)} &= p_3 + a_{11}^{(k)} a_{22}^{(k)} + a_{12}^{(k)} a_{21}^{(k)}, & b_4^{(k)} &= p_4 + a_{12}^{(k)} a_{22}^{(k)}.
\end{aligned}$$

Solving this system we obtain an iterative procedure of the form (2.6). This case ($s = 2$) reduces to Samelson's iteration.

Using the previous idea on polynomial factorization, J.J. Petrić and S.B. Prešić [32] treated a problem of simultaneous determination of all solutions of the system of algebraic equations

$$\begin{aligned}
J_1(x, y) &\equiv A_1 x^2 + 2B_1 xy + C_1 y^2 + 2D_1 x + 2E_1 y + F_1 = 0,\\
J_2(x, y) &\equiv A_2 x^2 + 2B_2 xy + C_2 y^2 + 2D_2 x + 2E_2 y + F_2 = 0.
\end{aligned}$$

## 3. Factorization $1-1-\cdots-1$

In the case $s = n$, i.e., $n_\nu = 1$ ($\nu = 1, 2, \dots, n$), the factors are linear

$$A_\nu(z) = z + a_{\nu 0} = z - z_\nu \qquad (\nu = 1, 2, \dots, n),$$

and (2.4) reduces to

$$(z - z_1^{(k)})(z - z_2^{(k)}) \cdots (z - z_n^{(k)}) \left( \sum_{\nu=1}^{n} \frac{z - z_\nu^{(k+1)}}{z - z_\nu^{(k)}} - n + 1 \right) \equiv P(z).$$

Then, the scalar form of (2.6) can be obtained easily as

$$z_\nu^{(k+1)} = z_\nu^{(k)} - \frac{P(z_\nu^{(k)})}{\prod_{\substack{j=1 \\ j\neq\nu}}^{n} \left(z_\nu^{(k)} - z_j^{(k)}\right)} \qquad (\nu = 1,2,\dots,n;\ k = 0,1,\dots). \tag{3.1}$$

Thus, in this important case, S. Prešić's factorization approach leads to the Weierstrass' formulas (3.1) (see [44]), which were not well-known in that period. These formulas were obtained several times in various ways by many authors. Weierstrass used them in a new constructive proof of fundamental theorem of algebra. In a book on numerical solution of algebraic equations from 1960, written by French mathematician E. Durand [8], one chapter was dedicated to iterative methods for simultaneous finding polynomial zeros, where the author obtained formulas (3.1) in an implicit form. It seems that Bulgarian mathematician K. Dočev [7] was the first who used these formulas in their original form for numerical calculation and who proved their quadratic converegence.

Introducing $Q(z) = \prod_{j=1}^{n} (z - z_j^{(k)})$, formulas (3.1) can be represented in the form (Newtonian type)

$$z_\nu^{(k+1)} = z_\nu^{(k)} - \frac{P(z_\nu^{(k)})}{Q'(z_\nu^{(k)})} \qquad (\nu = 1,2,\dots,n;\ k = 0,1,\dots). \tag{3.2}$$

Beside the polynomial $Q(z)$ we consider also polynomials $R_\nu(z)$ defined by

$$R_\nu(z) = \frac{Q(z)}{z - z_\nu^{(k)}} = \prod_{\substack{j=1 \\ j\neq\nu}}^{n} \left(z - z_j^{(k)}\right) \qquad (\nu = 1,2,\dots,n).$$

Their expanded forms are

$$Q(z) = z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \dots + (-1)^n \sigma_n,$$

$$R_\nu(z) = z^{n-1} - \sigma_1^{(\nu)} z^{n-2} + \sigma_2^{(\nu)} z^{n-3} - \dots + (-1)^{n-1} \sigma_{n-1}^{(\nu)},$$

where $\sigma_1, \sigma_2, \dots, \sigma_n$ are elementary symmetric functions of $z_1, z_2, \dots, z_n$ (see [23, Section 1.3.1]). For the sake of simplicity, we omit the upper index in $z_\nu^{(k)}$, and for $z_\nu^{(k+1)}$ we use the notation $\hat{z}_\nu$. Similarly, $\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \dots, \sigma_{n-1}^{(\nu)}$ are also such functions that do not involve $z_\nu$. It is easy to see that $Q'(z_\nu) = R_\nu(z_\nu)$ ($\nu \in I = \{1,2,\dots,n\}$). In the note [43], which was our first paper in mathematics inspired only by the S. Prešić paper [35], we showed: *If all zeros of $Q(z)$ are simple, then the inverse matrix of*

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_1^{(2)} & & \sigma_1^{(n)} \\ \vdots & & & \\ \sigma_{n-1}^{(1)} & \sigma_{n-1}^{(2)} & & \sigma_{n-1}^{(n)} \end{bmatrix} \tag{3.3}$$

*is given by*

$$(3.4)\qquad W^{-1}=\begin{bmatrix} D_1z_1^{n-1} & -D_1z_1^{n-2} & \cdots & (-1)^{n-1}D_1 \\ D_2z_2^{n-1} & -D_2z_2^{n-2} & \cdots & (-1)^{n-1}D_2 \\ \vdots & & & \\ D_nz_n^{n-1} & -D_nz_n^{n-2} & \cdots & (-1)^{n-1}D_n \end{bmatrix},$$

*with* $D_\nu = 1/Q'(z_\nu)$ $(\nu \in I)$.

The corresponding S. Prešić's form (2.6), i.e., a vector form of (3.2) can be written as

$$(3.5)\qquad \boldsymbol{z}^{(k+1)} = T(\boldsymbol{z}^{(k)}) \qquad (k=0,1,\ldots),$$

where $T(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{e}(\boldsymbol{z})$ and

$$\boldsymbol{z}=\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix},\quad \boldsymbol{e}(\boldsymbol{z})=\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix},\quad e_\nu = \frac{P(z_\nu)}{Q'(z_\nu)},\quad Q(z)=\prod_{j=1}^{n}(z-z_j).$$

Taking the system of Viète's formulas for polynomial $P(z)$, given by (1.1), $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$, where the $i$-th coordinate in the vector $\boldsymbol{f}(\boldsymbol{z})$ is equal to $\sigma_i + (-1)^{i-1}p_i$ $(i = 1,2,\ldots,n)$, and applying the known iterative procedure of Newton-Kantorovič,

$$(3.5)\qquad \boldsymbol{z}^{(k+1)} = \boldsymbol{z}^{(k)} - W^{-1}(\boldsymbol{z}^{(k)})\boldsymbol{f}(\boldsymbol{z}^{(k)}) \qquad (k=0,1,\ldots),$$

in order to solve the previous system of nonlinear equations, we obtain (3.5). Here, the Jacobi matrix is exactly given by (3.3) and its inverse by (3.4). It seems that Kerner [16] was the first who observed this fact. His proof was slightly different from ours.

Regarding to the iterative method (3.2), in 1980 Dirk P. Laurie [18] stated the following problem: *If* $\sum_{\nu=1}^{n} z_\nu = -p_1$, *prove that*

$$(3.6)\qquad \sum_{\nu=1}^{n} \hat{z}_\nu = -p_1.$$

It is a nice property of the method (3.2) and it was known earlier (see Dočev [7]).

Relation (3.6) holds regardless of the value of $\sum_{\nu=1}^{n} z_\nu$. We gave now a proof of that as an application of the Cauchy residue method and it was published in the book [26, pp. 347–348]. Indeed, since $\hat{z}_\nu = z_\nu - P(z_\nu)/Q'(z_\nu)$, $\nu = 1,2,\ldots,n$, we have

$$(3.7)\qquad \sum_{\nu=1}^{n} \hat{z}_\nu = \sum_{\nu=1}^{n} z_\nu - \sum_{\nu=1}^{n} \frac{P(z_\nu)}{Q'(z_\nu)}.$$

No doubt that the S. Prešić's work on this area is very important and that it has a great influence on the development of this field in our country. In the last thirty years several mathematicians in Serbia, especially those from the University of Niš and University of Novi Sad, have been very active in this field. For the references see, for instance, [28] and [29].

## References

[1] G. Alefeld and J. Herzberger, *On the convergence speed of some algorithms for the simultaneous approximation of polynomial roots*, SIAM J. Numer. Anal. **2** (1974), 237–243.

[2] L. Bairstow, *Investigations relating to the stability of the aeroplane*, Rep. & Memo. **154**, Advisory Committee for Aeronautics, 1914.

[3] F.L. Bauer and K. Samelson, *Polynomkerne und Iterationsverfahren*, Math. Z. **67** (1957), 93–98.

[4] C. Carstensen, *On Grau's method for simultaneous factorization of polynomials*, SIAM J. Numer. Anal. **29** (1992), 601–613.

[5] C. Carstensen and T. Sakurai, *Simultaneous factorization of a polynomial by rational approximation*, J. Comput. Appl. Math. **61** (1995), 165–178.

[6] S.P. Chung, *Generaliztion and acceleration of an algorithm of Sebastião e Silva and its duals*, Numer. Math. **25** (1976), 365–377.

[7] K. Dočev, *Videoizmenen metod na Newton za edinovremenno priblizitel'no presmyatane na vsichki koreni na dadeno algebrichno uravnenie*, Fiz.-Mat. Spis. Bulgar. Akad. Nauk. **5** (2) (1962), 136–139.

[8] E. Durand, *Solution Numérique des Équations Algébraique* (tome 1), Masson et Compagnie, Paris, 1960.

[9] J. Dvorčuk, *Factorization of a polynomial into quadratic factors by Newton method*, Apl. Mat. **14** (1969), 54–80.

[10] A.A. Grau, *The simultaneous Newton improvement of a complete set of approximate factors of a polynomial*, SIAM J. Numer. Anal. **8** (1971), 425–438.

[11] A.S. Householder, *Generalizations of an algorithm of Sebastião e Silva*, Numer. Math. **13** (1971), 38–46.

[12] A.S. Householder, *Postscript to: "Generalizations of an algorithm of Sebastião e Silva"*, Numer. Math. **20** (1972/73), 205–207.

[13] A.S. Householder and G.W. Stewart, *Comments on "Some iterations for factoring polynomials"*, Numer. Math. **13** (1969), 470–471.

[14] A.S. Householder and G.W. Stewart, *The numerical factorization of a polynomial*, SIAM Rev. **13** (1971), 38–46.

[15] B. Jovanović, *A method for obtaining iterative formulas of higher order*, Mat. Vesnik **9** (1972), 365–369.

[16] I.O. Kerner, *Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynommen*, Numer. Math. **8** (1966), 290–294.

[17] N. Kjurkchiev, *Some remarks on Weierstrass root-finding method*, C.R. Acad. Bulgare Sci. **46** (8) (1993), 17–20.

[18] D.P. Laurie, *Problem H-315*, The Fibonacci Quarterly **18** (1980), 190.

[19] Shin-nge Lin, *A method of successive approximations of evaluating the real and complex roots of cubic and higher-order equations*, J. Math. Phys. Mass. Inst. Tech. **20** (1941), 231–242.

[20] Shin-nge Lin, *A method for finding roots of algebraic equations*, J. Math. Phys. Mass. Inst. Tech. **22** (1943), 60–77.

[21] D. Marković, *O približnoj faktorizaciji polinoma*, Vesnik Druš. Mat. Fiz. NRS, **8** (1955), 53–58.

[22] G.V. Milovanović, *A method to accelerate iterative processes in Banach space*, Univ. Beograd. Publ. Electrotehn. Fak. Ser. Mat. Fiz. No 461 – No 495 (1974), 67–71.

[23] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore – New Jersey – London –Hong Kong, 1994.

[24] G.V. Milovanović and M.S. Petković, *On the convergence order of a modified method for simultaneous finding polynomial zeros*, Computing **30** (1983), 171–178.

[25] G.V. Milovanović and M.S. Petković, *On computational effeciency of the iterative methods for the simultaneous approximation of polynomial zeros*, ACM Trans. Math. Software **12** (1986), 295–306.

[26] D.S. Mitrinović and J.D. Kečkić, *The Cauchy Method of Residues – Theory and Applications*, Mathematics and Applications (East Europian Series), Vol. 9, D. Reidel Publishing Company, Dordrecht – Boston – Lancaster, 1984.

[27] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[28] M.S. Petković, *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*, Springer-Verlag, Berlin, 1989.

[29] M.S. Petković, Đ. Herceg, S. Ilić, *Point Estimation Theory and its Applications*, Institute of Mathematics, Novi Sad, 1997.

[30] M.S. Petković and G.V. Milovanović, *A note on some improvements of the simultaneous methods for determination of polynomial zeros*, J. Comput. Appl. Math. **9** (1983), 65–69.

[31] M.S. Petković, G.V. Milovanović, L.V. Stefanović, *On some higher-order methods for the simultaneous approximation of multiple polynomial zeros*, Comput. Math. Appl. **9** (1) (1986), 951–962.

[32] J.J. Petrić and S.B. Prešić, *An algorithm for the solution of* $2 \times 2$ *system of nonlinear algebraic equations*, Publ. Inst. Math. (Beograd) (N.S.) **12** (26) (1971), 85–94.

[33] M. Prešić: *Jedan iterativni postupak za određivanje k korena polinoma*, Doktorska disertacija, Beograd, 1971.

[34] S.B. Prešić, *Un procédé iteratif pour la factorisation polynomes*, C. R. Acad. Sci. Paris **262** (1966), 862–863.

[35] S.B. Prešić, *Jedan iterativni postupak za faktorizaciju polinoma*, Mat. Vesnik **5** (20) (1968), 205–216.

[36] K. Samelson, *Factorisierung von Polynomes durch funktionale Iteration*, Bayer. Akad. Wiss. Math. Natur. Kl. Abh. **95** (1959), 1–26.

[37] J. Schröder, *Factorization of polynomials by generalized Newton procedures*, In: Constructive Aspects of the Fundamental Theorem of Algebra (B. Dejon and P. Henrici, eds.), John Wiley, New York, 1969, 295–320.

[38] J. Sebastião e Silva, *Sur une méthode d'approximation semblable à celle de Gräffe* Portugal. Math. **2** (1941), 271–279.

[39] G.W. Stewart, *Some iterations for factoring a polynomial*, Numer. Math. **13** (1969), 458–470.

[40] G.W. Stewart, *On Samelson's iteration for factoring polynomials*, Numer. Math. **15** (1970), 306–314.

[41] G.W. Stewart, *On the convergence of Sebastião e Silva's method for finding a zero of a polynomial*, SIAM Rev. **12** (1970), 458–460.

[42] K. Tanabe, *Behaviour of the sequences around multiple zeros generated by some simultaneous methods for solving algebrac equations*, Tech. Rep. Inf. Procces. Numer. Anal. **4-2** (1983), 1–6 (in Japanese).

[43] D.Đ. Tošić and G.V. Milovanović. *An application of Newton's method to simultaneous determination of zeros of a polynomial*, Univ. Beograd. Publ. Electrotehn. Fak. Ser. Mat. Fiz. No 412 – No 460 (1973), 175–177.

[44] K. Weierstrass, *Neuer Beweis des Satzes, dass jede ganze rational Function einer Veränderlichen dargestellt werden kann als ein Product aus Linearen Functionen darselben Veräderlichen*, In: Gesammelten Werke, Vol. 3 (1903), Johnson, New York, 1967, 251–269.

# Contents