Orthogonal polynomials relative to weight functions of Prudnikov type

Walter Gautschi* Gradimir V. Milovanović†

Abstract

Moment-based methods are used to generate the three-term recurrence relation for polynomials orthogonal with respect to the Prudnikov, the generalized Prudnikov, and Prudnikov-type weight functions and their symmetric extensions. All procedures developed are implemented, and made available, in Matlab software.

Keywords: Orthogonal polynomials, Prudnikov-type weight functions, three-term recurrence relation

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1 Introduction

Let

$$\rho_\nu(x) = 2\,x^{\nu/2}K_\nu(2\sqrt{x}), \quad x > 0, \quad \nu \in \mathbb{R},$$

where K_ν is the second-kind modified Bessel function of order ν ([6, §10.25]). *Prudnikov polynomials* [10, §3] are the polynomials orthogonal with respect to the weight function

$$w_\nu(x) = \rho_\nu(x), \quad 0 < x < \infty, \quad \nu \geq 0. \tag{1.1}$$

*Department of Computer Science, Purdue University, West Lafayette, IN 47907-2066, USA (wgautschi@purdue.edu)

†Serbian Academy of Sciences and Arts, 11000 Belgrade, Kneza Mihaila 35, Serbia and University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia (gvm@mi.sanu.ac.rs)

A. P. Prudnikov (1927–1999) advocated their study in [7, Problem 2] and dealt with the case $\nu = 0$ in [*ibid.*, Problem 1]. S. Yakubovich, in [10], considered also *generalized Prudnikov polynomials* orthogonal with respect to the weight function

$$w_\nu^\alpha(x) = x^\alpha \rho_\nu(x), \quad 0 < x < \infty, \quad \alpha > -1, \tag{1.2}$$

as well as *Prudnikov-type polynomials*, in [9, §2] of type 1, orthogonal with respect to

$$w_\nu^+(x) = \mathrm{e}^{-x} \rho_\nu(x), \quad 0 < x < \infty, \quad \nu > -1, \tag{1.3}$$

and in [9, §3] of type 2, orthogonal with respect to

$$w_\nu^-(x) = x^{-1} \mathrm{e}^{-1/x} \rho_\nu(x), \quad 0 < x < \infty, \quad \nu \in \mathbb{R}. \tag{1.4}$$

Symmetric extensions of all these polynomials will also be considered, where x in the weight function is replaced by $|x|$, and the support interval is extended to the whole real line.

Multiple orthogonal polynomials relative to the pair $x^\alpha \rho_\nu(x)$, $x^\alpha \rho_{\nu+1}(x)$ of weight functions have been studied in [8]; see also [8], [1], [11, §IIA].

The object of this note is to develop the respective orthogonal polynomials and their symmetric extensions, in particular, to obtain the three-term recurrence relations they satisfy and to provide related Matlab software. The approach used in all cases is the classical Chebyshev algorithm, computing the recurrence coefficients from the moments of the weight function. Because of the underlying ill-conditioning, high-precision computation is required.

Weight functions involving modified Bessel functions $K_\nu(x)$ (rather than $K_\nu(2\sqrt{x})$) have been used previously in connection with wave functions for nonlocal potentials [5]; see also [2, Exercise 2.32], [3, §2.1.3], [4, §2.1.3].

2 Moments

The nth-order moment of the generalized Prudnikov weight function is

$$\mu_n^{\nu,\alpha} = \int_0^\infty x^n w_\nu^\alpha(x) \mathrm{d}x = \Gamma(n+\alpha+\nu+1)\Gamma(n+\alpha+1), \quad \nu \geq 0, \quad \alpha > -1, \tag{2.1}$$

as follows from [10, Eq. (2.4)] where μ is replaced by $n+\alpha$. For the first Prudnikov-type weight function, the moment of order n is

$$\mu_n^+ = \int_0^\infty x^n w_\nu^+(x)\,\mathrm{d}x = \int_0^\infty x^n \mathrm{e}^{-x}\rho_\nu(x)\,\mathrm{d}x = 2\int_0^\infty x^{n+\nu/2}\mathrm{e}^{-x}K_\nu(2\sqrt{x})\,\mathrm{d}x, \qquad \nu > -1.$$

Using Mathematica 12.3, one finds (cf. also [9, Eq. (1.8)])

$$\mu_n^+ = n!\Gamma(n+\nu+1)U(n+\nu+1,\nu+1,1),$$

where $U(a,b,x)$ is the confluent hypergeometric function (also known as Tricomi's function) or, in terms of generalized hypergeometric functions, $U(a,b,x) = x^{-a}\,{}_2F_0(a,a-b+1;-;-x^{-1})$ [6, §13.6(vi)]. Thus,

$$\mu_n^+ = n!\Gamma(n+\nu+1)\,{}_2F_0(n+\nu+1,n+1;-;-1). \tag{2.2}$$

For the second Prudnikov-type weight function, the moments have been given in [9, Eq. (1.9)], though involving (in the last line of the equation) the gamma function at a nonpositive integer, that is, ∞. We have, however,

$$\mu_n^- = \int_0^\infty x^n w_\nu^-(x)\mathrm{d}x = 2\int_0^\infty x^{n+\nu/2-1}\mathrm{e}^{-1/x}K_\nu(2\sqrt{x})\mathrm{d}x, \quad \nu\in\mathbb{R}. \tag{2.3}$$

Here, the second integral can be expressed in terms of the Meijer G-function, for which we use here the Matlab notation on the right of

$$G_{p,q}^{m,n}\left(z;\ \begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\right) = \texttt{meijerG}\,(\,[a_1,\ldots,a_n],[a_{n+1},\ldots,a_p],[b_1,\ldots b_m],[b_{m+1},\ldots,b_q],z) \qquad 0\le n\le p,\ 0\le m\le q.$$

(The content between brackets may be empty. For example, if $p=0$, the first two arguments of `meijerG` are empty, or only the second one if $p=n$.)

To begin with, the term on the far right of (2.3) can be written as a Mellin transform of the function $f(x) = 2\,\mathrm{e}^{-1/x}K_\nu(2\sqrt{x})$,

$$\mu_n^- = (\mathcal{M}\,f)(s) = \int_0^\infty x^{s-1}\,f(x)\mathrm{d}x, \quad s = n+\nu/2.$$

Using the Mathematica 12.2 command

```
MellinTransform[2 BesselK[v,2Sqrt[x]]Exp[-1/x],x,s]
```

yields

(2.4)
$$(\mathcal{M}\,f)(s) = \frac{2^{2s-2}}{\pi^{3/2}}\,\texttt{meijerG}\,(\,[1/2, 1, (2-2s-\nu)/4, (4-2s-\nu)/4,$$
$$(2-2s+\nu)/4, (4-2s+\nu)/4], [\ ], [\ ], [\ ], 64).$$

Letting

$$a_1 = 1, \quad a_2 = 1-s-\nu/2, \quad a_3 = 1-s+\nu/2 \quad \text{and} \quad p = n = 3,\ q = m = 0,$$

the right-hand side of (2.4) is

$$\frac{2^{p+1-n-a_1-a_2-a_3}}{\pi^{n-p/2}}\,\texttt{meijerG}\,(\,[\,a_1/2, (a_1+1)/2, a_2/2, (a_2+1)/2, a_3/2, (a_3+1)/2],$$
$$[\ ], [\ ], [\ ], 4^p),$$

which, by [6, Eq. 16.19.4] and $s = n + \nu/2$, equals

$$\texttt{meijerG}\,(\,[a_1, a_2, a_3], [\ ], [\ ], [\ ], 1) = \texttt{meijerG}\,(\,[1, 1-(n+\nu), 1-n], [\ ], [\ ], [\ ], 1).$$

Thus, simplifying by employing [6, Eq. 16.19.1], we get

(2.5)
$$\mu_n^- = \texttt{meijerG}\,(\,[\ ], [\ ], [0, n+\nu, n], [\ ], 1).$$

The moments of the symmetric extension of all the weight functions above are twice the moments stated, if n is even, and zero if n is odd.

3 Orthogonal polynomials and recurrence coefficients

It is well known that (monic) orthogonal polynomials π_k relative to a positive weight function $w(x)$ on some finite or infinite interval $[a, b]$ satisfy a three-term recurrence relation

(3.1)
$$\pi_{k+1}(x) = (x-\alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0, 1, 2, \ldots,$$
$$\pi_{-1} = 0, \quad \pi_0 = 1,$$

where the coefficients α_k are real and $\beta_k > 0$. Conventionally, β_0 is taken to be $\beta_0 = \int_a^b w(x)\mathrm{d}x$. For any computational work with orthogonal polynomials, knowledge of this recurrence relation, that is, of its recurrence coefficients, is indispensable.

There is Matlab software available that generates to any given accuracy the first N recurrence coefficients α_k, β_k, $k = 0, 1, 2, \ldots, N-1$, and places them into the first, respectively second, column of an $N \times 2$ array `ab`; see, e.g., [2, §1]. The procedure used here is the Chebyshev algorithm, generating the first N recurrence coefficients from the first $2N$ moments of the weight function. Given the ill-conditioned nature of this proposition, it is important to know how many working digits are required to obtain all N coefficients β_k (and thus, presumably, also all $\alpha_k \neq 0$) to a given relative accuracy. For the three weight functions of Prudnikov type, this is answered by the Matlab routines `dig_gprudnikov.m`, `dig_prudnikov_type.m`, and for the respective symmetric extensions by `dig_gprudnikov_symm.m`, `dig_prudnikov_type_symm.m`. These routines not only provide the desired number `dig` of working digits, but also the respective array `ab` of recurrence coefficients to the accuracy requested. Once this number `dig` of required working digits is known, the array `ab` can be generated directly by the routines `sr_gprudnikov.m`, `sr_prudnikov_type.m`, `sr_gprudnikov_symm.m`, `sr_prudnikov_type_symm.m`.

For all Matlab routines needed, visit

`https://www.cs.purdue.edu/archives/2002/wxg/codes/PRUD.html`.

3.1 Generalized Prudnikov polynomials

Our target precision for the recurrence coefficients, in this and the next two subsections, is 15-digit accuracy. For generalized Prudnikov polynomials, the results of the routine `dig gprudnikov.m` are shown in Table 1 for selected values of ν and for $\alpha = \pm 1/2$. (Other values of $\alpha > -1$, including $\alpha = 0$, in the range from $-.9$ to 10.6, have led to basically the same results, except, occasionally, somewhat larger ones, but never by more than 3 units.) It can be seen that, for each N shown, the results are more or less the same, which means that the underlying conditioning is essentially independent of the parameters ν and α. The results, in fact, suggest condition numbers of the order $10^{10} - 10^{12}$ when $N = 25$, $10^{28} - 10^{30}$ when $N = 50$, and $10^{63} - 10^{67}$ when $N = 100$.

N	ν	α	dig	N	ν	α	dig	N	ν	α	dig
25	0	$-1/2$	25	50	0	$-1/2$	43	100	0	$-1/2$	78
		$1/2$	25			$1/2$	44			$1/2$	79
	$1/3$	$-1/2$	27		$1/3$	$-1/2$	45		$1/3$	$-1/2$	81
		$1/2$	27			$1/2$	45			$1/2$	82
	$2/3$	$-1/2$	27		$2/3$	$-1/2$	45		$2/3$	$-1/2$	81
		$1/2$	27			$1/2$	45			$1/2$	82
	1	$-1/2$	25		1	$-1/2$	43		1	$-1/2$	80
		$1/2$	25			$1/2$	43			$1/2$	80
	3	$-1/2$	25		3	$-1/2$	44		3	$-1/2$	80
		$1/2$	26			$1/2$	43			$1/2$	81
	6	$-1/2$	26		6	$-1/2$	44		6	$-1/2$	81
		$1/2$	26			$1/2$	43			$1/2$	81

Table 1: The number `dig` of digits required in the Chebyshev algorithm to obtain the first N recurrence coefficients of the generalized Prudnikov polynomials to an accuracy of 15 decimal digits

We used our routine `sr_gprudnikov(dig,nofdig,N,nu,alpha)` with `dig` =30, `nofdig`=18, `N`=11, `nu`=`alpha`=0, to check Table 9.1 in [7] containing the values of $a_n = \sqrt{\beta_n}$ for $n = 1, 2, \ldots, 10$. Agreement to all digits was observed except for the last digit, which occasionally is off by one unit.

3.2 Prudnikov-type polynomials of the first type

Here the results of the routine `dig_prudnikov_type.m` with `type`=1 (for the weight function w_ν^+) are shown in Table 2 for selected values of $\nu > -1$. As before in the case of generalized Prudnikov polynomials, the results are practically independent of ν.

The condition numbers for $N = 25, 50, 100$ are now about 10^{20}, 10^{40}, and 10^{85}, that is, substantially larger than in the case of generalized Prudnikov weight functions.

N	ν	dig	N	ν	dig	N	ν	dig
25	$-2/3$	34	50	$-2/3$	55	100	$-2/3$	102
	$-1/3$	35		$-1/3$	55		$-1/3$	102
	0	35		0	54		0	100
	$1/3$	35		$1/3$	55		$1/3$	102
	$2/3$	35		$2/3$	55		$2/3$	102
	1	35		1	54		1	100
	3	35		3	54		3	101
	6	35		6	54		6	101

Table 2: The number `dig` of digits required in the Chebyshev algorithm to obtain the first N recurrence coefficients of the Prudnikov-type polynomials of type 1 to an accuracy of 15 decimal digits

N	ν	dig	N	ν	dig	N	ν	dig
25	-6	23	50	-6	42	100	-6	79
	-3	25		-3	42		-3	79
	-1	25		-1	43		-1	79
	$-2/3$	26		$-2/3$	44		$-2/3$	81
	$-1/3$	26		$-1/3$	44		$-1/3$	81
	0	24		0	42		0	79
	$1/3$	26		$1/3$	44		$1/3$	81
	$2/3$	26		$2/3$	44		$2/3$	81
	1	24		1	43		1	79
	3	26		3	43		3	80
	6	25		6	44		6	80

Table 3: The number `dig` of digits required in the Chebyshev algorithm to obtain the first N recurrence coefficients of the Prudnikov polynomials of type 2 to an accuracy of 15 decimal digits

3.3 Prudnikov-type polynomials of the second type

Here the results of the routine `dig_prudnikov_type.m` with `type`=2 (for the weight function w_ν^-) are shown in Table 3 for selected values of $\nu \in \mathbb{R}$. They are quite similar to the ones in §3.1 where applicable, and so is the degree of ill-contitioning.

Knowing the degree of ill-conditioning, it is easy to estimate the number of digits needed to get any desired accuracy. Thus, for example, when $N = 100$, to get 32-digit accuracy will require something like 32+67=99 digits in the case of generalized Prudnikov weight functions and Prudnikov weight functions of type 2, and 32+87=119 digits for Prudnikov weight functions of type 1. Both these numbers have been corroborated numerically.

3.4 Polynomials orthogonal relative to the symmetric extension of weight functions of Prudnikov type

Symmetry usually lowers condition numbers. This is the case here, where, compared with the case of generalized Prudnikov polynomials in subsection 3.1, the number `dig` of required digits is now about one half of those in Table 1 when N is 25 and 50, and even somewhat smaller when $N = 100$. More specifically, `dig` is never greater than 16, 22, 35 for, respectively, $N = 25, 50, 100$ and parameters ν and α as in Table 1. Similarly, for symmetric Prudnikov-type polynomials of type 1, the largest numbers `dig` are 20, 27, 45 for, respectively $N = 25, 50, 100$ and for ν as in Table 2, and for symmetric Prudnikov-type polynomials of type 2 they are 15, 21, 34 for ν as in Table 3.

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