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ZERO DISTRIBUTION OF POLYNOMIALS ORTHOGONAL ON THE RADIAL RAYS IN THE COMPLEX PLANE*

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This paper is dedicated to Professor D. S. Mitrinović

Abstract. In this paper we continue our investigations on polynomials orthogonal on the radial rays in the complex plane introduced and discussed in [1–4]. Here, we study zero distribution of these polynomials and locate the regions in which these zeros are contained. We also analyze the cases when the zeros are on the rays. Several numerical examples are included.

1. Introduction

One of us (see [1–4]) defined a new inner product as follows: For chosen lengths and angles

$$l_s \in (0, +\infty], \quad \theta_s \in [0, 2\pi), \qquad s = 0, 1, \ldots , m-1,$$

let

$$(f,g) = \sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} f(z)\overline{g(z)}|w_s(z)|\, dz, \quad \varepsilon_s = e^{i\theta_s}, \tag{1.1}$$

where $|w_s(z)|$ is a weight function on the radial ray L_s which connects the origin $z = 0$ and the point $z_s = l_s\varepsilon_s$ $(0 \le s \le m-1)$. This can be rewritten in the form

$$(f,g) = \sum_{s=0}^{m-1} \int_0^{l_s} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}|w_s(x\varepsilon_s)|\, dx,$$

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or

$$(f,g)=\int_0^1\sum_{s=0}^{m-1} l_s f(l_s\varepsilon_s x)\overline{g(l_s\varepsilon_s x)}|w_s(l_s\varepsilon_s x)|\,dx. \tag{1.2}$$

Because of

$$\|f\|^2=(f,f)=\sum_{s=0}^{m-1}\int_0^{l_s}|f(x\varepsilon_s)|^2|w_s(x\varepsilon_s)|\,dx>0$$

except for $f(z)\equiv 0$, we conclude that this inner product is positive-definite. Thus, one can construct the corresponding *orthogonal polynomial sequence* $\{\pi_N(z)\}_{N=0}^{+\infty}$, for example by using Gram-Schmidt orthogonalizing process. It implies the existence and uniqueness of such polynomial sequence.

In papers [1–4], we discussed the recurrence relations, associated matrix polynomials for these sequences of the polynomials, and in some special cases, we found generating functions, differential equations, some representations and connections with some standard polynomials orthogonal on the real line.

If we rotate the whole figure of the rays, we can notice some interesting properties.

Theorem 1.1. *Let α be an angle in $(-\pi,\pi]$ and let the rays L_0, L_1, $\dots$, L_{m-1}, after a rotation for the angle α, become L_0^α, $L_1^\alpha,\dots,L_{m-1}^\alpha$, respectively. Then, the sequence $\{\pi_N^\alpha(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product*

$$\big(f,g\big)_\alpha=\sum_{s=0}^{m-1}e^{-i\alpha}\varepsilon_s^{-1}\int_{L_s^\alpha}f(z)\overline{g(z)}|w_s(ze^{-i\alpha})|\,dz$$

can be expressed by

$$\pi_N^\alpha(z)=\pi_N(ze^{-i\alpha}),$$

where the polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ are orthogonal with respect to (1.1).

Proof. Let $\{\pi_N(z)\}_{N=0}^{+\infty}$ be orthogonal polynomials with respect to (1.1). Then, we have

$$\begin{aligned}&\big(\pi_K(e^{-i\alpha}z),\pi_N(e^{-i\alpha}z)\big)_\alpha\\&\qquad=\sum_{s=0}^{m-1}e^{-i\alpha}\varepsilon_s^{-1}\int_{L_s^\alpha}\pi_K(e^{-i\alpha}z)\overline{\pi_N(e^{-i\alpha}z)}|w_s(ze^{-i\alpha})|\,dz.\end{aligned}$$

After changing variable $u = ze^{-i\alpha}$, we get

$$\big(\pi_K(ze^{-i\alpha}), \pi_N(ze^{-i\alpha})\big)_\alpha = \sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} \pi_K(u)\overline{\pi_N(u)}|w_s(u)|\,du,$$

i.e.,

$$\big(\pi_K(ze^{-i\alpha}), \pi_N(ze^{-i\alpha})\big)_\alpha = \big(\pi_K(u), \pi_N(u)\big).$$

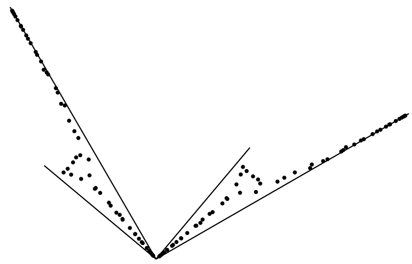
Because of the uniqueness of orthogonal polynomials (up to a multiplicative constant), we conclude that the statement is valid. □

Corollary 1.2. *The zeros of $\pi_N^\alpha(z)$ are obtained from the zeros of $\pi_N(z)$ by the rotation for the angle α.*

Proof. Let ζ be a zero of the polynomial $\pi_N(z)$, i.e., $\pi_N(\zeta) = 0$. According the previous theorem, we find

$$\pi_N^\alpha(e^{i\alpha}\zeta) = \pi_N(\zeta) = 0,$$

i.e, $\zeta e^{i\alpha}$ is a zero of the polynomial $\pi_N^\alpha(z)$. □



Zeros of polynomials $\pi_N(z)$ and $\pi_N^\alpha(z)$

Example 1.1. Consider two rays in the complex plane whose ends are at the points

$$z_0 = 3\,e^{i\pi/6} \quad \text{and} \quad z_1 = \frac{3}{2}\,e^{i5\pi/18} \quad \text{(see Figure 1.1)}.$$

The zeros of polynomials $\pi_N(z)$ orthogonal on these rays, with respect to the weights $w_s(z) = 1$ $(s = 0, 1)$, for $N = 1, 2, \ldots, 10$, are in the sector between these rays (see Figure 1.1). The case after a rotation of the rays for $\alpha = \pi/2$ is also presented in the same figure. The corresponding zeros of polynomials $\pi_N^\alpha(z)$, for $N = 1, 2, \ldots, 10$, can be obtained from the zeros of $\pi_N(z)$ by a rotation by the same angle $\alpha = \pi/2$.

The previous simple statements will be very useful in the next section.

2. Location of the Zeros

We start with a general case of orthogonality on the rays.

Theorem 2.1. *The polynomial* $\pi_N(z)$ $(N > 0)$ *orthogonal with respect to* (1.1) *has all zeros in the minimal rectangular cover of the radial rays with edges parallel with the coordinate axes,*

$$R = \{z \in \mathbb{C} : a_1 < \operatorname{Re}(z) < a_2 \ \wedge \ b_1 < \operatorname{Im}(z) < b_2\},$$

where

$$a_1 = \min_{\cos\theta_s \le 0} l_s \cos\theta_s, \quad a_2 = \max_{\cos\theta_s \ge 0} l_s \cos\theta_s,$$

and

$$b_1 = \min_{\sin\theta_s \le 0} l_s \sin\theta_s, \quad b_2 = \max_{\sin\theta_s \ge 0} l_s \sin\theta_s.$$

Proof. Suppose that ζ is a zero of $\pi_N(z)$. Then we can write

$$\pi_N(z) = (z-\zeta) r_{N-1}(z), \quad r_{N-1}(z) \in \mathcal{P}_{N-1}.$$

Because of the orthogonality, we have

$$0 = \big(\pi_N(z), r_{N-1}(z)\big) = \sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} (z-\zeta) r_{N-1}(z) \overline{r_{N-1}(z)} |w_s(z)|\, dz = 0,$$

i.e.,

$$\sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} (z-\zeta) |r_{N-1}(z)|^2 |w_s(z)|\, dz = 0.$$

Using notation as in (1.2), we yield

$$\int_0^1 \sum_{s=0}^{m-1} l_s (l_s\varepsilon_s x - \zeta) |r_{N-1}(l_s\varepsilon_s x)|^2 |w_s(l_s\varepsilon_s x)|\, dx = 0.$$

Since the real and imaginary part of the integral on the left must be equal to zero, we have

$$\int_0^1 \sum_{s=0}^{m-1} (x l_s \cos\theta_s - A) l_s |r_{N-1}(x l_s \varepsilon_s)|^2 |w_s(x l_s \varepsilon_s)|\, dx = 0$$

and

$$\int_0^1 \sum_{s=0}^{m-1} (x l_s \sin\theta_s - B) l_s |r_{N-1}(x l_s \varepsilon_s)|^2 |w_s(x l_s \varepsilon_s|\, dx = 0,$$

where $A = \mathrm{Re}(\zeta)$ and $B = \mathrm{Im}(\zeta)$. It means that the functions

$$F(x) = \sum_{s=0}^{m-1} (xl_s \cos\theta_s - A) l_s |r_{N-1}(xl_s\varepsilon_s)|^2 |w_s(xl_s\varepsilon_s)|$$

and

$$G(x) = \sum_{s=0}^{m-1} (xl_s \sin\theta_s - B) l_s |r_{N-1}(xl_s\varepsilon_s)|^2 |w_s(xl_s\varepsilon_s)|$$

change their signs in $x \in (0,1)$. In order to conclude that the zeros are in R, we suppose contrary.

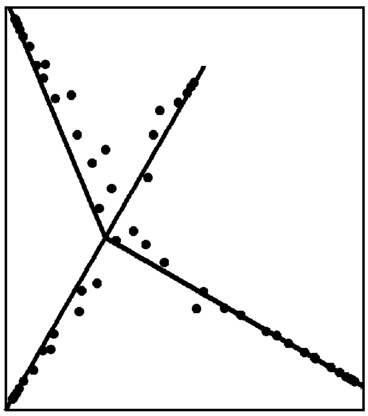
At first, we suppose that $A \geq a_2 = \max\{l_s \cos\theta_s : \ \cos\theta_s \geq 0\}$. Then, for every s and $x \in (0,1)$, we have $A \geq l_s \cos\theta_s > xl_s \cos\theta_s$. But, because of $A - xl_s \cos\theta_s x > 0$, we conclude that $F(x) < 0$, for $x \in (0,1)$, which gives a contradiction ($F(x)$ changes the sign in $(0,1)$). Thus, $A < a_2$.

In the same way we prove $A > a_1$ and $b_1 < B < b_2$. $\square$

Example 2.1. Consider a case with four rays determined by

$$z_0 = 2e^{i\pi/3}, \quad z_1 = \frac{5}{2}e^{i5\pi/18}, \quad z_2 = 2e^{i4\pi/3}, \quad z_3 = 3e^{i11\pi/6}.$$

The zeros of polynomials $\pi_N(z)$ orthogonal on these rays, with respect to the weights $w_s(z) = 1$ $(s = 0,1,2,3)$, for $N = 1,2,\ldots,10$, are in a rectangular cover (see Figure 2.1).



Zeros of polynomials $\pi_N(z)$, for $N = 1,2,\ldots,10$.

Theorem 2.2. *Let $z_0 = l_0 e^{i\theta_0}$, $z_1 = l_1 e^{i\theta_1}$ be the endpoints of the radial rays L_0 i L_1, respectively, and let*

$$(2.1) \qquad (f, g) = e^{-i\theta_0} \int_{L_0} f(z)\overline{g(z)}|w_0(z)|dz + e^{-i\theta_1} \int_{L_1} f(z)\overline{g(z)}|w_1(z)|dz,$$

where $|w_0(z)|$ and $|w_1(z)|$ are weight functions on these rays. Then all zeros of the polynomial $\pi_N(z)$, orthogonal with respect to the inner product (2.1), *are in the triangle Oz_0z_1.*

Proof. Using notation from Corollary 1.2 we find three rectangles which contain all zeros of $\pi_N(z)$.

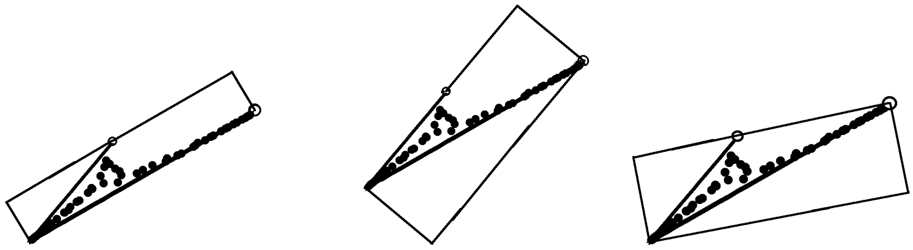
(a) If we rotate the whole figure of the rays L_0 and L_1 for the angle α_0, such that $z_0^{(\alpha_0)}$ belongs to the positive part of the real axes. Now, all zeros of the polynomial $\pi_N^{(\alpha_0)}(z)$ orthogonal on the new rays $L_0^{(\alpha_0)}$, $L_1^{(\alpha_0)}$ belong to the minimal rectangular cover of the rays $P_0^{(\alpha_0)}$, whose one of the edges contains $L_0^{(\alpha_0)}$.

(b) If we rotate the whole figure of the rays L_0 and L_1 for the angle α_1, such that $z_1^{(\alpha_1)}$ belongs to the positive part of the real axes. Now, all zeros of the polynomial $\pi_N^{(\alpha_1)}(z)$ orthogonal on the new rays $L_0^{(\alpha_1)}$, $L_1^{(\alpha_1)}$ belong to the minimal rectangular cover of the rays $P_1^{(\alpha_1)}$, whose one of the edges contains $L_1^{(\alpha_1)}$.

(c) Finally, if we rotate the whole figure of the rays L_0 and L_1 for the angle α_2, such that $z_0^{(\alpha_2)}z_1^{(\alpha_2)}$ belongs to the line parallel to the real axes. Now, all zeros of the polynomial $\pi_N^{(\alpha_2)}(z)$ orthogonal on the new rays $L_0^{(\alpha_2)}$, $L_1^{(\alpha_2)}$ belong to the minimal rectangular cover of the rays $P_2^{(\alpha_2)}$, whose one of the edges contains $z_0^{(\alpha_2)}z_1^{(\alpha_2)}$.

Rotating figures obtained in (1), (2) and (3) back to the starting position, we rotate also the zeros of the polynomials $\pi_N^{(\alpha_k)}(z)$, $k = 0, 1, 2$, and their covers to the corresponding rectangles P_0, P_1, P_2, which contain all zeros of the polynomial $\pi_N(z)$. An intersection of these rectangles is the interior of the triangle Oz_0z_1. $\square$

Example 2.2. We consider again the case of rays from Example 1.1. The rectangles P_0, P_1, P_2, mentioned in the proof of Theorem 2.2, are presented in Figures 2.2a, 2.2b, and 2.2c.



2.2a2.2b2.2c

Theorem 2.3. *All zeros of the polynomial* $\pi_N(z),\ N \in \mathbb{N}$, *orthogonal with respect to* (1.1) *lie in the convex hull which contains the rays.*

Proof. Denote the rays by $L_k,\ k = 0, 1, \dots, m-1$ and their endpoints by $z_k,\ k = 0, 1, \dots, m-1$.

Let L_{k_0} be the longest ray. If exists, the point z_{k_1} $(k_1 > k_0)$ such that the line $z_{k_0} z_{k_1}$ does not have an intersection with any ray from the set $L_j, j = 0, 1, \dots, m-1$, we rotate the whole ray-figure for the angle α_0 such that the line $z_{k_0}^{(\alpha_0)} z_{k_1}^{(\alpha_0)}$ is parallel with the real axes. Then, all zeros of the polynomial $\pi_N^{(\alpha_0)}(z)$ lie in rectangle $P_0^{(\alpha_0)}$ whose one edge contains $z_{k_0}^{(\alpha_0)} z_{k_1}^{(\alpha_0)}$.

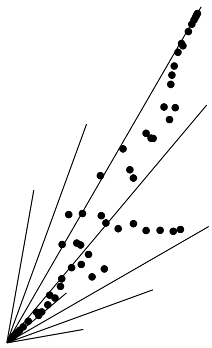
Also, if exists the endpoint z_{k_2} $(k_2 < k_0)$ such that the line $z_{k_0} z_{k_2}$ does not have an intersection with any ray from the set $L_j, j = 0, 1, \dots, m-1$, we rotate the whole ray-figure for the angle α_1 such that the line $z_{k_0}^{(\alpha_1)} z_{k_2}^{(\alpha_1)}$ be parallel with the real axes. Then, all zeros of the polynomial $\pi_N^{(\alpha_1)}(z)$ lie in rectangle $P_1^{(\alpha_1)}$ whose one edge contains $z_{k_0}^{(\alpha_1)} z_{k_2}^{(\alpha_1)}$.

We repeat the whole procedure for the points z_{k_1} and z_{k_2}, and the next points, respectively. So, we find rectangles $P_j^{(\alpha_j)},\ j = 0, 1, \dots$, which contain all zeros of the polynomials $\pi_N^{(\alpha_j)}(z),\ j = 0, 1, \dots$. By a rotation to the starting position of the figure, we yield the rectangles $P_j,\ j = 0, 1, 2, \dots$, which contain all zeros of the polynomial $\pi_N(z)$. Their intersection is a convex hull, containing all zeros of $\pi_N(z)$. The vertices of this convex hull are the tops of those rays which hold on its convexity. $\square$

Example 2.3. Consider the case of the radial rays L_s determined the points $z_s = l_s e^{i\theta_s}$ $(s = 0, 1, \dots 7)$, where

l_s	1	2	3	1	4	5	3	2
θ_s	$\pi/18$	$\pi/9$	$\pi/6$	$2\pi/9$	$5\pi/18$	$\pi/3$	$7\pi/18$	$4\pi/9$

Some zeros of the polynomials $\pi_N(z)$, $N = 1, 2, \ldots, 10$, are out of the regions which are determined by tops of successive rays (see Figure 2.3). But, all zeros lie in the convex hull over the rays.



We consider now the case of m rays on the real line determined by

$$z_s = l_s \quad (s = 0, 1, \ldots, \nu - 1) \quad \text{and} \quad z_s = -l_s \quad (s = \nu, \ldots, m - 1),$$

where $0 < \nu < m - 1$. Let $w(x)$ be a weight function on the real line. The polynomials $\pi_N(x)$ orthogonal with respect to the inner product

$$(f, g) = \sum_{s=0}^{\nu-1} \int_0^{l_s} f(x)\overline{g(x)}w(x)\, dx + \sum_{s=\nu}^{m-1} \int_{-l_s}^{0} f(x)\overline{g(x)}w(x)\, dx$$

can be treated in an usual sense. Namely, taking the intervals on the real line $\mathbb{R}$,

$$\Delta_s = [0, l_s] \quad (s = 0, 1, \ldots, \nu - 1) \quad \text{and} \quad \Delta_s = [-l_s, 0] \quad (s = \nu, \ldots, m - 1),$$

and the characteristic function of a set Δ_s, defined by

$$\chi(\Delta_s; t) = \begin{cases} 1, & t \in \Delta_s, \\ 0, & t \notin \Delta_s, \end{cases}$$

the previous inner product can be represented as

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, \Omega(x)\, dx,$$

where

$$\Omega(x) = w(x) \sum_{s=0}^{m-1} \chi(\Delta_s; x).$$

It is easy to see that polynomials $\pi_N(x)$ has all zeros in the interval (a, b), where

$$a = -\max_{\nu \le s \le m-1} l_s, \qquad b = \max_{0 \le s \le \nu-1} l_s.$$

Remark 2.1. Using the characteristic function of a set, the case of several rays with the same angle can be replaced only with one ray by the same angle.

3. The Case of Equidistant Angles

In this section, we study the case of equal angles between successive rays, i.e.,

$$\varepsilon_s = e^{i2\pi s/m}, \quad s = 0, 1, \dots, m-1,$$

with an inner product defined by

$$(f, g) = \sum_{s=0}^{m-1} l_s \int_0^1 f(xl_s\varepsilon_s)\overline{g(xl_s\varepsilon_s)}|w_s(xl_s\varepsilon_s)|\, dx. \tag{3.1}$$

Theorem 3.1. *Let $m = pq \quad (p, q \in \mathbb{N})$ and let the lengths and the weights of the radial rays be p-periodical, i.e.,*

$$l_{s+kp} = l_s, \qquad |w_{s+kp}(xl_{s+kp}\varepsilon_{s+kp})| = |w_s(xl_s\varepsilon_s)|, \tag{3.2}$$

where $0 \le s \le p-1$, $k = 1, 2, \dots, q-1$. Then the polynomial $\pi_N(z)$ $(N > 0)$ has the property

$$\pi_N(z\varepsilon_p) = \varepsilon_p^N \pi_N(z), \quad N = 0, 1, \dots .$$

Proof. Let $\pi_N(z)$ be the (monic) polynomial of degree N orthogonal with respect to the inner product (3.1). Then

$$(\pi_N, g) = \sum_{s=0}^{m-1} l_s \int_0^1 \pi_N(xl_s\varepsilon_s)\overline{g(xl_s\varepsilon_s)}|w_s(xl_s\varepsilon_s)|dx = 0,$$

for any $g(z) \in \mathcal{P}_{N-1}$.

For a given p, we put $Q_{N,p}(z) = \varepsilon_p^{-N}\pi_N(z\varepsilon_p)$ and $H_p(z) = g(z\varepsilon_p)$. Evidently, the polynomial $Q_{N,p}(z)$ is monic. We have

$$\begin{aligned}
(Q_{N,p}, H_p) &= \sum_{s=0}^{m-1} l_s \int_0^1 Q_{N,p}(xl_s\varepsilon_s)\overline{H_p(xl_s\varepsilon_s)}|w_s(xl_s\varepsilon_s)|\,dx \\
&= \sum_{s=0}^{m-1} l_s \int_0^1 \varepsilon_p^{-N}\pi_N(xl_s\varepsilon_p\varepsilon_s)\overline{g(xl_s\varepsilon_s\varepsilon_p)}|w_s(xl_s\varepsilon_s)|\,dx \\
&= \varepsilon_p^{-N} \sum_{k=p}^{m-1+p} l_k \int_0^1 \pi_N(xl_k\varepsilon_k)\overline{g(xl_k\varepsilon_k)}|w_s(xl_s\varepsilon_s)|\,dx \\
&= \varepsilon_p^{-N}(\pi_N, g) = 0,
\end{aligned}$$

because of $\varepsilon_{m+k} = \varepsilon_k$.

Since $H_p(z)$ can be every polynomial in $\mathcal{P}_{N-1}$, we conclude that $Q_{N,p}(z)$ is an orthogonal (monic) polynomial with respect to (3.1). Finally, from the uniqueness of $\pi_N(z)$ it follows that $\varepsilon_p^{-N}\pi_N(z\varepsilon_p) = \pi_N(z)$. $\square$

Theorem 3.2. *Under the assumptions* (3.2)*, if* ξ *is a zero of* $\pi_N(z)$*, then its zeros are also* $\xi\varepsilon_{kp}$*,* $k = 1, 2, \ldots, q-1$.

Example 3.1. Consider a case of $m = 16$ radial rays $L_s\colon z_s = l_s e^{i\theta_s}$ $(s = 0, 1, \ldots, 15)$ in the complex plane, whose lengths are given by $l_{4k+\nu} = 2\nu + 1$, $\nu, k = 0, 1, 2, 3$, and arguments by $\theta_s = s\pi/8$, $s = 0, 1, \ldots, 15$ (Figure 3.1).

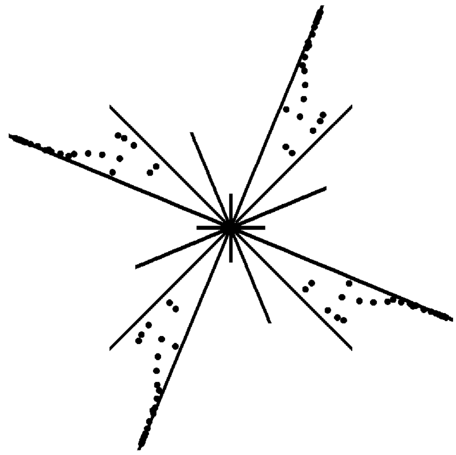
As we can see, the complete figure can be obtained by using only 4 successive rays. Namely, such a sub-figure constituted by 4 rays, should be rotated 3 times by the angles $k\pi/2$, $k = 1, 2, 3$. Also, we see that for a zero in the first sub-figure there exist the corresponding zeros in the other sub-figures obtained by rotations.

Now, we discuss the cases when the zeros stay on the rays.

Theorem 3.3. *Let the conditions* (3.2) *hold for* $p = 1$ *or* $p = 2$*. Then the polynomial* $\pi_N(z)$ $(N > 0)$ *orthogonal on the radial rays* L_s *with respect to the inner product* (1.2) *has all zeros on the rays* L_s $(s = 0, 1, \ldots, m-1)$.

Proof. Let $\zeta_0 = \rho e^{i\alpha}$ be a zero of $\pi_N(z)$. According to the previous theorem, its zeros are also $\zeta_k = \zeta_0\varepsilon_{kp}$, $k = 1, \ldots, q-1$, where $q = m/p$. Hence

$$\prod_{k=0}^{q-1}(z - \zeta_k) = z^q - \zeta_0^q.$$



Zeros of polynomials $\pi_N(z)$ orthogonal on equidistant rays

Then the polynomial $\pi_N(z)$ we can write in the form

$$\pi_N(z) = (z^q - \zeta_0^q) r_{N-q}(z),$$

where $r_{N-q}(z)$ is a polynomial from $\mathcal{P}_{N-q}$. Because of orthogonality, we have

$$0 = (\pi_N, r_{N-q}) = \sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} (z^q - \zeta_0^q) r_{N-q}(z) \overline{r_{N-q}(z)} |w_s(z)|\, dz,$$

i.e.,

$$0 = \sum_{s=0}^{m-1} l_s \int_0^1 \left[\left(x l_s \varepsilon_s \right)^q - \zeta_0^q \right] |r_{N-q}(x l_s \varepsilon_s)|^2 |w_s(x l_s \varepsilon_s)| dx.$$

The real and imaginary part of the previous integral become

$$\sum_{s=0}^{m-1} l_s \int_0^1 \Big[(x l_s)^q \cos\big(\frac{2\pi}{m} sq\big) - \rho^q \cos(\alpha q) \Big] |r_{N-q}(x l_s \varepsilon_s)|^2 |w_s(x l_s \varepsilon_s)|\, dx = 0$$

and

$$\sum_{s=0}^{m-1} l_s \int_0^1 \Big[(x l_s)^q \sin\big(\frac{2\pi}{m} sq\big) - \rho^q \sin(\alpha q) \Big] |r_{N-q}(x l_s \varepsilon_s)|^2 |w_s(x l_s \varepsilon_s)|\, dx = 0,$$

respectively. Now, we consider two cases: $p = 1$ and $p = 2$.

CASE $p = 1$. The previous relations become

$$\sum_{s=0}^{m-1} l_s \int_0^1 \Big((xl_s)^m - \rho^m \cos(\alpha m) \Big) |r_{N-m}(xl_s\varepsilon_s)|^2 |w_s(xl_s\varepsilon_s)| \, dx = 0$$

and

$$-\rho^m \sin(\alpha m) \sum_{s=0}^{m-1} l_s \int_0^1 |r_{N-m}(xl_s\varepsilon_s)|^2 |w_s(xl_s\varepsilon_s)| \, dx = 0.$$

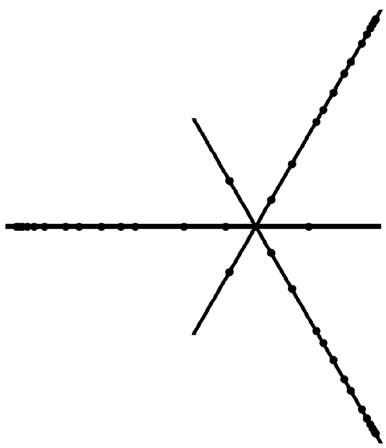
From the second one we find $\alpha = \pi\nu/m$, $\nu \in \mathbb{N}_0$. But, if ν is an odd number, the first relation reduces to

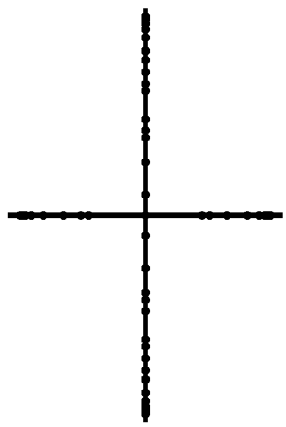
$$\sum_{s=0}^{m-1} l_s \int_0^1 \Big((xl_s)^m + \rho^m \Big) |r_{N-m}(xl_s\varepsilon_s)|^2 |w_s(xl_s\varepsilon_s)| \, dx = 0,$$

what is impossible because of positivity of the integrand. Thus, ν must be an even number ($\nu = 2s$), so that $\alpha = 2\pi s/m$, $s \in \mathbb{N}_0$.

CASE $p = 2$. The second relation immediately gives $\sin(\alpha m/2) = 0$, i.e., $\alpha = 2\pi s/m$, $s \in \mathbb{N}_0$.

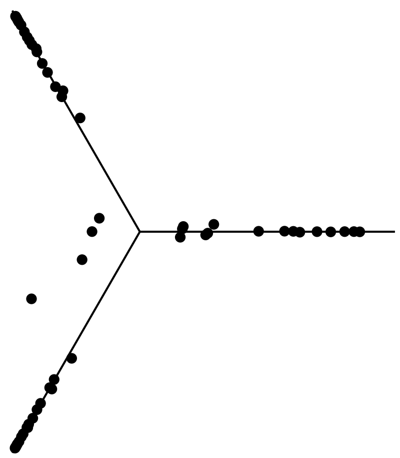
Thus, we can conclude that an arbitrary zero $\zeta_0 = \rho e^{i\alpha}$ of $\pi_N(z)$ may have the argument α from the set of the arguments of the rays only. Also, using Theorem 2.3 on the convex hull of the zeros, we conclude that the zeros of $\pi_N(z)$ are on the rays. $\square$





Example 3.2. Consider radial rays in the complex plane determined by $z_s = l_s e^{i\theta_s}$, $s = 0, 1, \dots 5$, where $l_{2s} = 1$, $l_{2s+1} = 2$, $s = 0, 1, 2$, and $\theta_s = s\pi/3$, $s = 0, 1, \dots 5$. Let $w(z) \equiv 1$. Zeros of the corresponding orthogonal polynomials $\pi_N(z)$, $N = 1, 2, \dots, 10$, are on the radial rays (see Figure 3.2).

Example 3.3. Let again $w(z) \equiv 1$. The zeros of polynomials $\pi_N(z)$, $N = 1, 2, \dots, 10$, orthogonal on the radial rays in the complex plane, whose ends are at the points $z_s = l_s e^{i\theta_s}$ $(s = 0, 1, 2, 3$, where $l_0 = l_2 = 3$, $l_1 = l_3 = 2$, and $\theta_s = \pi s/2$, $s = 1, 2, 3, 4$, are also on the rays (see Figure 3.3).



Some zeros of $\pi_N(z)$ are outside the rays

Example 3.4. Now, we consider three equidistant rays $L_s\colon z_s = e^{2is\pi/3}$, $s = 0, 1, 2$, but with different weights

$$|w_s(x\varepsilon_s)| = x^{\alpha_s}(1-x)^{\beta_s}, \qquad s = 0, 1, 2,$$

where

$$(\alpha_0, \beta_0) = (1, 2), \quad (\alpha_1, \beta_1) = (3, 4), \quad (\alpha_2, \beta_2) = (5, 6).$$

The zeros of polynomials $\pi_N(z)$, $N = 1, 2, \dots, 11$, orthogonal on these radial rays are displayed in Figure 3.4. Evidently, in this case, there are zeros outside the rays.

4. A Complete Symmetric Case

Now we consider a complete symmetric case, i.e., when $l_s = l$, $\varepsilon_s = e^{i2\pi s/m}$, $0 \le s \le m-1$, and

$$|w_s(x\varepsilon_s)| = w(x), \quad x \in (0,l),\ s = 0,1,\dots,m-1.$$

Then the inner product (1.1) reduces to

$$(f,g) = \int_0^l \left(\sum_{s=0}^{m-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\,dx. \tag{4.1}$$

For the zeros of $\pi_N(z)$ we can prove:

Theorem 4.1. *All zeros of the polynomial $\pi_N(z)$, orthogonal with respect to* (4.1)*, are simple and located on the radial rays, with possible exception of a multiple zero in origin $z = 0$ of the order ν $(0 < \nu < m)$, if $N \equiv \nu \pmod m$.*

Proof. In Theorem 3.3, we proved that the zeros of $\pi_N(z)$ are on the rays. In the paper [1], it was proved that the polynomial $\pi_N(z)$ can be expressed in the form $\pi_N(z) = z^\nu q_n^{(\nu)}(z^m)$, $\nu \in \{0,1,\dots,m-1\}$, where $q_n^{(\nu)}(t)$ is orthogonal on $(0,l^m)$ with respect to a positive weight. It is well known that the zeros of $q_n^{(\nu)}(t)$ are real and distinct and are located in $(0,l^m)$. Let $\tau_k^{(n,\nu)}$, $k = 1,\dots,n$, denote the zeros of $q_n^{(\nu)}(t)$ in an increasing order

$$\tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)}.$$

Each zero $\tau_k^{(n,\nu)}$ generates m zeros

$$z_{k,s}^{(n,\nu)} = \sqrt[m]{\tau_k^{(n,\nu)}}\, e^{i2\pi s/m}, \quad s = 0,1,\dots,m-1,$$

of $\pi_N(z)$. On every ray we have

$$|z_{1,s}^{(n,\nu)}| < |z_{2,s}^{(n,\nu)}| < \cdots < |z_{n,s}^{(n,\nu)}|, \quad s = 0,1,\dots,m-1.$$

Also, $\pi_{N+1}(z)$ and $\pi_N(z)$ separate their zeros on the rays. $\square$

5. Numerical Calculations

In the previous examples we used a method for numerical determination of polynomial zeros. The method was based on the following facts.

Let $\{\pi_N(z)\}_{N=0}^{+\infty}$ be a sequence of orthogonal polynomials with respect to the inner product (1.2). Such polynomials can be expressed by linear relations

$$z\pi_k(z) = \sum_{j=0}^{k} \beta_{kj}\pi_j(z) + \pi_{k+1}(z), \qquad k = 0, 1, \ldots, N-1,$$

where

$$\beta_{kj} = \frac{(z\pi_k, \pi_j)}{(\pi_j, \pi_j)}.$$

In the matrix form, it can be represented as

$$z\boldsymbol{\pi}_N(z) = B_N \boldsymbol{\pi}_N(z) + \pi_N(z)\boldsymbol{e}_N, \tag{5.1}$$

where

$$B_N = \begin{bmatrix} \beta_{00} & 1 & 0 & \cdots & 0 \\ \beta_{10} & \beta_{11} & 1 & & 0 \\ \vdots & & \ddots & & \\ \beta_{N-1,0} & \beta_{N-1,1} & & & \beta_{N-1,N-1} \end{bmatrix} \tag{5.2}$$

and

$$\boldsymbol{\pi}_N(z) = [\pi_0(z),\, \pi_1(z),\, \ldots,\, \pi_{N-1}(z)]^T, \qquad \boldsymbol{e}_N = [0,\, 0,\, \ldots,\, 0,\, 1]^T.$$

For the zeros ξ_j $(0 \le j \le n)$ of $\pi_N(z)$, (5.1) reduces to the eigenvalue problem

$$\xi_j \boldsymbol{\pi}_N(\xi_j) = B_N \boldsymbol{\pi}_N(\xi_j).$$

Thus, ξ_j are eigenvalues of the matrix B_N and $\boldsymbol{\pi}_N(\xi_j)$ are the corresponding eigenvectors.

In numerical evaluation of the inner product (1.2) we use, in general case, m Gaussian n-point quadrature rules with respect to the weight functions $\Omega_s(x) = |w(l_s\varepsilon_s x)|$ on $(0,1)$, i.e.,

$$\int_0^1 F(x)\Omega_s(x)\,dx \cong \sum_{i=1}^{n} W_i^{(s)} F\big(x_i^{(s)}\big) \qquad (s = 0, 1, \ldots, m-1),$$

in order to get a discretized approximation of this product. In this way, we obtain

$$
\begin{aligned}
(5.3) \qquad (f,g) &= \int_0^1 \sum_{s=0}^{m-1} l_s f(l_s\varepsilon_s x)\overline{g(l_s\varepsilon_s x)}\,\Omega_s(x)\,dx \\
&\cong \sum_{s=0}^{m-1} l_s \sum_{i=1}^{n} W_i^{(s)} f\big(l_s\varepsilon_s x_i^{(s)}\big)\overline{g\big(l_s\varepsilon_s x_i^{(s)}\big)}.
\end{aligned}
$$

The number of nodes n should be taken so that the elements β_{kj} in the matrix (5.2) can be computed exactly, except for rounding errors. For that, it is enough to take $n = N$.

Of course, in the simplest case (Legendre case) we take only one Gaussian formula, i.e., Gauss-Legendre rule on $(0,1)$. Then (5.3) becomes

$$
(f,g) \cong \sum_{i=1}^{n} W_i \sum_{s=0}^{m-1} l_s f(l_s\varepsilon_s x_i)\overline{g(l_s\varepsilon_s x_i)},
$$

where x_i and W_i $(i = 1,\ldots,n)$ are Gauss-Legendre nodes and weights on $(0,1)$, respectively.

Finally, for computing the eigenvalues of the upper Hessenberg matrix B_N^T (zeros of $\pi_N(z)$) we use the EISPACK routine COMQR [5, pp. 277–284].

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