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Quadrature formulae with multiple nodes and a maximal trigonometric degree of exactness

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In this paper we consider interpolatory quadrature formulae with multiple nodes that have the maximal trigonometric degree of exactness.

1 Introduction

Let a function w be integrable and nonnegative on the interval $(-\pi,\pi)$, vanishing there only on a set of a measure zero. We want to construct a quadrature formula of the type

$$\int_{-\pi}^{\pi} f(x)w(x)\,dx = \sum_{\nu=0}^{2n}\sum_{j=0}^{2s} A_{j,\nu} f^{(j)}(x_\nu) + R(f), \tag{1}$$

where s is nonnegative integer, which is exact when f is a trigonometric polynomial of degree less than or equal to $d = (2n+1)(s+1)-1$. Thus, d is the maximal trigonometric degree of exactness for quadrature formula (1). Obviously, for $s=0$ we obtain a quadrature formula with simple nodes and maximal trigonometric degree of exactness equals to $2n$. Firstly, such quadratures were considered by Turetzkii [7]. Nodes of the such quadrature formula are zeros from $[-\pi,\pi)$ of the trigonometric polynomial of semi–integer degree $n+1/2$

$$\sum_{\nu=0}^{n}\left(c_\nu \cos\left(\nu+\frac{1}{2}\right)x + d_\nu \sin\left(\nu+\frac{1}{2}\right)x\right), \quad |c_n|+|d_n|\neq 0,\ c_\nu, d_\nu \in \mathbb{R},\ \nu=0,1,\ldots,n,$$

which is orthogonal on $(-\pi,\pi)$ with respect to the weight function w to every trigonometric polynomial of semi–integer degree less than or equal to $n-1/2$. Such orthogonal trigonometric polynomials of semi–integer degree were considered in [7] and [3]. Numerical methods for construction of orthogonal trigonometric polynomials of semi–integer degree and the corresponding quadrature formulae with maximal trigonometric degree of exactness (Gaussian quadratures for trigonometric polynomials) were given in [3] (see also [4] and [6] for some special weights). In [5] s-orthogonal trigonometric polynomials of semi-integer degree $n+1/2$ with respect to the weight function w on $(-\pi,\pi)$ were defined and considered. It was proved that they have exactly $2n+1$ distinct simple zeros on $[-\pi,\pi)$.

Quadrature formulae of the form (1) for the constant weight function $w(x)=1$, $x\in(-\pi,\pi)$, were considered in [2]. In this paper we generalize that result considering the quadrature formula (1) for an arbitrary weight function w. Also, we give a numerical example.

2 Existence, uniqueness and constructions of quadrature formulae

Our approach is based on a procedure given by Ghizzeti and Ossicini in [2]. We consider the following boundary differential problem

$$E(f)=0, \quad f^{(j)}(x_\nu)=0, \quad j=0,1,\ldots,2s,\ \nu=0,\ldots,2n, \quad f\in AC^{N-1}[-\pi,\pi], \tag{2}$$

where $N=(2n+1)(2s+2)-1$ and E is the following differential operator of order N: $E = \dfrac{d}{dx}\displaystyle\prod_{k=1}^{(2n+1)(s+1)-1}\left(\frac{d^2}{dx^2}+k^2\right)$.

In the case $n>0$, $s\geq 0$, the boundary problem (2) has the following $2n$ linearly independent non trivial solutions (see [2, p. 139]): $U_\ell(x) = \left(\prod_{\nu=0}^{2n}\sin\frac{x-x_\nu}{2}\right)^{2s+1}\cos\left(\ell+\frac{1}{2}\right)x$, $V_\ell(x) = \left(\prod_{\nu=0}^{2n}\sin\frac{x-x_\nu}{2}\right)^{2s+1}\sin\left(\ell+\frac{1}{2}\right)x$, for $\ell=0,1,\ldots,n-1$.

Therefore, according to [2, Theorem 2.5.I], we can prove the following theorem.

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Theorem 2.1 *The quadrature formula* (1) *may hold if and only if the nodes* $x_0, \ldots, x_{2n}$ *are zeros of* $s-$*orthogonal trigonometric polynomial of semi-integer degree* $n + 1/2$ *with respect to the weight function* w *on* $(-\pi, \pi)$.

Using notations given in [2, Theorem 2.5.I] we have $m(N-p)-N+q = (2n+1)(2s+1)-((2n+1)(2s+2)-1)+2n = 0$. Therefore, if one of nodes is fixed, the quadrature formula (1) is unique. So, we can fixed one of the nodes in advance, e.g., $x_0 = -\pi$. Then, using a trigonometric interpolation polynomial, $s-$orthogonality conditions lead to the following system of nonlinear equations

$$\int_{-\pi}^{\pi} \left(\cos \frac{x}{2}\right)^{2s+1} \frac{\prod_{\nu=1}^{2n} \left(\sin \frac{x-x_\nu}{2}\right)^{2s+2}}{\sin \frac{x-x_k}{2}} w(x)dx = 0, \quad k = 1, \ldots, 2n,$$

with unknown nodes $x_1, \ldots, x_{2n}$. The previous system can be solved applying the Newton–Kantorovič method. For the starting iteration we use the zeros of orthogonal trigonometric polynomial of semi-integer degree $n + 1/2$, with respect to w on $(-\pi, \pi)$, such that $x_0 = -\pi$. All of the elements in the corresponding Jacobi matrix can be calculated exactly, except the rounding errors, using Gaussian quadratures for trigonometric polynomials ([7], [3]). Knowing nodes x_ν, $\nu = 0, 1, \ldots, 2n$, it is possible to calculate weights $A_{j,\nu}$, $\nu = 0, 1, \ldots, 2n$, $j = 0, 1, \ldots, 2s$. For calculation of the weights we make use of the fact that the quadrature rule (1) is of an interpolatory type and that it must be exact for $f(x) = 1, \cos \ell x, \sin \ell x$, with $1 \leq \ell \leq n + 2sn + s$. Thus, the problem of calculation of weights reduces to solving a system of linear equations with unknown weights.

Example 2.2 As an example we compute the parameters of the quadrature formula for weight function $w(x) = 1+\sin 10x$ in the case $n = 3$, $s = 2$. Nodes x_ν, $\nu = 0, 1, \ldots, 6$, are: $-3.141592653589793, -2.189915995648243, -1.333802859228839, -0.4033401484136716, 0.4754862551348574, 1.358705012321870, 2.284363686786429$, and weights $A_{j,\nu}$, $j = 0, 1, \ldots, 4$, $\nu = 0, 1, \ldots, 6$, are given in Table 1 (numbers in parentheses denote decimal exponents). Numerical results are obtained using described procedure. For all computations we use MATHEMATICA and a software package described in [1] in double precision arithmetic.

Table 1 Weights $A_{j,\nu}$, $j = 0, 1, \ldots, 4$, $\nu = 0, 1, \ldots, 6$, of quadrature formula for weight function $w(x) = 1 + \sin 10x$; $n = 3$, $s = 2$

j	$A_{j,0}$	$A_{j,1}$	$A_{j,2}$	$A_{j,3}$
0	8.776367422159444(−1)	8.858726978166469(−1)	9.480324022785087(−1)	8.189268866862769(−1)
1	2.158841531674652(−2)	−2.159583157957023(−2)	1.605776098735195(−2)	−1.296616147623097(−2)
2	2.149026489456589(−2)	2.226900049704332(−2)	2.794916886125064(−2)	1.568181701343437(−2)
3	−1.045164240397695(−4)	1.028835066929633(−4)	−6.760549913522855(−5)	7.260579143363442(−5)
4	8.464468301212081(−5)	8.732267554876372(−5)	1.083522991634796(−4)	6.694877964305861(−5)

j	$A_{j,4}$	$A_{j,5}$	$A_{j,6}$
0	9.840574997625922(−1)	8.142289291849445(−1)	9.544301492346729(−1)
1	9.739402015066227(−4)	1.074074566458135(−2)	−1.479914056660530(−2)
2	3.111608766772306(−2)	1.519279672877689(−2)	2.851742321403937(−2)
3	−3.710771945881639(−6)	−6.106049088241541(−5)	6.135966894055344(−5)
4	1.208140742282386(−4)	6.567694275712946(−5)	1.105633157898151(−4)

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References

[1] A. S. Cvetković and G. V. Milovanović, The Mathematica Package "OrthogonalPolynomials", Facta Univ. Ser. Math. Inform. **19**, 17–36 (2004).
[2] A. Ghizzeti and A. Ossicini, Quadrature Formulae, Academie–Verlag, Berlin, 1970.
[3] G. V. Milovanović, A. S. Cvetković and M. P. Stanić, Trigonometric orthogonal systems and quadrature formulae (submitted)
[4] G. V. Milovanović, A. S. Cvetković and M. P. Stanić, Explicit formulas for five-term recurrence coefficients of orthogonal trigonometric polynomials of semi-integer degree, Appl. Math. Comput. (2007), doi:10.1016/j.amc.2007.08.062
[5] G. V. Milovanović, A. S. Cvetković and M. P. Stanić, Trigonometric orthogonal systems and quadrature formulae with maximal trigonometric degree of exactness, NMA 2006, Lecture Notes in Comput. Sci. **4310**, 402–409 (2007).
[6] G. V. Milovanović, A. S. Cvetković and M. P. Stanić, A special Gaussian rule for trigonometric polynomials, Banach J. Math. Anal. Banach J. Math.Anal. **1**, 85–90 (2007).
[7] A. H. Turetzkii, On quadrature formulae that are exact for trigonometric polynomials, East J. Approx. **11**, 337–359 (2005) (translation in English from Uchenye Zapiski, Vypusk **1**(149), Seria math. Theory of Functions, Collection of papers, Izdatel'stvo Belgosuniversiteta imeni V.I. Lenina, Minsk, 31–54 (1959))