Generalized inverses of operators on Hilbert C^*-modules

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November 22, 2012

Let A be a C^*-algebra and let $\mathcal{M}$ be a right A-module. This means that $(\mathcal{M}, +)$ is an Abelian group, and there exists an exterior multiplication: if $x \in \mathcal{M}$ and $a \in A$, then $x \cdot a \in \mathcal{M}$. This multiplication satisfies the same axioms as the scalar multiplication in vector spaces.

Additionally, if A does not have the unit, we assume that the scalar multiplication of elements in $\mathcal{M}$ exists. If $\lambda \in \mathbb{C}$ and $x \in \mathcal{M}$, then we write equivalently $x\lambda = \lambda x \in \mathcal{M}$. If A has the unit, then the scalar multiplication follows easily from the multiplication by elements of A.

Definition 0.1. Let $\mathcal{M}$ be a module over a C^*-algebra A. Suppose that there exists an A-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A$, satisfying the following:
(1) $\langle x, x \rangle \geq 0$ in A for all $x \in \mathcal{M}$;
(2) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{M}$;
(3) $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in \mathcal{M}$ and all $z \in A$.
Then $\mathcal{M}$ is a Hilbert pre-module over A.

Definition 0.2. If M is a pre-Hilbert module over A, and $\mathcal{M}$ is complete with respect to the norm $\| \cdot \|_{\mathcal{M}}$, then $\mathcal{M}$ is a Hilbert C^*-module over A, or $\mathcal{M}$ is a Hilbert C^* A-module.

Example 0.1. If A is a C^*-algebra, then A is itself a Hilbert module, since the inner product is given by $\langle a, b \rangle = a^* b$ for all $a, b \in A$.

More generally, let J be a right ideal of A. Then J is a Hilbert module over A, if the inner product is given by $\langle a, b \rangle = a^* b$.

Example 0.2. Let $M^{m \times n}$ denotes the set of all complex matrices of the form $m \times n$. Then $A^{m \times n}$ is a right $M^{n \times n}$-module. The norm $\|.\|$ can be defined as $\|A\|_{A^{m \times n}} = \|AA^*\|$.

On the other hand, we can consider $A^{m \times n}$ as a left $A^{m \times m}$-module, and the natural norm is defined as $\|A\|_{A^{m \times n}} = \|A^* A\|$.

We know that both norms are the same!

Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a C^*-algebra A. A mapping $T : \mathcal{M} \to \mathcal{N}$ is called *operator* if T is a bounded $\mathbb{C}$-linear A-homomorphism from $\mathcal{M}$ to $\mathcal{N}$, i.e. T satisfies:

$$T(x+y) = T(x)+T(y),\ T(\lambda x) = \lambda T(x),\ T(xa) = T(x)a, \quad x, y \in \mathcal{M},\ a \in A,\ \lambda \in \mathbb{C},$$

and there exists some $M \geq 0$ such that

$$\|T(x)\|_{\mathcal{M}} \leq M\|x\|_{\mathcal{N}},\ x \in \mathcal{M}.$$

The norm of T is given by

$$\|T\| = \inf\{M \geq 0 : \|T(x)\|_{\mathcal{M}} \leq M\|x\|_{\mathcal{N}},\ \text{for all } x \in \mathcal{M}\}.$$

The set of all operators from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. Particularly, $\mathrm{End}_A(\mathcal{M}) = \mathrm{Hom}_A(\mathcal{M}, \mathcal{M})$.

Lemma 0.1. $\mathrm{End}_A(\mathcal{M})$ *is a Banach algebra.*

We shall see that the question of adjoint operators is not trivial.

Lemma 0.2. *Let $\mathcal{M}$ be a Hilbert A-module, and let $T : \mathcal{M} \to \mathcal{M}$ and $T^* : \mathcal{M} \to \mathcal{M}$ be A-linear mappings such that*

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \quad \text{for all } x, y \in \mathcal{M}.$$

Then $T, T^* \in \mathrm{End}_A(\mathcal{M})$.

Definition 0.3. An operator $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ is adjointable, if there exists and operator $T^* \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that for all $x \in \mathcal{M}$ and all $y \in \mathcal{N}$ the following holds:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

There exists operators that are not adjointable.

The set of all adjointable operators from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}^*_A(\mathcal{M}, \mathcal{N})$. We see that $\mathrm{End}^*_A(\mathcal{M})$ is a C^*-algebra.

Theorem 0.1. *For $T \in \mathrm{End}^*_A(\mathcal{M})$ the following conditions are equivalent:*

(1) *T is a positive element in the C^*-algebra $\mathrm{End}^*_A(\mathcal{M})$;*

(2) *For all $x \in \mathcal{M}$ the element Tx is positive in the C^*-algebra A.*

Theorem 0.2. *Let $T : \mathcal{M} \to \mathcal{N}$ be a linear map. Then the following statements are equivalent:*

(1) *T is bounded and A-homomorphism;*

(2) *There exists a constant $K \geq 0$ such that the inequality $\langle Tx, Tx\rangle \leq K\langle x, x\rangle$ holds in A for all $x \in \mathcal{M}$.*

Lemma 0.3. *Let A be a unital C^*-algebra and let $r : A \to A$ be a linear map such that for some constant $K \geq 0$ the inequality $r(a)^* r(a) \leq K a^* a$ holds for all $a \in A$. Then $r(a) = r(1)a$ for all $a \in A$.*

Example 0.3. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ be the orthogonal decomposition of Hilbert modules. Define $P : \mathcal{M} \to \mathcal{M}$ to be the projection from $\mathcal{M}$ onto $\mathcal{N}$ parallel to $\mathcal{L}$. Then P is bounded, $\|P\| = 1$ and $P^* = P$. Hence, $P \in \mathrm{End}_A^*(\mathcal{M})$.

Theorem 0.3. (Misčenko) *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, and let $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$ such that $R(T)$ is closed in $\mathcal{N}$. Then the following hold:*
(1) *$N(T)$ is a complemented submodule of $\mathcal{M}$ and $N(T)^\perp = R(T^*)$;*
(2) *$R(T)$ is a complemented module of $\mathcal{N}$ and $R(T)^\perp = N(T^*)$;*
(3) *T^* also has a closed range.*

Let $\mathcal{M}, \mathcal{N}$ be Hilbert modules, and let $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$, or $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$.
T is generalized invertible, if there exists some $T_1 \in \mathrm{Hom}(\mathcal{N}, \mathcal{M})$ such that $TT_1T = T$.
We can also require that S satisfies all Penrose eqautions, in order to obtain the Moore-Penrose inverse of T.

Outer inverse with prescribed range and null-module:
Let $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$, and let K and H be submodules of $\mathcal{M}$ and $\mathcal{N}$, respectively. Find $U \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that the following hold:

$$UTU = U,\ R(U) = K,\ N(U) = H.$$

If such U exists, then $U = T^{(2)}_{K,H}$.
Equivalent conditions (Xu, Zhang):

$$\mathcal{N} = A(K) \oplus H,\ N(T) \cap K = \{0\},\ \mathcal{M} = T^*(H^\perp) \oplus K^\perp),\ N(T^*) \cap H^\perp = \{0\}.$$

The notion for the commutators follows: $[U, V] = UV - VU$, for appropriate choice of operators U and V.

Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be Hilbert modules, and let $A \in \mathrm{Hom}^*(\mathcal{N}, \mathcal{L})$ and $B \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ have closed ranges, such that AB also has a closed range. Find necessary and sufficient conditions such that the reverse order law holds:

$$(AB)^\dagger = B^\dagger A^\dagger.$$

A new result follows.

Theorem 0.4. *If $A \in \mathrm{Hom}^*(\mathcal{N}, \mathcal{L})$, $B \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ and $AB \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ have closed ranges, then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $[A^\dagger A, BB^*] = 0$ *and* $[A^* A, BB^\dagger] = 0$;

(3) $R(A^*AB) \subset R(B)$ *and* $R(BB^*A^*) \subset R(A^*)$;

(4) A^*ABB^* *has a commuting Moore-Penrose inverse.*