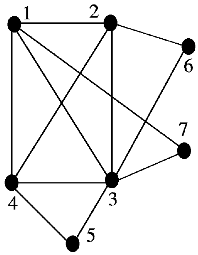
Nut Graphs in Extremal Singular Graphs



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http://www.impalayu.com/sga01.

University of Malta
Irene Sciriha

http://staff.um.edu.mt/isci1/

irene.sciriha-aquilina@um.edu.mt

Motivation

- What makes a graph singular?

- Which substructures determine that a graph is singular?

- To what extent can the nullity be increased by adding vertices to a graph, while preserving the original singular structure within the graph?

Plan

- **Substructures of Singular Graphs**
  - (i) Cores
  - (ii) Singular Configurations
  - (iii) Nut Graphs
- Minimal Basis for an Eigenspace
- Graph Invariant: Core Order Sequence
  - Nullity and Core Order exert mutual Control
- Extremal Singular Graphs
  - – Size of Substructures
  - – A Nut Subgraph has Maximum Size

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Singular Graph: Substructures

Singular Graph G: if $\exists$ $\mathbf{x}\neq\mathbf{0}$: $\mathbf{Ax}=\mathbf{0}$, $\mathbf{x}$: kernel eigenvector.

Core F

Label G:

$$\begin{pmatrix} \mathbf{A}(F) & \mathbf{C}' \\ (\mathbf{C}')^{\mathbf{t}} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{x}_F \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

each entry of $\mathbf{x}_F \neq 0$.

The induced subgraph F is a *core of G*.

If $\exists\mathbf{x} = \mathbf{x}_F$: G is a *core graph.*

Singular Configuration (SC)

A minimum # of columns of C' determines an induced subgraph of G.

Minimal Configuration(MC)

A singular configuration with $Q' = 0$.

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There are η SCs as induced subgraphs of G.

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The set of core vertices (CV): those vertices that lie on some core of G.

If a vertex does not lie on any core of G, then it is said to be *core forbidden* (CFV).

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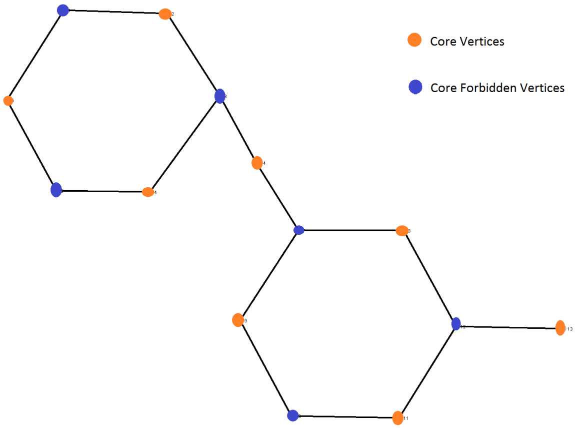
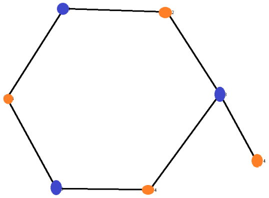


2 linearly independent kernel eigenvectors w. minimum support sum
{-2, 1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, -2, 1, 1, -1, 2, 0, 0, 0, 0, 0, 0}.

Singular Graph Nullity 2



Core Vertices

Core Forbidden Vertices

Singular Configuration for kernel eigenvector
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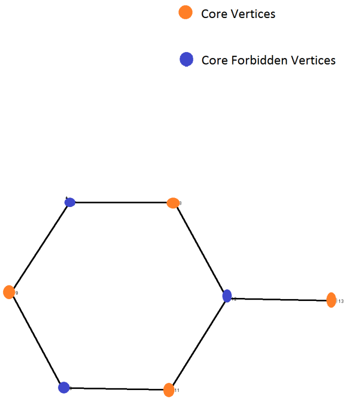


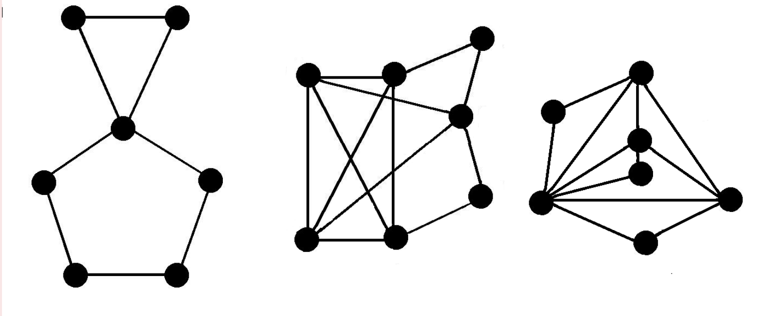
Singular Configuration for kernel eigenvector
$\{0, 0, 0, 0, -2, 1, 1, -1, 2, 0, 0, 0, 0, 0, 0\}$.

Nut Graph

A SC with no periphery.
The support is full.

Nut Graph

: A core graph of nullity one: connected & has no pendant vertex.
Nut graphs exist for all $n \geq 7$.



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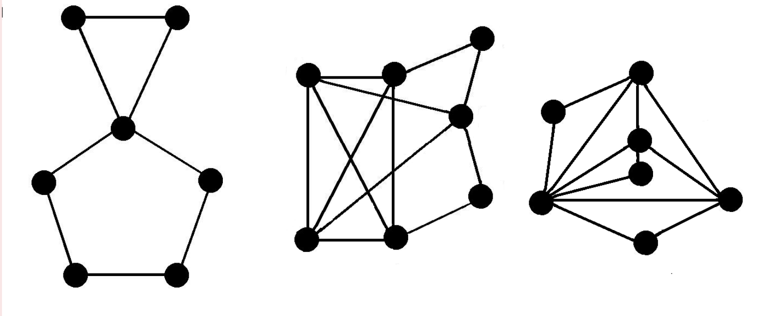

Minimal Basis

Support

The **support** $wt(x)$ of a vector is the number of non-zero elements in that vector.

Convention:

The vectors in a basis for a subspace are ordered according to the **monotonic non-decreasing sequence** of the support of its vectors.

Minimal Basis

The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\ell$ in a basis for W with the smallest *support sum* $\sum_{i=1}^{\ell} wt(\mathbf{x}_i)$, form a **minimal basis** B_{min} for W.

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New Vector Space Invariant

Minimal Basis Vectors Support Sequence

I. Sciriha *et al.* GTNNY 1996

For B_{min}: Support Sequence is $t_1, t_2, \ldots, t_q$
If for B: Support Sequence is $s_1, s_2, \ldots, s_q$,
then

$$\forall i, \quad t_i \leq s_i.$$

t_1	t_2	t_3	t_4	Support Sum	Type
2	5	5	7	19	Minimal Basis
2	5	7	7	21	Not Minimal
2	3	7	7	19	Impossible
2	4	6	7	19	Impossible

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Maximum Support for a Minimal Basis

If B_{min} for ker(A) is $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\eta)$, then the *core-width* τ is the weight of $\mathbf{x}_\eta$: the largest support.

Lemma

If $\mathbf{x} \in B_{min}$, then $\mathbf{x}$ has at least $\eta(G) - 1$ zero entries.

Proof.

- Write the η kernel eigenvectors of a basis B as the rows of the $\eta \times n$ matrix. : reduced by Gauss Row Reduction to $\mathbf{M}'$, with all entries in the columns above and below a pivot being zero.
- The set of vertices corresponding to the pivots is a **singular–configuration–vertex–representation**.

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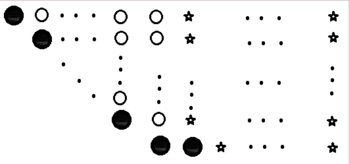

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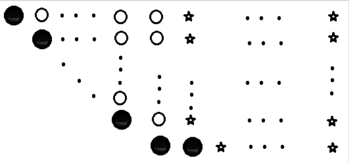

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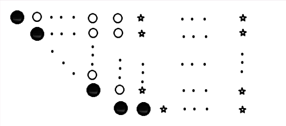

- But the weight–sequence of B_{min} is entry-wise less than that of B.

  Thus if $\mathbf{x} \in B_{min}$, then $\mathbf{x}$ has at least $\eta(G) - 1$ zero entries. □

Extremal Singular Graphs

Theorem

For a graph G on n vertices of nullity η and core width τ, $\tau + \eta \leq n + 1$.

For which graphs is the upper bound reached?

Definition

A singular graph G on n vertices with nullity η and core width τ is said to be **extremal singular** *if $\eta + \tau$ reaches $n + 1$.*

How large can a Core in a Fundamental System be?

Corollary

A singular graph G on n vertices of nullity η cannot have a core F_t of order t in $\mathcal{F}$ if $t > n + 1 - \eta$.

Nullity Controls Core size

Main Theorem

A graph G is extremal singular of nullity η, if and only if

- it is a core graph,
- the largest core in a fundamental system is a nut graph N and
- there are exactly $\eta - 1$ vertices of G not on N.

Nut Graph

A graph G is extremal singular of nullity one if and only if G is a nut graph on τ vertices.

In an Extremal Singular Graph:

If H is a singular configuration for a core in $\mathcal{F}$, then $\tau \geq |H|$.

By interlacing, a SC H is grown into G by adding at least $\eta(G) - 1$ vertices.

Thus $n \geq |H| + \eta(G) - 1$ vertices.

Since $\tau + \eta(G) = n + 1$, $|H| \leq \tau$.

τ controls 'atom' size

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Significance of τ

- The maximum core size τ for a minimal basis of the nullspace of G is an invariant of G.
- $\tau + \eta \leq n + 1$
- In extremal singular graphs:
  - Each vertex is a core vertex (core graph);
  - The largest core, of size τ in a fundamental system is a nut graph N;
  - There are exactly $\eta - 1$ vertices of G not on N;
  - Not only are core orders in $\mathcal{F}$ bounded above by τ; the orders of a singular configuration 'grown' from any core of $\mathcal{F}$ is also bounded above by τ.

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Back

Back

Back

Back