New Graph Invariants

Nut Graphs in Extremal Singular Graphs

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http://www.impalayu.com/sga01.php

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Motivation

• What makes a graph singular?
• Which substructures determine that a graph is singular?
• To what extent can the nullity be increased by adding vertices to a graph, while preserving the original singular structure within the graph?
Plan

- **Substructures of Singular Graphs**
  - (i) Cores
  - (ii) Singular Configurations
  - (iii) Nut Graphs
- Minimal Basis for an Eigenspace
- Graph Invariant: Core Order Sequence
- Nullity and Core Order exert mutual Control
- Extremal Singular Graphs
  - Size of Substructures
  - A Nut Subgraph has Maximum Size
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Singular Graph $G$: if $\exists \ x \neq 0: \ Ax = 0$, $x$: kernel eigenvector.

Core $F$

Label $G$:

$$
\begin{pmatrix}
A(F) & C' \\
(C')^t & Q
\end{pmatrix}
\begin{pmatrix}
x_F \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

Each entry of $x_F \neq 0$.

The induced subgraph $F$ is a core of $G$.

If $\exists x = x_F : G$ is a core graph.

Singular Configuration (SC)

A minimum # of columns of $C'$ determines an induced subgraph of $G$.

Minimal Configuration (MC)

A singular configuration with $Q' = 0$. 

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$\eta$ linearly independent kernel eigenvectors with minimum support sum determine a fundamental system of $\eta$ cores of $G$.

The ‘Atoms’ of Singular Graphs

There are $\eta$ SCs as induced subgraphs of $G$.

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A Structural Graph Invariant

The set of core vertices (CV): those vertices that lie on some core of $G$.

If a vertex does not lie on any core of $G$, then it is said to be core forbidden (CFV).
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2 linearly independent kernel eigenvectors w. minimum support sum
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The support is full.

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: A core graph of nullity one: connected & has no pendant vertex.
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The support $wt(x)$ of a vector is the number of non-zero elements in that vector.

Convention:

The vectors in a basis for a subspace are ordered according to the monotonic non-decreasing sequence of the support of its vectors.

Minimal Basis

The vectors $x_1, x_2, \ldots, x_\ell$ in a basis for $W$ with the smallest support sum $\sum_{i=1}^\ell wt(x_i)$, form a minimal basis $B_{\text{min}}$ for $W$. 
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New Vector Space Invariant

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I. Sciriha et al. GTNNY 1996

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Maximum Support for a Minimal Basis

If $B_{\text{min}}$ for $\ker(A)$ is $(x_1, x_2, \ldots, x_\eta)$, then the core-width $\tau$ is the weight of $x_\eta$: the largest support.
Lemma

If $x \in B_{min}$, then $x$ has at least $\eta(G) - 1$ zero entries.

Proof.

- Write the $\eta$ kernel eigenvectors of a basis $B$ as the rows of the $\eta \times n$ matrix.
  - reduced by Gauss Row Reduction to $M'$, with all entries in the columns above and below a pivot being zero.
- The set of vertices corresponding to the pivots is a singular--configuration--vertex--representation.
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- The set of vertices corresponding to the pivots is a singular-configuration-vertex-representation.
Lemma

If \( x \in B_{min} \), then \( x \) has at least \( \eta(G) - 1 \) zero entries.

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- But the weight-sequence of $B_{\min}$ is entry-wise less than that of $B$.

Thus if $x \in B_{\min}$, then $x$ has at least $\eta(G) - 1$ zero entries.
Theorem

For a graph $G$ on $n$ vertices of nullity $\eta$ and core width $\tau$, $\tau + \eta \leq n + 1$.

For which graphs is the upper bound reached?

Definition

A singular graph $G$ on $n$ vertices with nullity $\eta$ and core width $\tau$ is said to be extremal singular if $\eta + \tau$ reaches $n + 1$. 

Irene Sciriha
How large can a Core in a Fundamental System be?

Corollary

A singular graph $G$ on $n$ vertices of nullity $\eta$ cannot have a core $F_t$ of order $t$ in $\mathcal{F}$ if $t > n + 1 - \eta$. 
Main Theorem

A graph $G$ is extremal singular of nullity $\eta$, if and only if

- it is a core graph,
- the largest core in a fundamental system is a nut graph $N$ and
- there are exactly $\eta - 1$ vertices of $G$ not on $N$. 
A graph $G$ is extremal singular of nullity one if and only if $G$ is a nut graph on $\tau$ vertices.

In an Extremal Singular Graph:

If $H$ is a singular configuration for a core in $\mathcal{F}$, then $\tau \geq |H|$.

By interlacing, a SC $H$ is grown into $G$ by adding at least $\eta(G) - 1$ vertices.

Thus $n \geq |H| + \eta(G) - 1$ vertices.

Since $\tau + \eta(G) = n + 1$, $|H| \leq \tau$. 

$\tau$ controls 'atom' size
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### Nut Graph

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The maximum core size \( \tau \) for a minimal basis of the nullspace of \( G \) is an invariant of \( G \).

\[ \tau + \eta \leq n + 1 \]

In extremal singular graphs:

- Each vertex is a core vertex (core graph);
- The largest core, of size \( \tau \) in a fundamental system is a nut graph \( N \);
- There are exactly \( \eta - 1 \) vertices of \( G \) not on \( N \);
- Not only are core orders in \( \mathcal{F} \) bounded above by \( \tau \); the orders of a singular configuration 'grown' from any core of \( \mathcal{F} \) is also bounded above by \( \tau \).
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