Cospectrality of graphs with respect to distance matrices

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Joint work with

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Outline

- Definitions
- Cospectrality of adjacency based matrices \((A, L, Q)\)
- Cospectrality of distance based matrices \((D, D^L, D^Q)\)
- Examples of graphs defined by their spectra
A-spectrum

Let $G = (V, E)$ be a graph on $n$ vertices.

- The adjacency matrix: $A = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{if } ij \notin E \end{cases}$$

- The $A$-spectrum of $G$ is the spectrum of $A$ and denoted $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

For more details see:
Definitions

$L$-spectrum

- The Laplacian: $L = \text{Diag} - A$, where $\text{Diag}$ is the diagonal matrix of $G$’s degrees and $A$ the adjacency matrix.
- The $L$-spectrum of $G$ is the spectrum of $L$ and denoted $(\mu_1, \mu_2, \ldots, \mu_n)$, with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$.

For a survey see:
Q-spectrum

- The signless Laplacian: $Q = \text{Diag} + A$, where $\text{Diag}$ is the diagonal matrix of $G$’s degrees and $A$ the adjacency matrix.
- The $Q$-spectrum of $G$ is the spectrum of $Q$ and denoted $(q_1, q_2, \ldots, q_n)$, with $q_1 \geq q_2 \geq \ldots \geq q_n$.

For more details see:
Definitions

**$D$-spectrum**

- The distance matrix of $G$ is $D = (d_{ij})$, where $d_{ij}$ is the distance (length of a shortest path) between the vertices $i$ and $j$.
- The $D$-spectrum of $G$ is the spectrum of $D$ and denoted $(\partial_1, \partial_2, \ldots, \partial_n)$, with $\partial_1 \geq \partial_2 \geq \ldots \geq \partial_n$.

For a survey see:

[M. Aouchiche, P. Hansen, Linear Algebra Appl. 458, 2014]
The transmission $T(v)$ of a vertex $v$ is

$$T(v) = \sum_{u \in V} d(u, v)$$

The distance Laplacian of $G$ is $D^L = Tr - D$, where $Tr$ is the diagonal matrix of transmissions in $G$.

The $D^L$-spectrum of $G$ is the spectrum of $D$ and denoted $(\partial^L_1, \partial^L_2, \ldots, \partial^L_n)$, with $\partial^L_1 \geq \partial^L_2 \geq \ldots \geq \partial^L_n$.

For more details see:
[M. Aouchiche, P. Hansen, Two Laplacians for the Distance Matrix of a Graph. Linear Algebra Appl. 439 (2013) 21-33]
The distance Laplacian of $G$ is $D^Q = Tr + D$, where $Tr$ is the diagonal matrix of transmissions in $G$.

The $D^Q$-spectrum of $G$ is the spectrum of $D$ and denoted $(\partial_1^Q, \partial_2^Q, \ldots, \partial_n^Q)$, with $\partial_1^Q \geq \partial_2^Q \geq \ldots \geq \partial_n^Q$.

For more details see:
[M. Aouchiche, P. Hansen, Two Laplacians for the Distance Matrix of a Graph. Linear Algebra Appl. 439 (2013) 21-33]
[M. Aouchiche, P. Hansen, On the distance signless Laplacian of a graph, Linear and Multilinear Algebra, 64 (2016) 1113-1123]
Cospectrality

For a given graph matrix $M$

- two graphs are $M$-cospectral if they share the same $M$-spectrum
- isomorphic graphs are trivially $M$-cospectral, for any graph matrix $M$
- two $M$-cospectral non isomorphic graphs are called $M$-cospectral mates or $M$-mates

If $M_1, M_2, \ldots, M_k$ are graph matrices, two graphs are $(M_1, M_2, \ldots, M_k)$-mates if they are $M_i$-mates for all $i = 1, 2, \ldots, k$. 
A-Cospectrality

- Günthard and Primas conjectured: **There are no $A$-mates**
- Refutation over the class of trees in [Collatz, Sinogowitz, Abh. Math. Sem. Univ. Hamburg 21, 1957]

Two $A$–cospectral trees.
A-Cospectrality


Two \(A\)-cospectral graphs.
A-Cospectrality


Two $A$-cospectral connected graphs.
A-Cospectrality

First infinite family of trees with a $A$-mate was proposed in [Schwenk, In: Harary (Ed.), New Directions in the Theory of Graphs, 1973]

Other constructions in
[van Dam, Haemers, Linear Algebra Appl. 373, 2003]
[van Dam, Haemers, Koolen, Linear Algebra Appl. 423, 2007]
[Godsil, McKay, Aequationes Mathematicae 25, 1982]
[Haemers, Spence, European J Combinatorics 25, 2004]
A-Cospectrality

Surprise: asymptotically every tree has a $A$-mate


Number of graphs with a $A$-mate was studied and the following table was taken from [Brouwer, Spence, Elec J of Combinatorics 16, 2009]

<table>
<thead>
<tr>
<th>Nb of vertices</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb of graphs</td>
<td>34</td>
<td>156</td>
<td>1044</td>
<td>12346</td>
<td>274668</td>
<td>12005168</td>
<td>1018997864</td>
<td>165091172592</td>
</tr>
<tr>
<td>Nb of graphs with a mate</td>
<td>2</td>
<td>10</td>
<td>110</td>
<td>1722</td>
<td>51039</td>
<td>2560606</td>
<td>215331676</td>
<td>31067572481</td>
</tr>
</tbody>
</table>
**L-Cospectrality**

For more about $L$-cospectrality, see

[Fujii, Katsuda, Discrete Math. 207, 1999]
[Haemers, Spence, European J Combinatorics 25, 2004]
[Halbeisen, Hungerbühler, J. Graph Theory 31, 1999]
[Merris, Linear and Multilinear Algebra 43, 1997]
[Merris, Linear Algebra Appl. 197/198, 1994]
[Tan, Interdisciplinary Information Sciences 4, 1998]
Q-Cospectrality

For more about Q-cospectrality, see

[Cvetković, Simić, Publ. Inst. Math. (Beograd) 85(99), 2009]
[van Dam, Haemers, Linear Algebra Appl. 373, 2003]
[Haemers, Spence, European J. Combinatorics 25, 2004]
The study of $D$-Cospectrality evolved in a similar way as for $A$-cospectrality.

In [McKay, Ars Combinatoria 3, 1977]:

- The smallest $D$-mates (17 vertices)
- Construction of an infinite family of trees with a mate
- Proof of asymptotically every tree has a $D$-mate
We used Nauty to enumerate all trees on up to 20 vertices (https://cs.anu.edu.au/~bdm/nauty)

We used AutoGraphiX III to evaluate their $D$-spectra (https://www.gerad.ca/~gillesc/)

- 2 $D$-mates over 48629 trees on 17 vertices
- 2 pairs $D$-mates over 123867 trees on 18 vertices (can be obtained using Mckay’s method)
- 6 pairs $D$-mates over 317955 trees on 19 vertices (4 can be obtained using Mckay’s method)
- 14 pairs $D$-mates over 823065 trees on 20 vertices (9 can be obtained using Mckay’s method)
$D$-Cospectrality

Example of $D$-mates that cannot be obtained using McKay’s Method

$D$-cospectral trees on 19 vertices.
D-Cospectrality

Pairs, by column, of $D$-mates that can be obtained using McKay’s Method
**D-Cospectrality**


**Conjecture** There exists no pair of distance noncospectral trees $T_1$ and $T_2$, such that $\partial_1(T_1) = \partial_1(T_2)$

It was tested, in the same paper, on trees on up to 20 vertices and chemical trees on up to 24 vertices.
Cospectrality of distance based matrices

**$D$-Cospectrality**

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no $D$-mates with fewer than 7 vertices (141 graphs)
- 22 $D$-mates over 853 graphs on 7 vertices
- 658 $D$-mates over 11117 graphs on 8 vertices (8 triplets)
- 25058 $D$-mates over 261080 graphs on 9 vertices (up to 10 graphs with the same $D$-spectrum)
- 1389984 $D$-mates over 11716571 graphs on 10 vertices (up to 21 graphs with the same $D$-spectrum)
D\textsuperscript{L}-Cospectrality

The only triplet of D\textsuperscript{L}-mates on 7 vertices
Cospectrality of distance based matrices

**$D^L$-Cospectrality**

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no $D^L$-mates with fewer than 7 vertices (141 graphs)
- 43 $D^L$-mates over 853 graphs on 7 vertices (1 triplet)
- 745 $D^L$-mates over 11117 graphs on 8 vertices (up to 4 graphs with the same $D^L$-spectrum)
- 19778 $D^L$-mates over 261080 graphs on 9 vertices (up to 8 graphs with the same $D^L$-spectrum)
- 787851 $D^L$-mates over 11716571 graphs on 10 vertices (up to 16 graphs with the same $D^L$-spectrum)
Cospectrality of distance based matrices

$D^Q$-Cospectrality

The only pair of $D^Q$-mates on 5 vertices
**$D^Q$-Cospectrality**

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no $D^Q$-mates with fewer than 5 vertices (8 graphs)
- 2 $D^Q$-mates over 21 graphs on 5 vertices
- 6 $D^Q$-mates over 112 graphs on 6 vertices
- 38 $D^Q$-mates over 853 graphs on 7 vertices
- 453 $D^Q$-mates over 11117 graphs on 8 vertices (11 triplets)
- 8168 $D^Q$-mates over 261080 graphs on 9 vertices (up to 4 graphs with the same $D^Q$-spectrum)
- 319324 $D^Q$-mates over 11716571 graphs on 10 vertices (up to 9 graphs with the same $D^Q$-spectrum)
Comparison

For all graphs on 10 vertices, repartition of the size of families sharing the same $M$-spectrum

<table>
<thead>
<tr>
<th>Family size</th>
<th>$\mathcal{D}$</th>
<th>$\mathcal{D}^L$</th>
<th>$\mathcal{D}^Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>583922</td>
<td>345065</td>
<td>148101</td>
</tr>
<tr>
<td>3</td>
<td>46300</td>
<td>20010</td>
<td>5978</td>
</tr>
<tr>
<td>4</td>
<td>14369</td>
<td>6947</td>
<td>1138</td>
</tr>
<tr>
<td>5</td>
<td>1905</td>
<td>819</td>
<td>87</td>
</tr>
<tr>
<td>6</td>
<td>1714</td>
<td>580</td>
<td>26</td>
</tr>
<tr>
<td>7</td>
<td>288</td>
<td>138</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>283</td>
<td>82</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>45</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>64</td>
<td>17</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>33</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1389984</strong></td>
<td><strong>787851</strong></td>
<td><strong>319324</strong></td>
</tr>
</tbody>
</table>
Comparison

Proportions of mates among all graphs on up to 10 vertices

<table>
<thead>
<tr>
<th>n</th>
<th>$D$</th>
<th>$D^L$</th>
<th>$D^Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0.095238095</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0.053571429</td>
</tr>
<tr>
<td>7</td>
<td>0.025791325</td>
<td>0.050410317</td>
<td>0.044548652</td>
</tr>
<tr>
<td>8</td>
<td>0.05918863</td>
<td>0.067014482</td>
<td>0.040748403</td>
</tr>
<tr>
<td>9</td>
<td>0.095978244</td>
<td>0.075754558</td>
<td>0.03128543</td>
</tr>
<tr>
<td>10</td>
<td>0.118634027</td>
<td>0.067242455</td>
<td>0.027254049</td>
</tr>
</tbody>
</table>
### Comparison

**Number of spectra for graphs on up to 10 vertices**

<table>
<thead>
<tr>
<th>$n$</th>
<th># of graphs</th>
<th># of $\mathcal{D}$-spectra</th>
<th># of $\mathcal{D}^L$-spectra</th>
<th># of $\mathcal{D}^Q$-spectra</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>112</td>
<td>112</td>
<td>112</td>
<td>109</td>
</tr>
<tr>
<td>7</td>
<td>853</td>
<td>842</td>
<td>831</td>
<td>834</td>
</tr>
<tr>
<td>8</td>
<td>11117</td>
<td>10784</td>
<td>10730</td>
<td>10885</td>
</tr>
<tr>
<td>9</td>
<td>261080</td>
<td>247984</td>
<td>251007</td>
<td>256900</td>
</tr>
<tr>
<td>10</td>
<td>11716571</td>
<td>10975532</td>
<td>11302429</td>
<td>11552583</td>
</tr>
</tbody>
</table>
### Cospectrality involving 2 or 3 matrices

For all graphs on up to 10 vertices

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N(D, D^L)$</th>
<th>$M(D, D^L)$</th>
<th>$N(D, D^Q)$</th>
<th>$M(D, D^Q)$</th>
<th>$N(D^L, D^Q)$</th>
<th>$M(D^L, D^Q)$</th>
<th>$N(D, D^L, D^Q)$</th>
<th>$M(D, D^L, D^Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 – 7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>90</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>32</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1965</td>
<td>(7×) 3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>9449</td>
<td>(15×) 3</td>
<td>7712</td>
<td>(4×) 3</td>
<td>61909</td>
<td>(343×) 3</td>
<td>7622</td>
<td>(4×) 3</td>
</tr>
</tbody>
</table>
Cospectrality of distance based matrices

Cospectrality involving 2 or 3 matrices

The 4 triplets of \((D, D^L, D^Q)\)-mates on 10 vertices (columns)
Graphs determined by their $D^L$-spectrum

**Theorem**  From distance Laplacian spectrum of $G$, we can deduce the following:

(i) The number $n$ of vertices of $G$.
(ii) The Wiener index of $G$.
(iii) The number of connected components of the complement $\overline{G}$.

**Corollary**  The following graphs are determined by their distance Laplacian spectra:

the complete graph $K_n$;
the graph $K_n - e$ obtained from $K_n$ by the deletion of an edge;
the path $P_n$;
the comet $Co_{n,3}$.

**Theorem**  The $k$-partite graph on $n = n_1 + n_2 \cdots + n_k$ vertices, $K_{n_1, n_2, \ldots, n_k}$, is determined by its distance Laplacian spectrum.
Theorem. From signless distance Laplacian spectrum of a $G$, we can deduce the following:

(i) The number $n$ of vertices of $G$.
(ii) The Wiener index of $G$.
(iii) Whether $G$ is transmission regular.

Corollary. The following graphs are determined by their distance signless Laplacian spectra:

a) the complete graph $K_n$;
b) the graph $K_n - e$ obtained from $K_n$ by the deletion of an edge;
c) the path $P_n$;
d) the comet $C_{n,3}$.

Theorem. The cycle $C_n$ is determined by its signless Laplacian distance spectrum.