Cospectrality of graphs with respect to distance matrices

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Outline

- Definitions
- Cospectrality of adjacency based matrices (A, L, Q)
- Cospectrality of distance based matrices (D, D^L, D^Q)
- Examples of graphs defined by their spectra

Definitions

A-spectrum

Let $G = (V, E)$ be a graph on n vertices.

- The adjacency matrix: $A = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{if } ij \notin E \end{cases}$$

- The A-spectrum of G is the spectrum of A and denoted $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

For more details see:
[D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs-Theory and Applications, Verlag, Heidelberg-Leipzig, 1995]

Definitions

L-spectrum

- The Laplacian: $L = Diag - A$, where $Diag$ is the diagonal matrix of G's degrees and A the adjacency matrix.
- The L-spectrum of G is the spectrum of L and denoted $(\mu_1, \mu_2, \ldots, \mu_n)$, with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$.

For a survey see:
[R. Merris, Laplacian matrices of graphs: a survey. Linear Algebra Appl. 197/198 (1994) 143-176]

Q-spectrum

- The signless Laplacian: $Q = Diag + A$, where $Diag$ is the diagonal matrix of G's degrees and A the adjacency matrix.
- The Q-spectrum of G is the spectrum of Q and denoted $(q_1, q_2, \ldots, q_n)$, with $q_1 \geq q_2 \geq \ldots \geq q_n$.

For more details see:
[D. Cvetković, S. K. Simić, Towards a Spectral Theory of Graphs Based on the Signless Laplacian. I. Publ. Inst. Math. (Beograd) 85(99) (2009) 19-33]
[D. Cvetković, S. K. Simić, Towards a Spectral Theory of Graphs Based on the Signless Laplacian. II. Linear Algebra Appl. 432 (2010) 2257-2272]
[D. Cvetković, S. K. Simić, Towards a Spectral Theory of Graphs Based on the Signless Laplacian. III. Appl. Anal. Discrete Math. 4 (2010) 156-166]

Definitions

D-spectrum

- The distance matrix of G is $D = (d_{ij})$, where d_{ij} is the distance (length of a shortest path) between the vertices i and j.
- The D-spectrum of G is the spectrum of D and denoted $(\partial_1, \partial_2, \ldots, \partial_n)$, with $\partial_1 \geq \partial_2 \geq \ldots \geq \partial_n$.

For a survey see:
[M. Aouchiche, P. Hansen, Linear Algebra Appl. 458, 2014]
[D. Stevanović, A. Ilić, in: Math. Chem. Monogr., vol.12, Univ. of Kragujevac, Kragujevac, 2010]

Definitions

D^L-spectrum

- The transmission $T(v)$ of a vertex v is

$$T(v) = \sum_{u \in V} d(u, v)$$

- The distance Laplacian of G is $D^L = Tr - D$, where Tr is the diagonal matrix of transmissions in G
- The D^L-spectrum of G is the spectrum of D and denoted $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$, with $\partial_1^L \geq \partial_2^L \geq \ldots \geq \partial_n^L$.

For more details see:
[M. Aouchiche, P. Hansen, Two Laplacians for the Distance Matrix of a Graph. Linear Algebra Appl. 439 (2013) 21-33]
[M. Aouchiche, P. Hansen, Some properties of the distance Laplacian eigenvalues of a graph. Czech. Math. Journal, 64 (2014) 751-761]

Definitions

D^Q-spectrum

- The distance Laplacian of G is $D^Q = Tr + D$, where Tr is the diagonal matrix of transmissions in G
- The D^Q-spectrum of G is the spectrum of D and denoted $(\partial_1^Q, \partial_2^Q, \ldots, \partial_n^Q)$, with $\partial_1^Q \geq \partial_2^Q \geq \ldots \geq \partial_n^Q$.

For more details see:
[M. Aouchiche, P. Hansen, Two Laplacians for the Distance Matrix of a Graph. Linear Algebra Appl. 439 (2013) 21-33]
[M. Aouchiche, P. Hansen, On the distance signless Laplacian of a graph, Linear and Multilinear Algebra, 64 (2016) 1113-1123]

Cospectrality

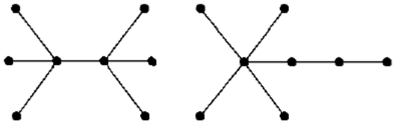
For a given graph matrix M

- two graphs are M-cospectral if they share the same M-spectrum
- isomorphic graphs are trivially M-cospectral, for any graph matrix M
- two M-cospectral non isomorphic graphs are called M-cospectral mates or M-mates

If $M_1, M_2, \ldots, M_k$ are graph matrices, two graphs are $(M_1, M_2, \ldots, M_k)$-mates if they are M_i-mates for all $i = 1, 2, \ldots, k$.

A-Cospectrality

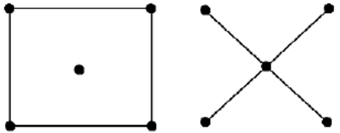
- Which graphs are determined by their A–spectrum? posed in [Günthard, Primas, Helv. Chim. Acta 39, 1956]
- Günthard and Primas conjectured: There are no A-mates
- Refutation over the class of trees in [Collatz, Sinogowitz, Abh. Math. Sem. Univ. Hamburg 21, 1957]



Two A–cospectral trees.

A-Cospectrality

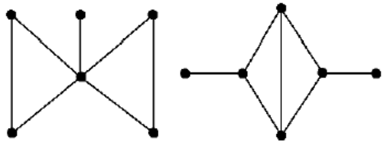
- Refutation for general graphs in Cvetković's Thesis, Univ. Beograd, 1971



Two A–cospectral graphs.

A-Cospectrality

- Refutation for connected graphs in [Baker, J. Math. Phys. 7, 1966]



Two A–cospectral connected graphs.

A-Cospectrality

First infinite family of trees with a A-mate was proposed in [Schwenk, In: Harary (Ed.), New Directions in the Theory of Graphs, 1973]

Other constructions in
[van Dam, Haemers, Linear Algebra Appl. 373, 2003]
[van Dam, Haemers, Koolen, Linear Algebra Appl. 423, 2007]
[Godsil, McKay, Lecture Notes in Math., Vol. 560, 1976]
[Godsil, McKay, Aequationes Mathematicae 25, 1982]
[Haemers, Spence, European J Combinatorics 25, 2004]
[Johnson, Newman, J. Combin. Theory B 28, 1980]

A-Cospectrality

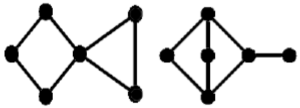
Surprise : asymptotically every tree has a A-mate
Proved in [Schwenk, In: Harary (Ed.), New Directions in the Theory of Graphs, 1973]

Number of graphs with a A-mate was studied and the following table was taken from [Brouwer, Spence, Elec J of Combinatorics 16, 2009]

Nb of vertices	5	6	7	8	9	10	11	12
Nb of graphs	34	156	1044	12346	274668	12005168	1018997864	165091172592
Nb of graphs with a mate	2	10	110	1722	51039	2560606	215331676	31067572481

Cospectrality of adjacency based matrices

L-Cospectrality



Two L–cospectral graphs.

For more about L-cospectrality, see

[Fujii, Katsuda, Discrete Math. 207, 1999]
[Haemers, Spence, European J Combinatorics 25, 2004]
[Halbeisen, Hungerbühler, J. Graph Theory 31, 1999]
[Merris, Linear and Multilinear Algebra 43, 1997]
[Merris, Linear Algebra Appl. 197/198, 1994]
[Tan, Interdisciplinary Information Sciences 4, 1998]

Q-Cospectrality

Two Q–cospectral graphs.

For more about Q-cospectrality, see

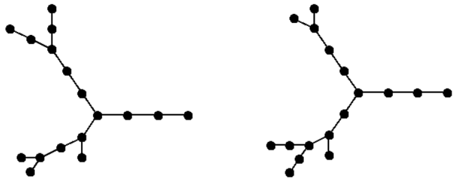
[Cvetković, Simić, Publ. Inst. Math. (Beograd) 85(99), 2009]
[van Dam, Haemers, Linear Algebra Appl. 373, 2003]
[Haemers, Spence, European J. Combinatorics 25, 2004]

D-Cospectrality

The study of D-Cospectrality evolved in a similar way as for A-cospectrality

In [McKay, Ars Combinatoria 3, 1977]:

- The smallest D-mates (17 vertices)



- Construction of an infinite family of trees with a mate
- Proof of asymptotically every tree has a D-mate

Cospectrality of distance based matrices

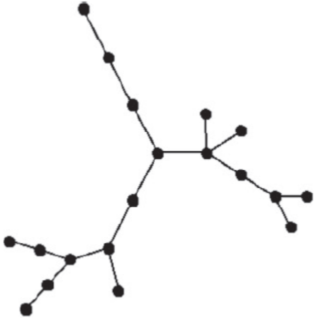
D-Cospectrality

We used *Nauty* to enumerate all trees on up to 20 vertices (https://cs.anu.edu.au/∼bdm/nauty)
We used *AutoGraphiX III* to evaluate their D-spectra (https://www.gerad.ca/∼gillesc/)

- 2 D-mates over 48629 trees on 17 vertices
- 2 pairs D-mates over 123867 trees on 18 vertices (can be obtained using Mckay's method)
- 6 pairs D-mates over 317955 trees on 19 vertices (4 can be obtained using Mckay's method)
- 14 pairs D-mates over 823065 trees on 20 vertices (9 can be obtained using Mckay's method)

D-Cospectrality

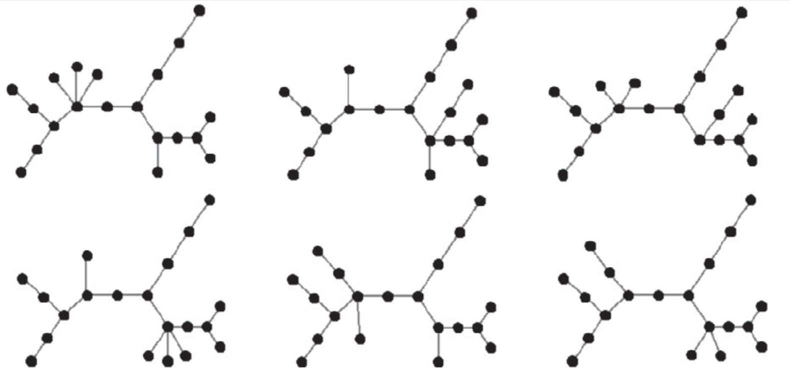
Example of D-mates that cannot be obtained using McKay's Method



$\mathcal{D}$-cospectral trees on 19 vertices.

D-Cospectrality

Pairs, by column, of D-mates that can be obtained using McKay's Method



D-Cospectrality

[Stevanović, Ilić, in: Math. Chem. Monogr., 12, Univ. of Kragujevac, 2010] suggests

Conjecture There exists no pair of distance noncospectral trees T_1 and T_2, such that $\partial_1(T_1) = \partial_1(T_2)$

It was tested, in the same paper, on trees on up to 20 vertices and chemical trees on up to 24 vertices

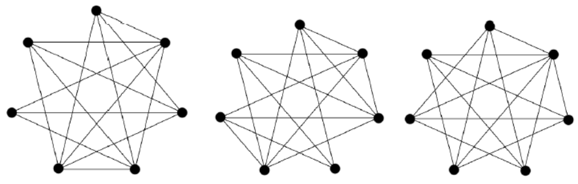
D-Cospectrality

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no D-mates with fewer than 7 vertices (141 graphs)
- 22 D-mates over 853 graphs on 7 vertices
- 658 D-mates over 11117 graphs on 8 vertices (8 triplets)
- 25058 D-mates over 261080 graphs on 9 vertices (up to 10 graphs with the same D-spectrum)
- 1389984 D-mates over 11716571 graphs on 10 vertices (up to 21 graphs with the same D-spectrum)

D^L-Cospectrality

The only triplet of D^L-mates on 7 vertices



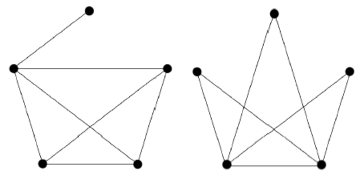
D^L-Cospectrality

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no D^L-mates with fewer than 7 vertices (141 graphs)
- 43 D^L-mates over 853 graphs on 7 vertices (1 triplet)
- 745 D^L-mates over 11117 graphs on 8 vertices (up to 4 graphs with the same D^L-spectrum)
- 19778 D^L-mates over 261080 graphs on 9 vertices (up to 8 graphs with the same D^L-spectrum)
- 787851 D^L-mates over 11716571 graphs on 10 vertices (up to 16 graphs with the same D^L-spectrum)

D^Q-Cospectrality

The only pair of D^Q-mates on 5 vertices



D^Q-Cospectrality

Based on the enumeration (Nauty for generation, AutoGraphiX III for evaluation) of connected graphs on up to 20 vertices

- no D^Q-mates with fewer than 5 vertices (8 graphs)
- 2 D^Q-mates over 21 graphs on 5 vertices
- 6 D^Q-mates over 112 graphs on 6 vertices
- 38 D^Q-mates over 853 graphs on 7 vertices
- 453 D^Q-mates over 11117 graphs on 8 vertices (11 triplets)
- 8168 D^Q-mates over 261080 graphs on 9 vertices (up to 4 graphs with the same D^Q-spectrum)
- 319324 D^Q-mates over 11716571 graphs on 10 vertices (up to 9 graphs with the same D^Q-spectrum)

Comparison

For all graphs on 10 vertices, repartition of the size of families sharing the same M-spectrum

Family size	$\mathcal{D}$	$\mathcal{D}^L$	$\mathcal{D}^Q$
2	583922	345065	148101
3	46300	20010	5978
4	14369	6947	1138
5	1905	819	87
6	1714	580	26
7	288	138	4
8	283	82	1
9	45	30	1
10	64	17	0
11	33	6	0
12	10	5	0
13	2	5	0
14	4	2	0
15	3	1	0
16	2	2	0
21	1	0	0
Total	1389984	787851	319324

Cospectrality of distance based matrices

Comparison

Proportions of mates among all graphs on up to 10 vertices

n	$\mathcal{D}$	$\mathcal{D}^L$	$\mathcal{D}^Q$
3	0	0	0
4	0	0	0
5	0	0	0.095238095
6	0	0	0.053571429
7	0.025791325	0.050410317	0.044548652
8	0.05918863	0.067014482	0.040748403
9	0.095978244	0.075754558	0.03128543
10	0.118634027	0.067242455	0.027254049

Comparison

Number of spectra for graphs on up to 10 vertices

n	# of graphs	# of $\mathcal{D}$–spectra	# of $\mathcal{D}^L$–spectra	# of $\mathcal{D}^Q$–spectra
3	2	2	2	2
4	6	6	6	6
5	21	21	21	20
6	112	112	112	109
7	853	842	831	834
8	11117	10784	10730	10885
9	261080	247984	251007	256900
10	11716571	10975532	11302429	11552583

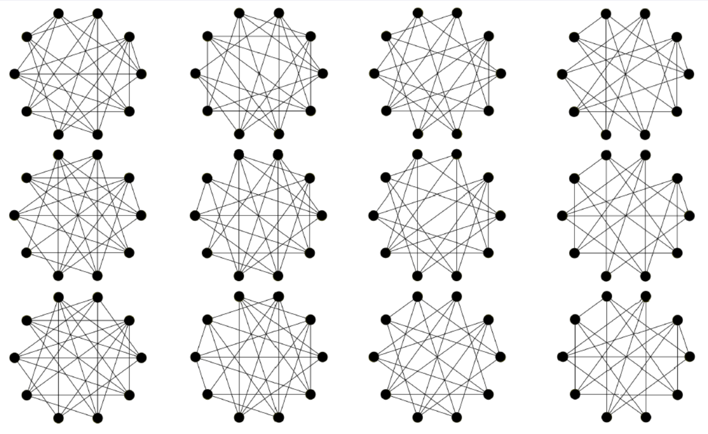
Cospectrality involving 2 0r 3 matrices

For all graphs on up to 10 vertices

n	$N(\mathcal{D},\mathcal{D}^L)$	$M(\mathcal{D},\mathcal{D}^L)$	$N(\mathcal{D},\mathcal{D}^Q)$	$M(\mathcal{D},\mathcal{D}^Q)$	$N(\mathcal{D}^L,\mathcal{D}^Q)$	$M(\mathcal{D}^L,\mathcal{D}^Q)$	$N(\mathcal{D},\mathcal{D}^L,\mathcal{D}^Q)$	$M(\mathcal{D},\mathcal{D}^L,\mathcal{D}^Q)$
3 – 7	0	1	0	1	0	1	0	1
8	0	1	0	1	90	2	0	1
9	32	2	0	1	1965	(7×) 3	0	1
10	9449	(15×) 3	7712	(4×) 3	61909	(343×) 3 (19×) 4	7622	(4×) 3

Cospectrality involving 2 0r 3 matrices

The 4 triplets of (D, D^L, D^Q)-mates on 10 vertices (columns)



Graphs determined by their D^L-spectrum

Theorem *From distance Laplacian spectrum of G, we can deduce the following:*

(i) *The number n of vertices of G.*

(ii) *The Wiener index of G.*

(iii) *The number of connected components of the complement $\overline{G}$.*

Corollary *The following graphs are determined by their distance Laplacian spectra:*

the complete graph K_n;

the graph $K_n - e$ obtained from K_n by the deletion of an edge;

the path P_n;

the comet $Co_{n,3}$.

Theorem *The k-partite graph on $n = n_1 + n_2 \cdots + n_k$ vertices, $K_{n_1,n_2,\dots,n_k}$, is determined by its distance Laplacian spectrum.*

Graphs determined by their D^Q-spectrum

Theorem *From signless distance Laplacian spectrum of a G, we can deduce the following:*

(i) *The number n of vertices of G.*
(ii) *The Wiener index of G.*
(iii) *Whether G is transmission regular.*

Corollary *The following graphs are determined by their distance signless Laplacian spectra:*

a) *the complete graph K_n;*
b) *the graph $K_n - e$ obtained from K_n by the deletion of an edge;*
c) *the path P_n;*
d) *the comet $Co_{n,3}$.*

Theorem *The cycle C_n is determined by its signless Laplacian distance spectrum.*