The curvature of Lorentzian $\mathcal{S}$-manifolds with an Osserman-type condition

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ABSTRACT. On a Lorentzian $\mathcal{S}$-manifold, which is a subclass of Lorentzian $g.f.f$-manifolds, a new Osserman-type condition is considered: the φ-null Osserman one. Here we briefly review some recent results, obtained by the authors, on the behaviour of the Jacobi operators and on a decomposition of the curvature tensor of such manifolds.

1. Introduction

The study of the curvature properties of a semi-Riemannian manifold is one of the most interesting and widely treated topics in geometry. Some of these properties can be deduced from the behaviour of certain curvature operators, among which there are the Jacobi operators.

Let (M, g) be a semi-Riemannian manifold; let us denote by ∇ the Levi-Civita connection and by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ the curvature operator of ∇. Fixed a point $p \in M$, if $z \in T_pM$ is a unit vector, the subspace $z^\perp \subset T_pM$ is invariant with respect to the action of the operator $R_z(\cdot) = R_p(\cdot, z)z$, and its restriction to $z^\perp$ is known as the (restricted) *Jacobi operator* with respect to z.

The Jacobi operator is a self-adjoint endomorphism of $z^\perp$, and the study of its eigenvalues, in the Riemannian case, or more generally the study of its characteristic polynomial, in the semi-Riemannian case, is of very special interest.

Indeed, for a Riemannian manifold (M, g), let us consider the unit sphere bundle $S(M)$, with fiber $S_p(M) = \{z \in T_pM \mid g_p(z, z) = 1\}$ at every $p \in M$. The manifold is said to be *pointwise Osserman* if, for every $p \in M$, the eigenvalues of the Jacobi operators R_z, with their algebraic multiplicities, are independent of $z \in S_p(M)$. If the manifold is pointwise Osserman and the eigenvalues of the R_z, with their algebraic multiplicities, do not vary from point to point, then the manifold is said to be *globally Osserman*, or simply *Osserman* (see [18]).

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A Riemannian space form is clearly an Osserman manifold, with Jacobi operators having exactly one constant eigenvalue. Any locally flat or rank-one symmetric space (two-point homogeneous space) is Osserman, with Jacobi operators having more than one constant eigenvalue. The converse statement is known as the Osserman Conjecture (see [23], [24]). This Conjecture has been proved to be true in almost any case ([11],[12],[13],[20],[21],[22]), except for some particularities in dimensions 4 and 16.

A condition of Osserman-type can be also considered in the semi-Riemannian context, where many counterexamples to the Osserman Conjecture have been found, except for the Lorentzian case. Indeed, it is known that any (connected) Lorentzian pointwise Osserman manifold is a space-form (see [4],[16],[17] and [18] for a simpler proof).

So, for a Lorentzian manifold, in [17] a new Osserman-type condition is introduced and studied: the *null Osserman condition*, which we shall recall shortly. It is interesting to see what this condition yields on manifolds endowed with a Lorentzian Sasakian structure, since this kind of structure carries a natural globally defined unit timelike vector field (the Reeb vector field) with respect to which the null Osserman condition may be considered. It is known that any Lorentzian Sasakian manifold with constant φ-sectional curvature is null Osserman with respect to the Reeb vector field.

Lorentzian $\mathcal{S}$-structures constitute a generalization of the Lorentzian Sasakian ones, and so it becomes natural to study the null Osserman condition on Lorentzian $\mathcal{S}$-manifolds. The first author proved in [6] that the null Osserman condition does not hold for Lorentzian $\mathcal{S}$-manifolds with constant φ-sectional curvature. Therefore, a specialized version of the null Osserman condition, the φ-null Osserman condition, has been introduced in [6], where some of the first properties have been also considered.

Carrying on our investigations, we were able to obtain in [8] some results concerning the relationships among the classical Osserman condition, the null Osserman one and the φ-null Osserman one, considering a principal torus bundle classically constructed on a Lorentzian $\mathcal{S}$-manifold (see [9] for this construction, in the case of indefinite metrics, and [3] for the Riemannian case).

At the same time, we obtained new results on the behaviour of the Jacobi operators on a Lorentzian $\mathcal{S}$-manifold, giving a characterization for the φ-null Osserman condition. Moreover, we generalized some curvature decompositions preliminarly provided in [6] for special cases. Here we give a synthetic review of these further results, referring the reader to [7] for more details.

2. Preliminaries and a first result

Let us recall (see [15],[25]) that a $(2n+1)$-dimensional manifold M $(n \geqslant 1)$ is called *indefinite Sasakian* if it is endowed with a structure (φ, ξ, η, g) made of a $(1,1)$-type tensor field φ of constant rank $2n$, a globally defined unit vector field ξ with its dual 1-form η and a semi-Riemannian metric tensor g, satisfying the following three conditions:

(1) $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$;
(2) $g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$, for any $X, Y \in \Gamma(TM)$;
(3) $(\nabla_X \varphi)(Y) = g(X, Y)\xi - \varepsilon\eta(Y)X$, for any $X, Y \in \Gamma(TM)$,

where $\varepsilon = g(\xi, \xi) = \pm 1$ according to the causal character of the vector field ξ, called the *Reeb vector field* of the structure. In particular, if g is a Lorentzian metric, one has necessarily $\varepsilon = -1$ and the manifold M with structure (φ, ξ, η, g) is said to be a *Lorentz Sasakian manifold*. We note that, if we use the Nijenhuis tensor field $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]$ of φ, then one can prove that the condition (3) is equivalent to both the conditions $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ and $d\eta(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$. The reader can also consult [2] or [5] for more details about contact structure in the Riemannian case.

A generalization of indefinite (and Lorentz) Sasakian manifolds is given by indefinite (and Lorentzian) $\mathcal{S}$-manifolds. Following [1] and [10], a $(2n+s)$-dimensional manifold M ($n \geqslant 1$ and $s \geqslant 1$) is said to be an *indefinite $\mathcal{S}$-manifold* if it is endowed with a structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$, with $\alpha \in \{1, \dots, s\}$, made of a $(1,1)$-type tensor field φ of constant rank $2n$, s globally defined unit vector fields ξ_α with their dual 1-forms η^α and a semi-Riemannian metric tensor g, satisfying the following three conditions:

(1) $\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha$ and $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$;
(2) $g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)$, for any $X, Y \in \Gamma(TM)$;
(3) $(\nabla_X \varphi)(Y) = g(X, Y)\tilde{\xi} - \tilde{\eta}(Y)X$, for any $X, Y \in \Gamma(TM)$,

where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$ according to the causal character of the vector field ξ_α, called *characteristic vector field* of the structure, $\tilde{\xi} = \sum_{\alpha=1}^{s} \xi_\alpha$ and $\tilde{\eta} = \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta_\alpha$. Clearly, for $s = 1$ we have an indefinite Sasakian manifold. If M is an $(2n + s)$-dimensional indefinite $\mathcal{S}$-manifold, with structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, some of the most immediate properties are listed below.

- $TM = \operatorname{Im}(\varphi) \oplus \langle \xi_1, \dots, \xi_s \rangle$ (orthogonal splitting).
- φ acts as an almost complex structure on the subbundle $\operatorname{Im}(\varphi)$.
- The signature of g on $\operatorname{Im}(\varphi)$ is of type $(2p, 2q)$, with $p + q = n$.
- The signature of g on TM is of type $(2p + h, 2q + k)$, with $h + k = s$.

If the metric tensor g is Lorentzian, then one can easily see that the signature of g on $\operatorname{Im}(\varphi)$ has to be Riemannian and exactly one of the characteristic vector fields has to be unit timelike. From now on, without loss of generality, we always assume that it is ξ_1, so that $\varepsilon_1 = -1$ and $\varepsilon_\alpha = +1$ for any $\alpha \neq 1$. A manifold M with an indefinite $\mathcal{S}$-structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$ where g is a Lorentzian metric is said to be a *Lorentzian $\mathcal{S}$-manifold*.

We note that, using the Nijenhuis tensor field of φ, then one can prove that the condition (3) of the definition is equivalent to both the conditions $[\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha = 0$ and $d\eta^\alpha(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$ and any $\alpha \in \{1, \dots, s\}$.

Following [17] (see also [18]), let us recall the notion of null Osserman condition. First, we need the notion of Jacobi operator with respect to a lightlike vector. Let (M, g) be a Lorentzian manifold, $p \in M$ and $u \in T_pM$ a lightlike vector, that is $u \neq 0$ and $g_p(u, u) = 0$. Then $\operatorname{span}(u) \subset u^\perp$ and we put $\bar{u}^\perp = u^\perp/\operatorname{span}(u)$. The canonical projection is $\pi\colon u^\perp \to \bar{u}^\perp$. A positive definite inner product $\bar{g}$ can be defined on $\bar{u}^\perp$ by putting $\bar{g}(\bar{x},\bar{y}) = g_p(x,y)$, where $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$, so that $(\bar{u}^\perp, \bar{g})$ becomes an Euclidean vector space. The *Jacobi operator with respect to* u is the endomorphism $\bar{R}_u\colon \bar{u}^\perp \to \bar{u}^\perp$ defined by $\bar{R}_u(\bar{x}) = \pi(R_p(u,x)x)$, for all $\bar{x} = \pi(x) \in \bar{u}^\perp$. One sees that $\bar{R}_u$ is self-adjoint, hence diagonalizable. Now, we can define the null Osserman condition. If $z \in T_pM$ is a unit timelike vector, the *null congruence set* of z at p is $N(z) = \{u \in T_pM \,|\, g_p(u,u) = 0,\ g_p(u,z) = -1\}$.

Definition 2.1 ([17],[18]). A Lorentzian manifold (M,g) is said to be *null Osserman with respect to* z, $z \in T_pM$ being a unit timelike vector, if the eigenvalues of $\bar{R}_u$, counted with multiplicities, are independent of $u \in N(z)$.

As we said, it is known that Lorentz Sasakian manifolds with constant φ-sectional curvature are null Osserman with respect to the characteristic vector field. In contrast to this, we have proved the following.

Proposition 2.1 ([7]). *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be a Lorentzian $\mathcal{S}$-manifold, with* $\dim(M) = 2n+s$, $s \geqslant 2$. *Then M does not verify the null Osserman condition with respect to $(\xi_1)_p$, for all $p \in M$.*

The above result is based on the fact that the null congruence set $N(\xi_1)$ contains null vectors of the following two types

- $u = \xi_1 + \xi_\beta$, with $\beta \neq 1$,
- $v = \xi_1 + x$, with unit $x \in \mathrm{Im}(\varphi)$,

and one can prove that $\bar{R}_u = 0$ but $\bar{R}_v \neq 0$.

3. The φ-null Osserman condition and some consequences

Motivated by the previous result, in [6] it has been considered the subset

$$N_\varphi((\xi_1)_p) = \{u = \xi_1 + x \in T_pM \,|\, \text{with unit } x \in \mathrm{Im}(\varphi)\},$$

called the *φ-null congruence set* of ξ_1, and the following definition has been given.

Definition 3.1 ([6]). A Lorentzian $\mathcal{S}$-manifold $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, with $\alpha \in \{1,\dots,s\}$, is said to be *φ-null Osserman* at a point $p \in M$ if and only if the eigenvalues of $\bar{R}_u$, counted with multiplicities, are independent of $u \in N_\varphi((\xi_1)_p)$.

The notion of φ-null Osserman condition seems to be suitable for Lorentzian $\mathcal{S}$-manifold. Indeed, it generalizes the null Osserman condition, since in a Lorentz Sasakian manifold one has clearly $N_\varphi(\xi) = N(\xi)$ and the φ-null Osserman condition reduces to the null Osserman one. Moreover, in [6] is proved that any Lorentzian $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman with respect to $(\xi_1)_p$, at any point $p \in M$, thus recovering the similar known result for Lorentz Sasakian space forms.

Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha \in \{1,\dots,s\}$, be a $(2n+s)$-dimensional Lorentzian $\mathcal{S}$-manifold, $s \geqslant 1$. Fix $p \in M$ and consider $u \in N_\varphi((\xi_1)_p)$. Then $u = (\xi_1)_p + x$, with unit $x \in \mathrm{Im}(\varphi_p)$, and we can consider the Jacobi operator $R_x\colon x^\perp \to x^\perp$ corresponding to $\bar{R}_u\colon \bar{u}^\perp \to \bar{u}^\perp$, and vice-versa. Studying the links between these two operators we get the following.

Proposition 3.1 ([7]). *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, be a Lorentzian $\mathcal{S}$-manifold, with $\dim(M) = 2n + s$, $s \geqslant 1$. For any $p \in M$, M is φ-null Osserman at p if and only if the eigenvalues of R_x, counted with multiplicities, are independent of any unit $x \in \mathrm{Im}(\varphi_p)$.*

If $V = x^\perp \cap \mathrm{Im}(\varphi_p)$ and $U = \mathrm{span}(\xi_2, \dots, \xi_s)$, then we have

- $\bar{u}^\perp \cong V \oplus U$ (or simply $\bar{u}^\perp \cong V$, if $s = 1$),
- $x^\perp \cong V \oplus \mathrm{span}(\xi_1) \oplus U$ (or simply $x^\perp \cong V \oplus \mathrm{span}(\xi_1)$, if $s = 1$),

where V and U are both invariant subspaces with respect to the action of $\bar{R}_u$. The previous result is based on the decomposition $\bar{R}_u = \bar{R}_u|_V \circ p_V + \bar{R}_u|_U \circ p_U$, where p_V and p_U are the projections of $\bar{u}^\perp$ onto V and U, respectively. We see that $\bar{R}_u|_U \circ p_U$ always admits the eigenvalues $\lambda_0 = 0$, with multiplicity $m_0 = s - 2$, and $\lambda_1 = s - 1$, with multiplicity $m_1 = 1$, independent of $u \in N((\xi_1)_p)$. Furthermore, we have found the following.

Proposition 3.2 ([7]). *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, be a Lorentzian $\mathcal{S}$-manifold, with $\dim(M) = (4m + 2) + s$, $s \geqslant 1$, where $\mathrm{rank}(\mathrm{Im}(\varphi)) = 4m + 2$. If M is φ-null Osserman at a point $p \in M$, then only one of the following two cases can occur:*

(i) *$\bar{R}_u|_V \circ p_V$ admits exactly one eigenvalue with multiplicity $4m + 1$;*
(ii) *$\bar{R}_u|_V \circ p_V$ admits exactly two eigenvalues with multiplicities 1 and $4m$.*

The proof is similar to those of Q.S. Chi ([11]), based on some well-known results about the maximal dimensions of distributions on spheres.

4. Curvature properties

Note that if $\bar{R}_u|_V \circ p_V$ admits exactly one eigenvalue, M has constant φ-sectional curvature at the point p, and if we suppose that M is φ-null Osserman at each $p \in M$, and that the eigenvalues of $\bar{R}_u$ do not depend on the point p, then M is a Lorentzian $\mathcal{S}$-space form.

Now, we see what happens if $\bar{R}_u|_V \circ p_V$ admits exactly two different eigenvalues, one of which having multiplicity 1. Namely, we are able to give a curvature characterization for those φ-null Osserman Lorentzian $\mathcal{S}$-manifolds satisfying the above condition. The result is mainly based on the following two tools:

- a known construction for Osserman algebraic curvature tensors having two distinct eigenvalues;
- a special curvature characterization for Lorentzian $\mathcal{S}$-manifolds, obtained by the authors in [7], after a special result contained in [6].

Let us quote the first of these two tools, a purely algebraic result, according to the formulation given by P. Gilkey in [19, Lemma 3.5.1, p. 204].

Proposition 4.1. *Let $(V, \langle \cdot, \cdot \rangle)$ be a Riemannian vector space and R an Osserman algebraic curvature tensor on V. Suppose that the Jacobi operator $\mathcal{J}_R(x)$ has exactly two different eigenvalues $\{c_1, c_2\}$, where c_1 has multiplicity 1. Then,*

(i) *There exists an Hermitian structure ϕ on V assigning to each unit vector $x \in V$ the unit eigenvector ϕx of $\mathcal{J}_R(x)$ with respect to c_1;*

(ii) $R = c_2 R_{Id} + \frac{1}{3}(c_1 - c_2) R_\phi$;

where

$$R_{Id}(x,y)z = \langle y,z\rangle x - \langle x,z\rangle y,$$
$$R_\phi(x,y)z = \langle \phi y,z\rangle \phi x - \langle \phi x,z\rangle \phi y - 2\langle \phi x,y\rangle \phi z.$$

In order to provide the second tool, we need to recall a remarkable characterization for algebraic curvature tensors on a semi-Riemannian manifold.

Proposition 4.2 ([14]). *Let (M,g) be a semi-Riemannian manifold, F an algebraic curvature tensor on M and $p \in M$. The following two conditions are equivalent.*

(a) $F(x,y,z,w) = k\{g(x,z)g(y,w) - g(y,z)g(x,w)\}$, $k \in \mathbb{R}$, $x,y,z,w \in T_pM$;
(b) $F(x,y,x,y) = 0$, *on any degenerate plane* $\pi = \mathrm{span}\{x,y\}$ *in* T_pM.

In particular, this characterizes the curvature of semi-Riemannian manifolds by means of the behaviour on all the degenerate 2-planes but, in the context of Lorentzian $\mathcal{S}$-manifolds, we can only work with very special degenerate 2-planes. Therefore, using the above result, we were able to provide the following new curvature characterization.

Theorem 4.1 ([7]). *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha \in \{1,\dots,s\}$, $s \geqslant 1$, be a Lorentzian $\mathcal{S}$-manifold, $p \in M$ and R the Riemannian curvature tensor on M. The following two conditions are equivalent.*

(a) $R(u,y,u,y) = 0$, *on any degenerate plane* $\pi = \mathrm{span}\{u,y\}$ *in* T_xM, *with* $u \in N_\varphi(\xi_1)$ *and* $y \in u^\perp \cap \mathrm{Im}(\varphi)$;
(b) $R(x,y,z,w) = g(S(x,y)z,w) - g(T(x,y)z,w)$, *for all* $x,y,z,w \in T_xM$;

where S and T are the algebraic curvature operators on M defined by

$$S(x,y)z = g(\varphi x,\varphi z)\varphi^2 y - g(\varphi y,\varphi z)\varphi^2 x;$$
$$T(x,y)z = g(\varphi y,\varphi z)\tilde\eta(x)\tilde\xi - g(\varphi x,\varphi z)\tilde\eta(y)\tilde\xi - \tilde\eta(y)\tilde\eta(z)\varphi^2 x + \tilde\eta(x)\tilde\eta(z)\varphi^2 y;$$

with $\tilde\eta = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$ *and* $\tilde\xi = \sum_{\alpha=1}^s \xi_\alpha$.

It is worth noting that the above result also holds for any algebraic curvature tensor F on M, provided that it satisfies the identities

$$F(x,\xi_\beta,y,\xi_\gamma) = \varepsilon_\beta\varepsilon_\gamma g(\varphi x,\varphi y) \qquad \text{and} \qquad F(\varphi x,\varphi y,\varphi z,\xi_\beta) = 0$$

for all $x,y,z \in T_pM$ and all $\beta,\gamma \in \{1,\dots,s\}$.

Anyway, combining together Theorem 4.1 and Proposition 4.1 we finally get the following curvature characterization for φ-null Osserman Lorentzian $\mathcal{S}$-manifolds.

Theorem 4.2 ([7]). *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha \in \{1,\dots,s\}$, $s \geqslant 1$, be a Lorentzian $\mathcal{S}$-manifold, with $\dim(M) = 2n+s$. The following statements are equivalent.*

(a) *M is φ-null Osserman at a point $p \in M$, and $\bar R_u|_V \circ p_V$ has exactly two different eigenvalues $\{c_1,c_2\}$ for any $u \in N_\varphi((\xi_1)_p)$, where c_1 has multiplicity 1;*

(b) *there exists an Hermitian structure J on* $\mathrm{Im}(\varphi_p)$, $c_1, c_2 \in \mathbb{R}$ *such that*

$$R = T - S + c_2 R'_{Id} + \frac{1}{3}(c_1 - c_2) R'_J,$$

where R'_{Id} and R'_J are the algebraic curvature operators on M defined by

$$R'_{Id}(x, y)z = g(y', z')x' - g(x', z')y'$$
$$R'_J(x, y)z = g(Jy', z')Jx' - g(Jx', z')Jy' - 2g(Jx', y')z',$$

and $x' = -\varphi^2 x$ is the projection on $\mathrm{Im}(\varphi_p)$ *of any $x \in T_pM$.*

When $s = 1$, that is when the manifold is a null Osserman Lorentz Sasakian one, by direct calculation one has

$$T(x, y)z - S(x, y)z = g(x, z)y - g(y, z)x$$

for all $x, y, z \in T_pM$, and we obtain the following final result as a corollary.

Corollary 4.1 ([7]). *Let $(M, \varphi, \xi, \eta, g)$ be a Lorentz Sasakian manifold. The following statements are equivalent.*

(a) *M is null Osserman at a point $p \in M$, and $\bar{R}_u|_V \circ p_V$ has exactly two different eigenvalues $\{c_1, c_2\}$ for any $u \in N_\varphi((\xi)_p)$, with c_1 having multiplicity 1;*

(b) *there exists an Hermitian structure J on* $\mathrm{Im}(\varphi_p)$, $c_1, c_2 \in \mathbb{R}$ *such that*

$$R = -R_{Id} + c_2 R'_{Id} + \frac{1}{3}(c_1 - c_2) R'_J,$$

where R_{Id} is the algebraic curvature operator on M defined by

$$R_{Id}(x, y)z = g(y, z)x - g(x, z)y$$

for all $x, y, z \in T_pM$.

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