

Spectral Characterizations of Graphs; the generalized adjacency spectrum

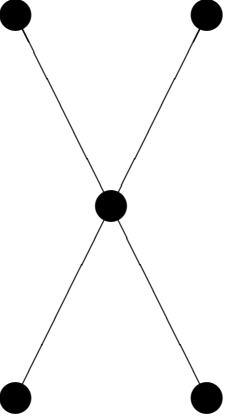
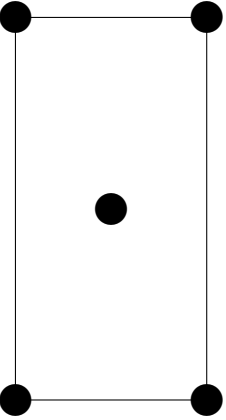
Willem H. Haemers

Tilburg University, The Netherlands.

In honour of Dragoš Cvetković







$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

spectrum

$$-2, 0^3, 2$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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spectrum $-2, 0^3, 2$

complement

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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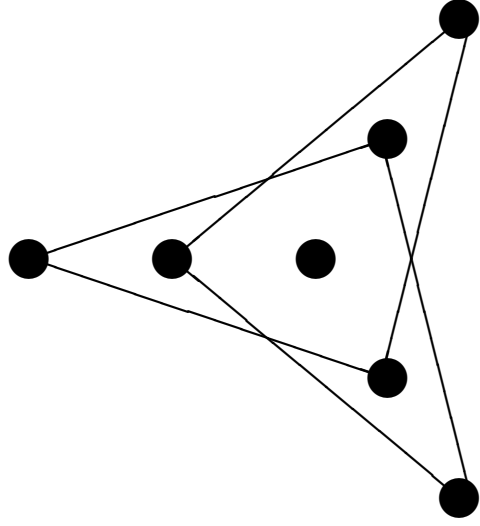
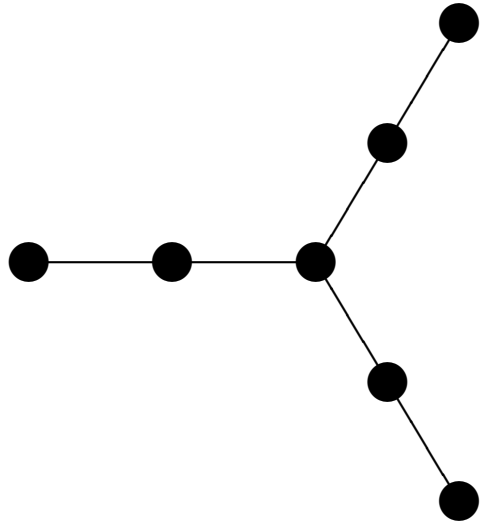
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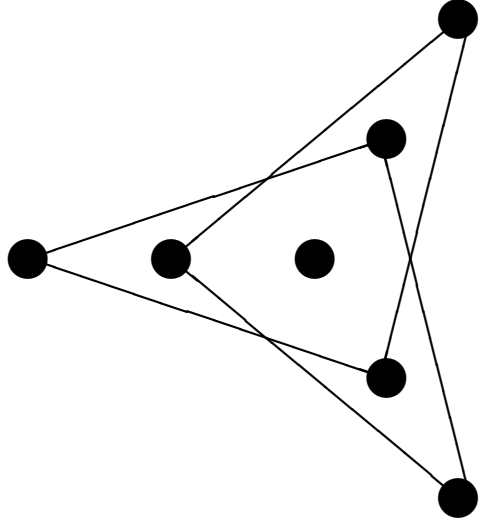
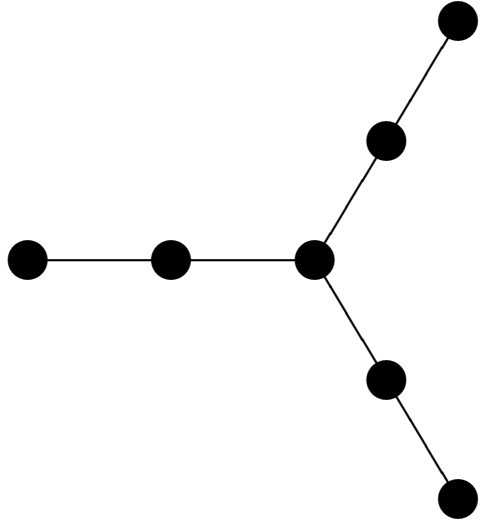
spectra

$$-1^2, 1, \frac{1}{2} \pm \frac{1}{2}\sqrt{17} \qquad -1^3, 0, 3$$



spectrum

$$0, \pm 1^2, \pm 2$$



spectrum

$$0, \pm 1^2, \pm 2$$

spectrum complement

$$-2^2, 1, 0^2, \frac{3}{2} \pm \frac{1}{2}\sqrt{33}$$

the **generalized adjacency matrix**

$$A(\alpha, \beta, \gamma) = \alpha A + \beta J + \gamma I \quad (\alpha \neq 0)$$

$$A(\alpha, \beta, \gamma)_{i,j} = \begin{cases} \alpha + \beta & \text{if } \{i, j\} \in E \\ \beta & \text{if } \{i, j\} \notin E, \ i \neq j \\ \beta + \gamma & \text{if } i = j \end{cases}$$

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$$A(1, 0, 0) = A, \quad A(-1, 1, -1) = \overline{A}, \quad A(-2, 1, -1) = S$$

$$A(\alpha, \beta, \gamma) \text{ cospectral with } A'(\alpha, \beta, \gamma)$$

$$A + \frac{\beta}{\alpha} J \text{ cospectral with } A' + \frac{\beta}{\alpha} J$$

THEOREM (Johnson and Newman 1980)

The following are equivalent:

- $A + yJ$ is cospectral with $A' + yJ$ for **all** $y \in \mathbb{R}$,
- $A + yJ$ and $A' + yJ$ are cospectral for **two** distinct values of y ,
- There exists a regular orthogonal matrix Q such that $Q^\top A Q = A'$.

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Two graphs are cospectral with respect to every generalized adjacency matrix if and only if

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The *generalized spectrum* of a graph G is the adjacency spectrum of G together with the adjacency spectrum of the complement of G .

CONJECTURE 1

Almost all graphs are determined by their adjacency spectrum (DAS)

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CONJECTURE 2

Almost all graphs are determined by their generalized adjacency spectrum (DGS)

For regular graphs DGS is the same as DAS

Ways to prove that a given graph is DAS or DGS?

- Find structural properties from the spectrum and hope that it determines the graph.
- Generate all graphs with the same number of vertices (and edges and triangles) and check.
- (Wang and Xu) Find all regular orthogonal matrices Q , such that $Q^{\top}AQ$ is a $(0, 1)$ matrix.

Method of Wang and Xu

Graph G with adjacency matrix A is *controllable* if the walk matrix

$$W = \begin{bmatrix} \mathbf{1} & A\mathbf{1} & A^2\mathbf{1} & \dots & A^{n-1}\mathbf{1} \end{bmatrix}$$

is nonsingular.

Method of Wang and Xu

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$$W = \begin{bmatrix} \mathbf{1} & A\mathbf{1} & A^2\mathbf{1} & \dots & A^{n-1}\mathbf{1} \end{bmatrix}$$

is nonsingular.

Suppose G is controllable and $Q^\top A Q$ is a $(0, 1)$ -matrix A' (say), for some regular orthogonal matrix Q , then

- Q is unique (for fixed A'),
- Q is rational,
- $\text{SNF}(W)$ gives an integer ℓ such that ℓQ is integral,
- In many cases $\ell = 2$ in which case Q is characterized.

THEOREM
O'Rourke and Turi (arXiv 2015)
Almost all graphs are controllable

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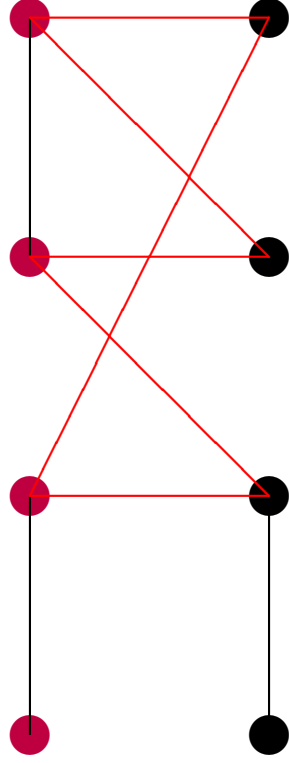
Wang (arXiv 2014)

If $\det(W) / \lfloor 2^{n/2} \rfloor$ is odd and square free, then G is DGS.

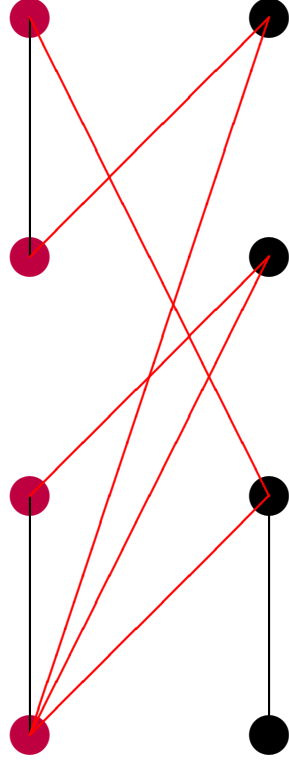
Fractions of graphs on n vertices which are not DAS, and not DGS.

n	number of graphs	not DAS	not DGS	reference
1	1	0	0	
2	2	0	0	
3	4	0	0	
4	11	0	0	
5	34	0.059	0	
6	156	0.064	0	
7	1044	0.105	0.038	
8	12346	0.139	0.094	
9	274668	0.186	0.160	Godsil, McKay (1976)
10	12005168	0.213	0.201	H, Spence (2004)
11	1018997864	0.211	0.208	H, Spence (2004)
12	165091172592	0.188		Brouwer, Spence (2009)

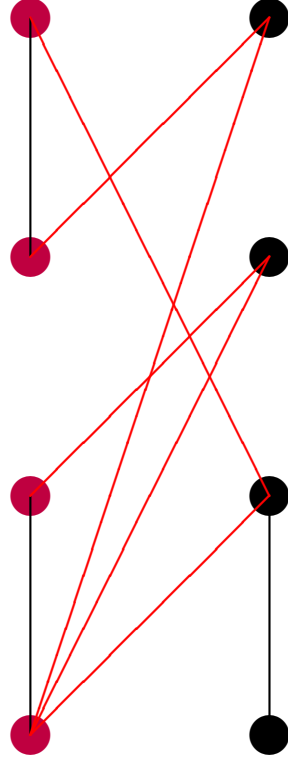
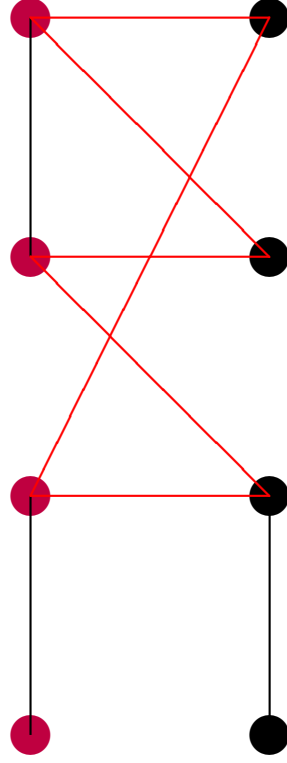
Godsil-McKay switching produces graphs which are cospectral with respect to the generalized adjacency matrix.



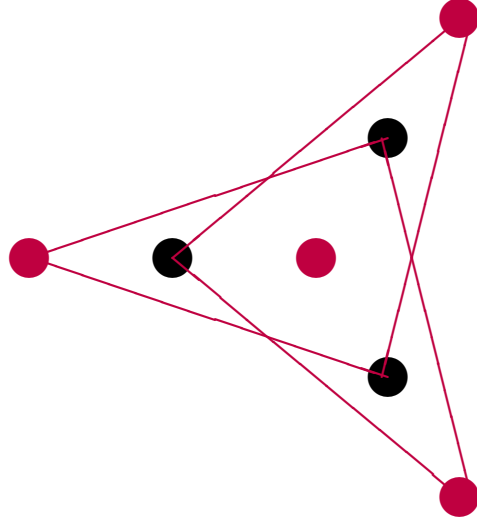
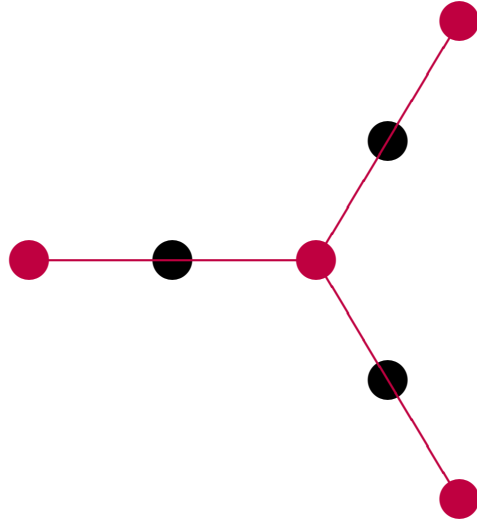
Godsil-McKay switching produces graphs which are cospectral with respect to the generalized adjacency matrix.



Godsil-McKay switching



Godsil-McKay switching



Fractions of graphs on n vertices which are not DAS,
not DGS, and with a proper GM switching set.

n	number of graphs	not DAS	not DGS	GM
1	1	0	0	0
2	2	0	0	0
3	4	0	0	0
4	11	0	0	0
5	34	0.059	0	0
6	156	0.064	0	0
7	1044	0.105	0.038	0.038
8	12346	0.139	0.094	0.085
9	274668	0.186	0.160	0.139
10	12005168	0.213	0.201	0.171
11	1018997864	0.211	0.208	0.174
12	165091172592	0.188		

$$\begin{bmatrix}
 1 & 1 & 1 & 1 & 1 \\
 1 & & 1 & & 1 \\
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 \end{bmatrix}
 \begin{bmatrix}
 1 & 1 & 1 & 1 & 1 \\
 & 1 & & 1 & \\
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 & & & & 1
 \end{bmatrix}
 \begin{bmatrix}
 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & & 1 \\
 & & 1 & 1 & 1 \\
 & & 1 & 1 & 1 \\
 & & 1 & 1 & 1
 \end{bmatrix}$$

GM-switch

Isomorphic after GM-switching: Petersen graph

$$A = \left[\begin{array}{c|c} \textcolor{red}{X} & N^\top \\ \hline N & B \end{array} \right] \quad \text{GM-switch} \quad \left[\begin{array}{c|c} \textcolor{red}{X} & N'^\top \\ \hline N' & B \end{array} \right] = A'$$

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$$N = \begin{bmatrix} H \\ O \\ J \end{bmatrix}, \quad N' = \begin{bmatrix} J-H \\ O \\ J \end{bmatrix}, \quad H\mathbf{1} = (J-H)\mathbf{1} = \frac{|X|}{2}\mathbf{1}.$$

$$A = \left[\begin{array}{c|c} \textcolor{red}{X} & N^\top \\ \hline N & B \end{array} \right] \quad \text{GM-switch} \quad \left[\begin{array}{c|c} \textcolor{red}{X} & N'^\top \\ \hline N' & B \end{array} \right] = A'$$

Suppose there exist permutation matrices P_1 and P_2 such that

$$P_1^\top X P_1 = \textcolor{red}{X}, \quad P_2^\top B P_2 = B, \quad P_2^\top N P_1 = N',$$

$$\text{then} \quad P^\top A P = A' \quad \text{with } A = \begin{bmatrix} P_1 & O \\ O & P_2 \end{bmatrix}.$$

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$$P_1^\top X P_1 = X, \quad P_2^\top B P_2 = B, \quad P_2^\top N P_1 = N',$$

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EXAMPLES: case $|X| = 2$, Petersen graph, grid.

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EXAMPLES: case $|X| = 2$, Petersen graph, grid.

THEOREM (Abiad, Brouwer, H)

The converse is not true.

Useful easy necessary conditions for isomorphism after GH-switching.

- Same degree sequence.
- Same numbers of 3-vertex configurations.

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Does not work for strongly regular graphs.

Symplectic graph $\text{Sp}(2, 2\nu)$. Vertex set $V = \mathbf{IF}_2^{2\nu} \setminus \{\mathbf{0}\}$.

$\mathbf{v} = [v_1 \dots v_{2\nu}]^\top$ adjacent to $\mathbf{w} = [w_1 \dots w_{2\nu}]^\top$ whenever

$$(v_1 w_2 + v_2 w_1) + (v_3 w_4 + v_4 w_3) + \dots + (v_{2\nu-1} w_{2\nu} + v_{2\nu} w_{2\nu-1}) = 1.$$

Equivalently, $\mathbf{v}^\top K \mathbf{w} = 1$ with

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Equivalently, $\mathbf{v}^\top K \mathbf{w} = 1$ with

$$K = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & \cdot \\ & & & & & & & 1 \\ & & & & & & & & 1 \end{bmatrix}$$

The symplectic graph $\text{Sp}(2, 2\nu)$ is strongly regular with parameters

$$n = 2^{2\nu} - 1, \quad k = 2^{2\nu-1}, \quad \lambda = 2^{2\nu-2}, \quad \mu = 2^{2\nu-2}.$$

and adjacency matrix $A = MKM^\top$ (over $\mathbf{F}_2$), where M consists of all nonzero vectors in $\mathbf{F}_2^{2\nu}$.

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Note that $2\text{-rank}(A) = 2\nu$.

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Note that $2\text{-rank}(A) = 2\nu$.

Any linear combination (over $\mathbf{F}_2$) of columns of A is again a column of A , or $\mathbf{0}$.

Godsil-McKay switching set in $\text{Sp}(2, 2\nu)$ ($\nu \geq 3$, $\mathbf{z} \in \mathbf{IF}_2^{2\nu-6}$):

$\mathbf{v}$	$\mathbf{w}$	$\mathbf{x}$	$\mathbf{y}$
$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \mathbf{z} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ \mathbf{z} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ \mathbf{z} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ \mathbf{z} \end{bmatrix}$

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$\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}$ are mutually nonadjacent, and for $\mathbf{u} \notin \{\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}\}$

$$\mathbf{u}^\top K \mathbf{v} + \mathbf{u}^\top K \mathbf{w} + \mathbf{u}^\top K \mathbf{x} + \mathbf{u}^\top K \mathbf{y} = \mathbf{u}^\top K(\mathbf{v} + \mathbf{w} + \mathbf{x} + \mathbf{y}) = \mathbf{u}^\top \mathbf{0} = 0.$$

$$A =$$

1	1	0	0
1	0	1	0
row 1			
row 2			

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

row 1

row 2

row 1 + row 2

The sum of these three rows equals $[0\ 0\ 0\ 0\ | \ 0\ \dots\ 0]$.

$$A' = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \text{row } 1 \\ \text{row } 2 \\ \text{row } 1 + \text{row } 2 \end{array}$$

The sum of these three rows equals $[\ 1 \ 1 \ 1 \ 1 \mid 0 \ \dots \ 0 \]$.

A is not isomorphic to A' . $2\text{-rank}(A') = 2\nu + 2$.

THEOREM Abiad and H (2015)

There exist strongly regular graphs with the parameters of $\text{Sp}(2, 2\nu)$ and 2-rank

$$2\nu, 2\nu + 2, \dots, 2\nu + 12 \lfloor \frac{\nu}{3} \rfloor$$

CONGRATULATIONS DRAGOŠ