# Spectral and combinatorial properties of lexicographic powers of graphs

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Joint work with Nair Abreu[2], Paula Carvalho[1], Cybele Vinagre[3]

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Spectra of graphs and applications 2016 - A Conference in honor of Professor Dragos Cvetković for his 75th birthday

# Outline

# Notation

- $G = (V(G), E(G))$ is a simple graph of order $n$ and size $m$, vertex set $V(G)$ and edge set $E(G)$.
- $A_G = (a_{ij})$ is the adjacency matrix of $G$, that is, is the $n \times n$ matrix with $a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \textit{otherwise} \end{cases}$;
- $D_G$ is the diagonal matrix whose diagonal elements are the degrees $d_1, \ldots, d_n$ of the vertices of $G$;
- $L_G = D_G - A_G$ is the Laplacian matrix of $G$.

# Notation (cont.)

- The eigenvalues of $A_G$ and $L_G$ are indexed in nonincreasing order, i.e.,
  - $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$;
  - $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$.

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- The eigenvalue $\mu_{n-1}(G)$ is the *algebraic connectivity* of $G$.

- The all one vector $j$ is the eigenvector of $L_G$ associated to $\mu_n(G)$ and the multiplicity of $\mu_n(G)$ is equal to the number of components of $G$.

# Lexicographic product

## Definition

The *lexicographic product* of two graphs $H$ and $G$ is the graph $H \cdot G$ (also called the graph *composition* and denoted $H[G]$) where

- its vertex set is the cartesian product $V(H) \times V(G)$
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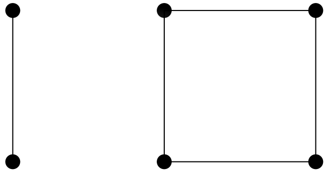
This operation was introduced by Harary(1959) and Sabidussi (1959).



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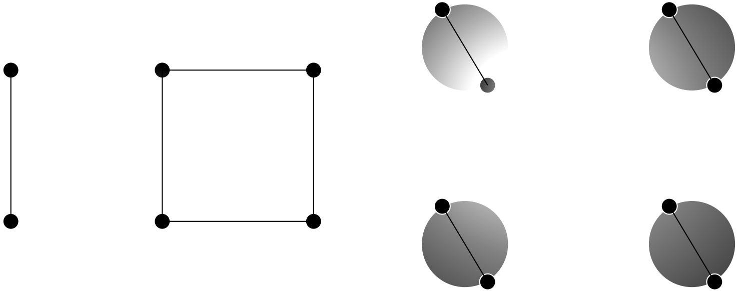
The lexicographic product is associative but it is not commutative.



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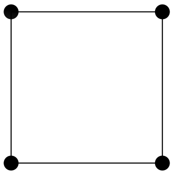

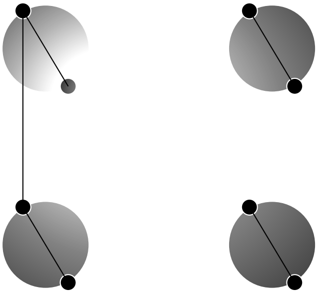

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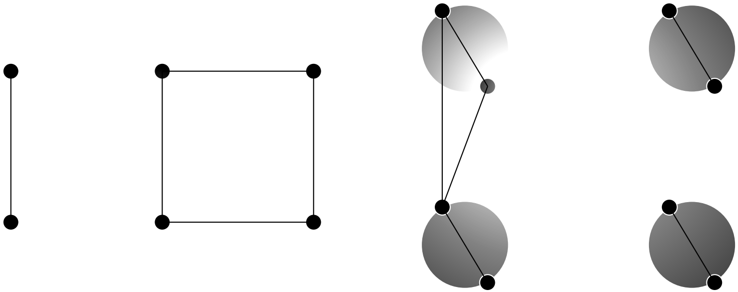

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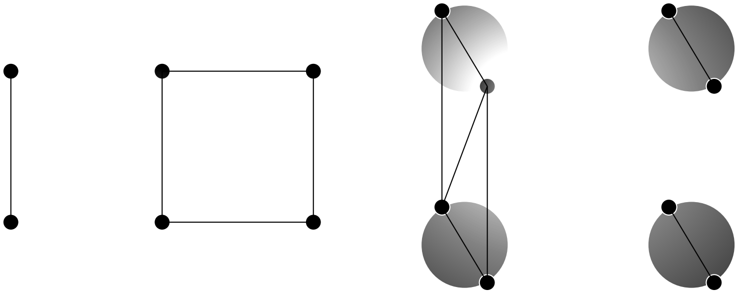

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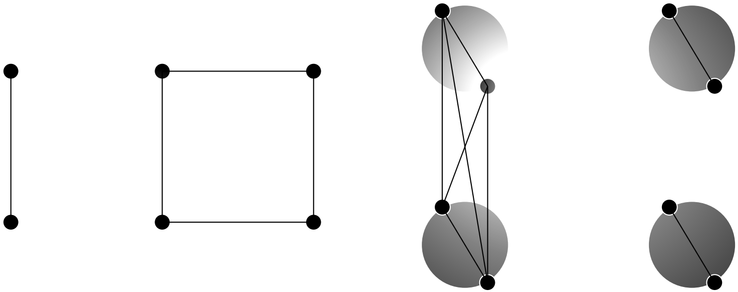

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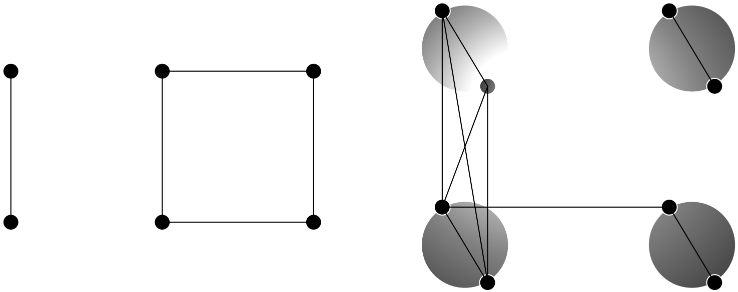

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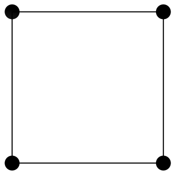

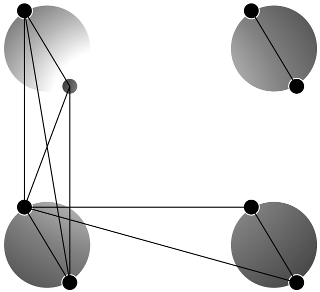

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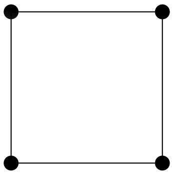

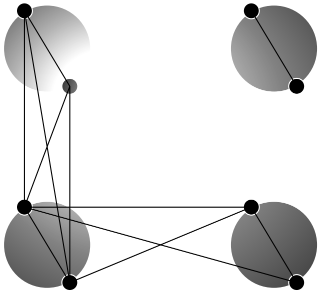

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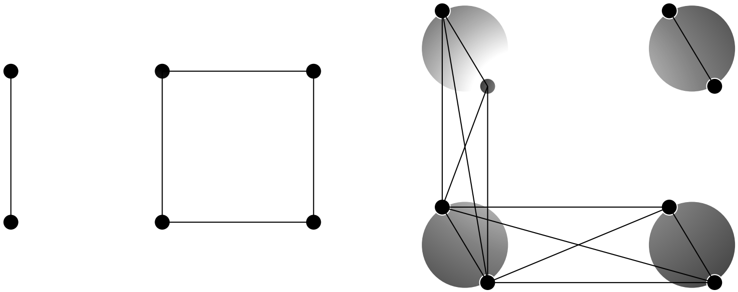

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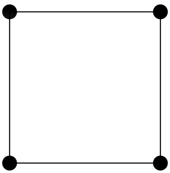

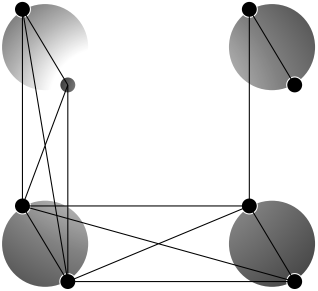

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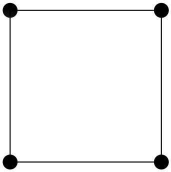

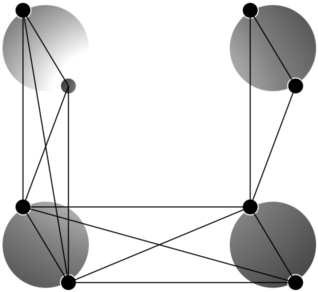

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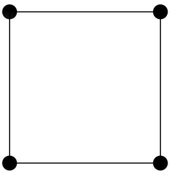

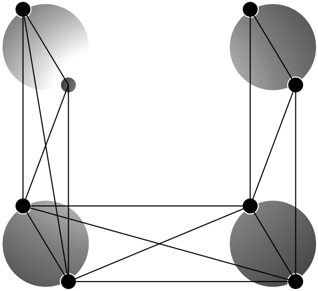

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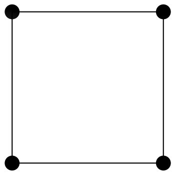

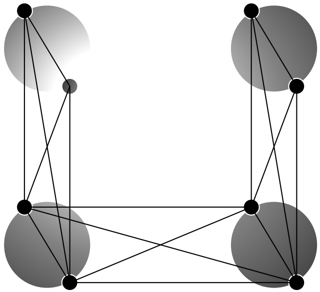

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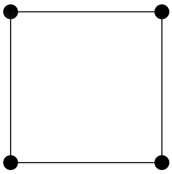

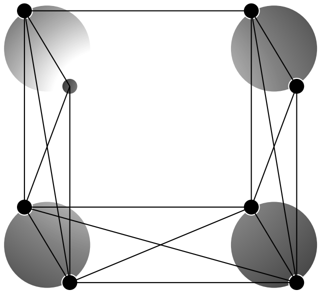

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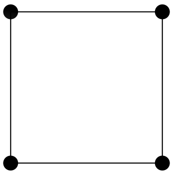

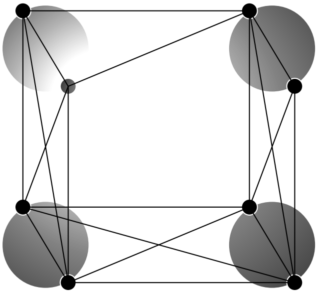

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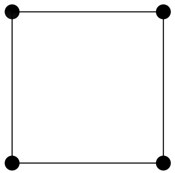

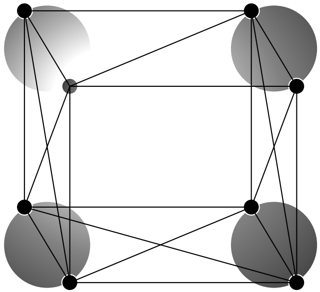

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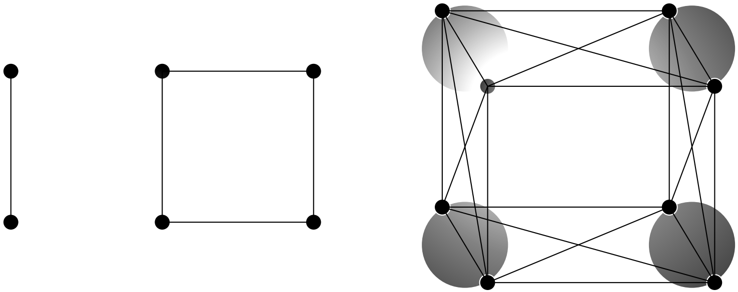

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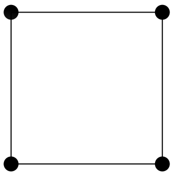
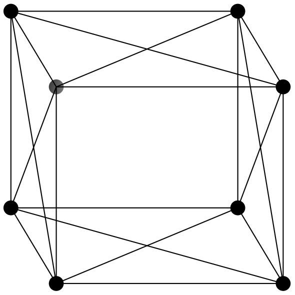


$C_4 \cdot K_2 = C_4[K_2]$



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**Theorem**

If $G$ is a $p$-regular graph of order $m$ and $H$ is graph of order $n$, then

$$\sigma_A(H[G]) = n\,(\sigma_A(G) \setminus \{p\}) \cup \sigma(mA_H + pI),$$

where $n\,(\sigma_A(G) \setminus \{p\})$ means that the multiset $(\sigma_A(G) \setminus \{p\})$ is repeated $n$ times.

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In our example:

$\sigma_A(K_2) = \{1, -1\}$
$\sigma_A(C_4) = \{2, 0, 0, -2\}$
$m = 2$
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If $G$ is a $p$-regular graph of order $m$ and $H$ is graph of order $n$, then

$$\sigma_A(H[G]) = n\,(\sigma_A(G) \setminus \{p\}) \cup \sigma(mA_H + pI),$$

where $n\,(\sigma_A(G) \setminus \{p\})$ means that the multiset $(\sigma_A(G) \setminus \{p\})$ is repeated $n$ times.

In our example:

$\sigma_A(K_2) = \{1, -1\}$

$\sigma_A(C_4) = \{2, 0, 0, -2\}$

$m = 2$

$n = 4$

$p = 1$

So

$\sigma_A(K_2) \setminus \{1\} = \{-1\}$ with multiplicity $n = 4$

$2 \times 2 + 1 = 5$

$2 \times 0 + 1 = 1$ (twice)

$2 \times (-2) + 1 = -3$

## The spectra of the lexicographic product

From [C., Freitas, Martins, Robbiano, 2013] we can deduce the theorem.

**Theorem**

If $G$ is a $p$-regular graph of order $m$ and $H$ is graph of order $n$, then

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So

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$2 \times 2 + 1 = 5$
$2 \times 0 + 1 = 1$ (twice)
$2 \times (-2) + 1 = -3$

$$\sigma_A(C_4[K_2]) = \{5, 1, 1, -3, -1, -1, -1, -1\}$$

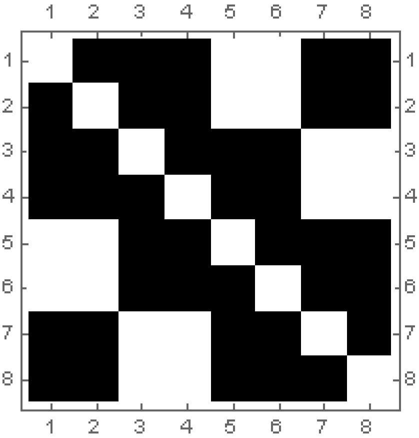
# Arbitrary number of iterations of the lexicographic product

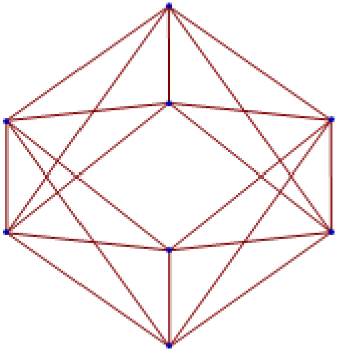
$$
\begin{array}{rcl}
H^0[G] & = & G, \\
H^1[G] & = & H[G] \\
H^2[G] & = & H[H[G]], \\
 & \vdots & \\
H^k[G] & = & H[H^{k-1}[G]] \\
 & \vdots &
\end{array}
$$

( we assume that the graph $H$ is connected, and then, for $k \geq 1$, $H^k[G]$ is connected).

# Example: Adjacency Matrix and graph for $C_4^k[K_2]$

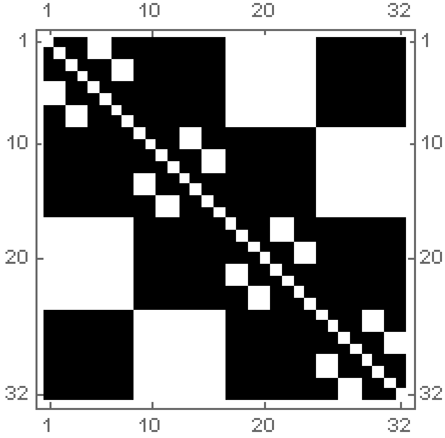
k=1

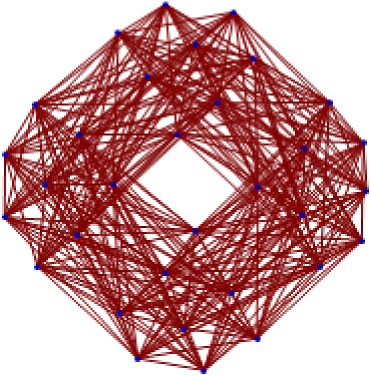


# Example: Adjacency Matrix and graph for $C_4^k[K_2]$

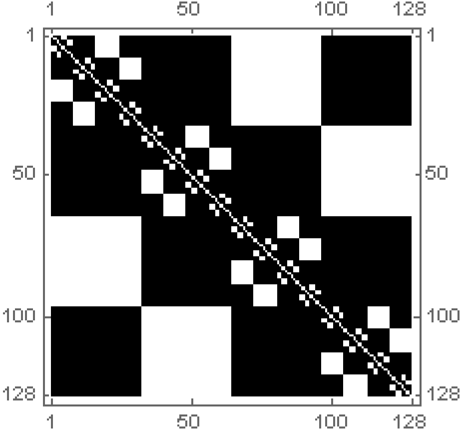
k=2

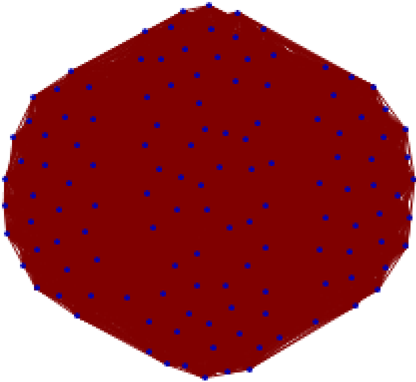


# Example: Adjacency Matrix and graph for $C_4^k[K_2]$

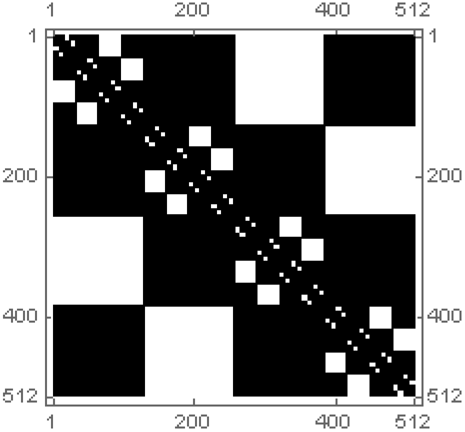
k=3

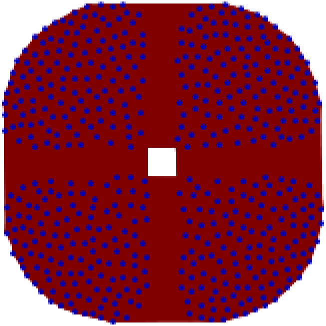


# Example: Adjacency Matrix and graph for $C_4^k[K_2]$

k=4



# Iterated lexicographic products of graphs

## Theorem

Let $G$ be a connected $p$-regular graph of order $m$ with $s$ distinct eigenvalues such that $\sigma_A(G) = \{p, \lambda_2^{[g_2]}(G), \ldots, \lambda_s^{[g_s]}(G)\}$ and let $H$ be a connected $q$-regular graph of order $n$ with $t$ distinct eigenvalues such that $\sigma_A(H) = \{q, \lambda_2^{[h_2]}(H), \ldots, \lambda_t^{[h_t]}(H)\}$, then $H^k[G]$ is a $r_k$-regular graph of order $\nu_k$, such that

$$r_k = mq\frac{n^k - 1}{n-1} + p, \qquad \nu_k = mn^k,$$

$$\sigma_A(H^k[G]) = \left\{\lambda_2^{[n^k g_2]}(G), \ldots, \lambda_s^{[n^k g_s]}(G)\right\} \cup \left\{mq\frac{n^k - 1}{n-1} + p\right\} \cup \Lambda_k$$

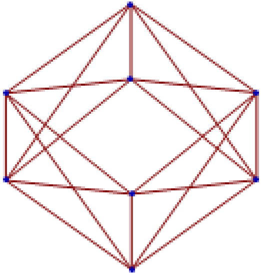
where, for $k \geq 1$,

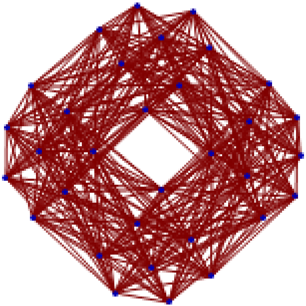
$$\Lambda_k = \bigcup_{i=0}^{k-1} \left\{(mn^i\lambda_2(H) + r_i)^{[n^{k-1-i}h_2]}, \ldots, (mn^i\lambda_t(H) + r_i)^{[n^{k-1-i}h_t]}\right\}.$$

# Example: the spectrum of $H^k[G] = C_4^k[K_2]$.

Notice that $H = C_4$ and $G = K_2$($m = 2$, $n = 4$, $p = 1$ and $q = 2$)

$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$





| $k=0$ | $k=1$ | $k=2$ |
|---|---|---|
| $\nu_0 = 2$ | $\nu_1 = 8$ | $\nu_2 = 32$ |
| $r_0 = 1$ | $r_1 = 5$ | $r_2 = 21$ |
| $\sigma : \{1, -1\}$ | $\{5, (1)^{[2]}, (-1)^{[4]}, (-3)\}$ | $\{21, (5)^{[2]}, (1)^{[8]}, (-1)^{[16]}, (-3)^{[4]}, -11\}$ |

# Spectra of powers of lexicographic products of graphs

## Corollary

Let $H$ be a connected $q$-regular graph of order $n$ with $t$ distinct eigenvalues such that $\sigma_A(H) = \{q, \lambda_2^{[h_2]}(H), \ldots, \lambda_t^{[h_t]}(H)\}$.
Then $H^k$ is a $r_k$-regular graph of order $\nu_k$, such that

$$\begin{aligned} r_k &= q\frac{n^k - 1}{n-1}, \\ \nu_k &= n^k, \end{aligned}$$

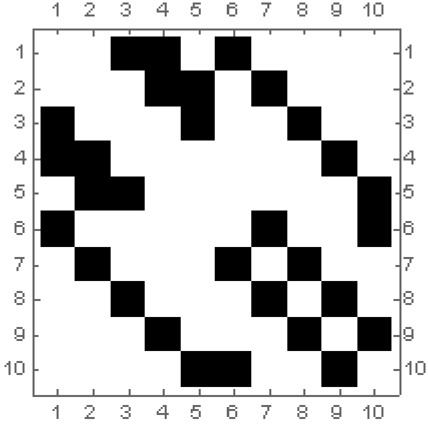
$$\sigma_A(H^k) = \left(\bigcup_{i=0}^{k-1} \left\{(n^i\lambda_2(H) + r_i)^{[n^{k-1-i}h_2]}, \ldots, (n^i\lambda_t(H) + r_i)^{[n^{k-1-i}h_t]}\right\}\right) \cup \{r_k\},$$

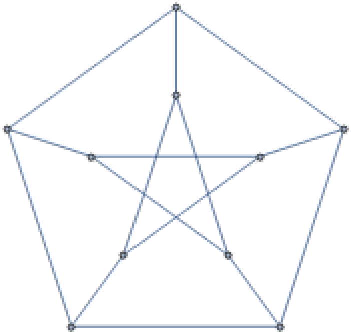
for $k \geq 1$.

The number of distinct eigenvalues of $H^k$ is not greater that $(t-1)k+1$.

# Powers of the Petersen Graph $P$: $\sigma_A(P) = \{-2^{[4]}, 1^{[5]}, 3\}$

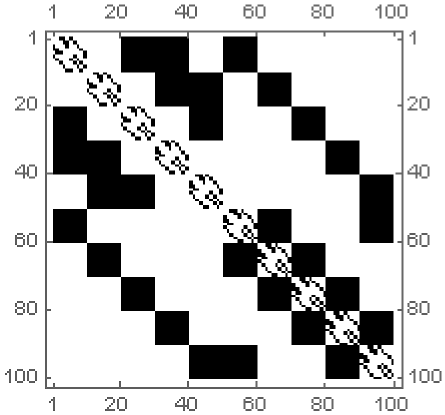
$k = 1$ ; $n = 10$

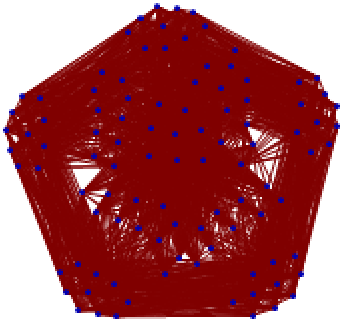


# Powers of the Petersen Graph $P$: $\sigma_A(P) = \{-2^{[4]}, 1^{[5]}, 3\}$

$k = 2$ ; $n = 10^2$

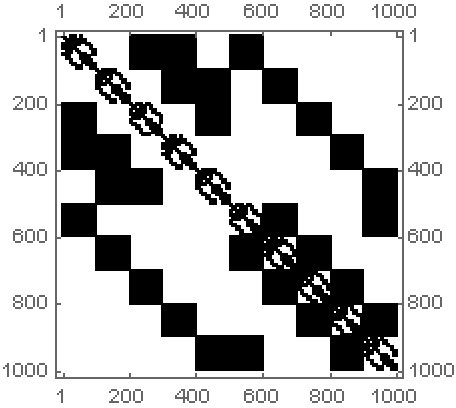


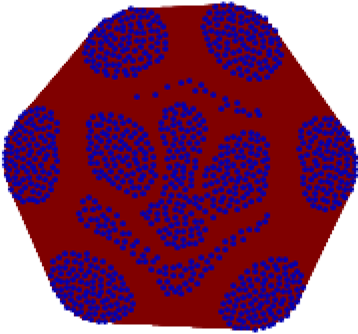



$$\sigma_A(P^2) = \{33, -2^{[40]}, 1^{[50]}, -17^{[4]}, 13^{[5]}\}$$

# Powers of the Petersen Graph $P$: $\sigma_A(P) = \{-2^{[4]}, 1^{[5]}, 3\}$

$k = 3$ ; $n = 10^3$



$$\sigma_A(P^3) = \{333, -2^{[400]}, 1^{[500]}, -17^{[40]}, 13^{[50]}, -167^{[4]}, 133^{[5]}\}$$

# Powers of the Petersen Graph $P$: $\sigma_A(P) = \{-2^{[4]}, 1^{[5]}, 3\}$

And, ... for $k = 10$ ; $n = 10^{10}$

$$\begin{aligned}\sigma_A(P^{10}) \;=\; \Big\{ & 3333333333, -2^{[4000000000]}, 1^{[5000000000]}, \\ & -17^{[400000000]}, 13^{[500000000]}, \\ & -167^{[40000000]}, 133^{[50000000]}, \\ & -1667^{[4000000]}, 1333^{[5000000]}, \\ & -16667^{[400000]}, 13333^{[500000]}, \\ & -166667^{[40000]}, 133333^{[50000]}, \\ & -1666667^{[4000]}, 1333333^{[5000]}, \\ & -16666667^{[400]}, 13333333^{[500]}, \\ & -166666667^{[40]}, 133333333^{[50]}, \\ & -1666666667^{[4]}, 1333333333^{[5]} \Big\}\end{aligned}$$

# Powers of the Petersen Graph $P$: $\sigma_A(P) = \{-2^{[4]}, 1^{[5]}, 3\}$

| $k$ | Spectrum of $P^k$ |
|---|---|
| $k=1$ | $3, \quad 1^{[5]}, \quad -2^{[4]}$ |
| $k=2$ | $33, \quad 13^{[5]}, \ 1^{[50]}, \quad -2^{[40]}, \ -17^{[4]}$ |
| $k=3$ | $333, \ 133^{[5]}, \ 13^{[50]}, 1^{[500]}, \quad -2^{[400]}, \ -17^{[40]}, \ -167^{[4]}$ |
| $k=100$ | $3 \times \sum_{i=0}^{99} 10^i, \quad 1^{[5\times 10^{99}]}, \ -2^{[4\times 10^{99}]},$ $\left(10^m + 3\sum_{i=0}^{m-1} 10^i\right)^{[5\times 10^{99-m}]}, \ m = 1, \ldots, 99,$ $-\left(7 + 10^m + 6\sum_{i=1}^{m-1} 10^i\right)^{[4\times 10^{99-m}]}, \ m = 1, \ldots, 99.$ |

Notice that the graph $P^k$ has $10^k$ vertices, in particular $P^{100}$ has the googol number of vertices $10^{100}$. All the computations were done by Mathematica and lasted just a few seconds.

# The Laplacian spectra of lexicographic compositions

From [C., Freitas, Martins, Robbiano, 2013] we can also deduce the following result.

**Theorem**

Let $G$ be a graph of order $m$ and $H$ a graph of order $n$, then

$$\sigma_L(H[G]) = \left( \bigcup_{j=1}^{n} (m d_H(j) + (\sigma_L(G) \setminus \{0\})) \right) \cup m\sigma_L(H).$$

# The Laplacian spectra of lexicographic compositions

## Theorem

Let $G$ be a connected graph of order $m$ and $H$ be a connected graph of order $n$. Then $H^k[G]$ is a graph of order $\nu_k = mn^k$, and

$$\sigma_L(H^k[G]) = \Omega_G^k \cup \Omega_H^k,$$

where

$$\Omega_G^k = \bigcup_{(j_1,j_2,\ldots,j_k)\in[n]^k} \left(\sigma_L(G) \setminus \{0\} + m\sum_{i=1}^{k} n^{i-1} d_H(j_i)\right)$$

$$\Omega_H^k = \bigcup_{i=2}^{k} \left( \bigcup_{(j_i,\ldots,j_k)\in[n]^{k-i+1}} \left(mn^{i-2}\sigma_L(H)\setminus\{0\} + m\sum_{r=i}^{k} n^{r-1} d_H(j_r)\right)\right) \bigcup mn^{k-1}\sigma_L(H)$$

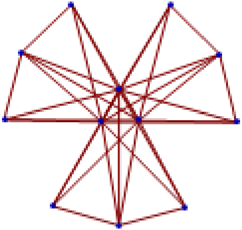
for $k \geq 1$.

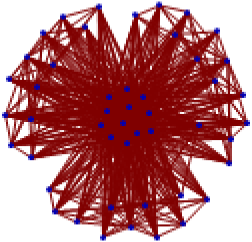
# The Laplacian spectra of lexicographic compositions

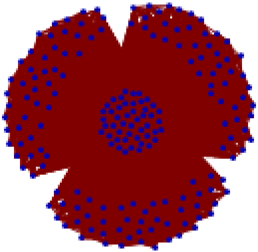
Example: $H^k[G]$ with $H = K_{1,3}$, $G = P_3$

$\sigma_L(K_{1,3}) = \{4, 1, 1, 0\}$; $\sigma_L(P_3) = \{3, 1, 0\}$; $n = 4$; $m = 3$

| $k = 1$ | $k = 2$ | $k = 3$ |
|---|---|---|
| $\nu_1 = 12$ | $\nu_2 = 48$ | $\nu_3 = 192$ |





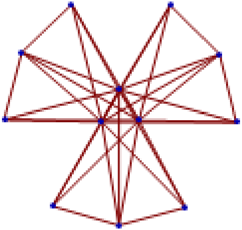


# The Laplacian spectra of lexicographic compositions

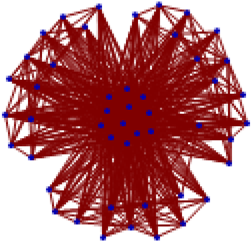
Example: $H^k[G]$ with $H = K_{1,3}$, $G = P_3$

$\sigma_L(K_{1,3}) = \{4, 1, 1, 0\}$; $\sigma_L(P_3) = \{3, 1, 0\}$; $n = 4$; $m = 3$

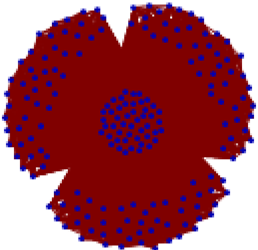
$k = 1$   $\nu_1 = 12$



$k = 2$   $\nu_2 = 48$



$k = 3$   $\nu_3 = 192$



$$\sigma_L(H^3[G]) = \Big\{192^{[4]}, 190, 186^{[3]}, 184^{[3]}, 183^{[2]}, 168^{[6]}, 166^{[3]}162^{[9]}, 160^{[9]}, 159^{[6]}, 156^{[2]}, 96^{[9]}, 94^{[3]}, 90^{[9]}, 88^{[9]}, 87^{[6]}, 72^{[18]}, 70^{[9]}, 66^{[27]}, 64^{[27]}, 63^{[18]}, 60^{[6]}, 48^{[2]}, 0\Big\}$$

# Lexicographic powers of graphs

Now, let us consider the case $G = H$.

## Corollary

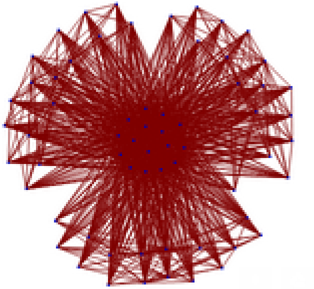
Let $H$ be a connected graph of order $n$. Then $H^k$ is a graph of order $\nu_k = n^k$, such that

$$\sigma_L(H^k) = \bigcup_{i=1}^{k-1} \left( \bigcup_{(j_i,\ldots,j_{k-1}) \in [n]^{k-i}} \left( n^{i-1}\sigma_L(H) \setminus \{0\} + \sum_{r=i}^{k-1} n^r d_H(j_r) \right) \right) \bigcup n^{k-1}\sigma_L(H)$$

for $k \geq 1$.

# Lexicographic powers of graphs

Example: $H = K_{1,3}$:
$n = 4$,
$\sigma_L(K_{1,3}) = \{4, 1, 1, 0\}$.



$H^3 = K_{1,3}^3$ is a graph of order

$$\nu_3 = 4^3 = 64$$

with

$$\sigma_L(K_{1,3}^3) = \bigcup_{i=1}^{2} \left( \bigcup_{(j_i, j_2) \in [4]^{3-i}} \left( 4^{i-1} \sigma_L(K_{1,3}) \setminus \{0\} + \sum_{r=i}^{2} 4^r d_H(j_r) \right) \right) \bigcup 4^2 \sigma_L(K_{1,3})$$

which gives $\sigma_L(K_{1,3}^3) =$

$$\left\{ 64^{[3]}, 61^{[2]}, 56^{[2]}, 53^{[6]}, 52^{[2]}, 32^{[6]}, 29^{[6]}, 24^{[9]}, 21^{[18]}, 20^{[6]}, 16^{[2]}, 0 \right\}$$

# Some combinatorial properties of $H^k$

- **Laplacian index**

$$\mu_1(H^k) = n^{k-1}\mu_1(H).$$

- **Algebraic connectivity**

$$\mu_{n^k-1}(H^k) = n^{k-1}\mu_{n-1}(H).$$

- **Minimum and maximum degree**

$$\delta(H^k) = \delta(H)\frac{n^k-1}{n-1} \text{ and } \Delta(H^k) = \Delta(H)\frac{n^k-1}{n-1}.$$

# Some combinatorial properties of $H^k$

- **Stability number**

  As $\alpha(H[G]) = \alpha(H)\,\alpha(G)$ [Geller, 1975], where $G$ is an arbitrary graph

  $$\alpha(H^k) = \alpha(H)^k.$$

  Furthermore, from the spectral upper bound [(Godsil (2008) and Lu (2007)] for an arbitrary graph $G$

  $$\alpha(G) \leq n \; \frac{\mu_1(G) - \delta(G)}{\mu_1(G)},$$

  we obtain,

  $$\begin{aligned} \alpha(H^k) &\leq n^k \frac{\mu_1(H^k) - \delta(H^k)}{\mu_1(H^k)} \\ &= n^k \frac{\mu_1(H) - \delta(H)}{\mu_1(H)}. \end{aligned}$$

# Some combinatorial properties of $H^k$

- **Vertex connectivity**

  As the lexicographic product $H[G]$ is connected if and only if $H$ is a connected graph [Harary and Wilcox, 1967] if both $G$ and $H$ are not complete [Geller and Stahl, 1975 ]

$$\upsilon(H[G]) = m\,\upsilon(H),$$

  where $\upsilon(H)$ denotes the vertex connectivity of $H$. So,

$$\upsilon(H^k) = n^{k-1}\upsilon(H).$$

  Furthermore, we may conclude that when $H$ is connected not complete (and then $H^k$ is also connected not complete),

$$n^{k-1}\mu_{n-1}(H) \leq \upsilon(H^k) \leq \delta(H)\frac{n^k - 1}{n - 1}.$$

# Some combinatorial properties of $H^k$

- **Chromatic number**

It is well known the following lower bound due to Hoffman

$$\chi(G) \;\; \geq \;\; 1 - \frac{\lambda_1(G)}{\lambda_n(G)}.$$

As direct consequence, if a graph $H$ is $q$-regular of order $n$,

$$\begin{aligned} \chi(H^k) &\geq 1 - \frac{r_k}{\lambda_{n^k}(H^k)} \\ &= 1 - \frac{n^k - 1}{n^{k-1}\left((n-1)\frac{\lambda_n(H)}{q} + 1\right) - 1}. \end{aligned}$$

# Some combinatorial properties of $H^k$

## Proposition

Let $H$ be a connected not complete graph and let $G$ be an arbitrary graph of order $m$. For every $k \in \mathbb{N}$

$$diam(H^{k+1}) = diam(H^k[G]) = \mathrm{diam}(H).$$

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# THANK YOU!

Ten years after, congratulations again Professor Dragos Cvetković

This work was supported in part by the Portuguese Foundation for Science and Techonology (FCT - Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.