Resolvent Energy of Graphs

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Introduction

I. Gutman, B. Furtula, E. Zogić, E. Glogić, Resolvent energy of graphs, *MATCH Communications in Mathematical and in Computer Chemistry* **75** (2016) 279 - 290.

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- M-square matrix of order n; Resolvent matrix of M:

$$\mathcal{R}_M(z) = (zI_n - M)^{-1} \tag{1}$$

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- G-simple graph; A-adjacency matrix; $\lambda_1, \lambda_2, ..., \lambda_n$ eigenvalues
- The k-th spectral moment of G:

$$M_k(G) = \sum_{i=1}^{n} (\lambda_i)^k \tag{2}$$

$M_0 = n, M_1 = 0, M_2 = 2m,\ M_k = 0$ for all odd values of k if and only if G is bipartite.

1. D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs Theory and Application, 1980.
2. T. S. Shores, Linear Algebra and Matrix Analysis, 2007.

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- Energy of graph G

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{3}$$

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- Resolvent matrix of adjency matrix A

$$\mathcal{R}_A(z) = (zI_n - A)^{-1} \tag{4}$$

$$\frac{1}{z - \lambda_i}, i = 1, 2, ..., n - \text{eigenvalues of } \mathcal{R}_A(z) \tag{5}$$

$\lambda_i \leq n - 1$ is satisfied by all eigenvalues of all n-vertex graphs.

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Resolvent energy of graph G

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i} \tag{6}$$

Introduction

Resolvent Estrada index

$$EE_r(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{(n-1)^k} \tag{7}$$

$$EE_r(G) = \sum_{i=1}^{n} \frac{n-1}{n-1-\lambda_i} \tag{8}$$

EE_r is undefined in the case of the complete graph K_n.

- X. Chen, J. Qian, Bounding the resolvent Estrada index of a graph, (2012)
- X. Chen, J. Qian, On resolvent Estrada index, (2015)
- I. Gutman, B. Furtula, X. Chen, J. Qian, Graphs with smallest resolvent Estrada indices (2015)
- I. Gutman, B. Furtula, X. Chen, J. Qian, Resolvent Estrada index - computational and mathematical studies, (2015)

Basic properties of resolvent energy

Theorem 1.

If G is graph of order n and $M_k(G)$, $k = 0, 1, 2, \ldots$ its spectral moments, then

$$ER(G) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{M_k(G)}{n^k}. \qquad (9)$$

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Corollary 1.

If e is a edge of the graph G, denote by $G - e$ the subgraph obtained by deleting e from G. Then for any edge e of G,

$$ER(G - e) < ER(G).$$

Basic properties of resolvent energy

Corollary 2.

Let K_n be the complete graph of order n and $\overline{K_n}$ the edgeless graph of order n. Then for any graph G of order n, different from K_n and $\overline{K_n}$,

$$ER(\overline{K_n}) < ER(G) < ER(K_n), \tag{10}$$

$$1 \leq ER(G) \leq \frac{2n}{n+1}. \tag{11}$$

$ER(G) = 1$ if and only if $G \cong \overline{K_n}$ while $ER(G) = \frac{2n}{n+1}$ if and only if $G \cong K_n$.

Basic properties of resolvent energy

Corollary 3.

Among connected graphs of order n, the graph with the smallest resolvent energy is a tree.

Theorem 2.

Amog trees of order n, the path P_n has the smallest and the star S_n has the greatest resolvent energy.

- H. Deng, A proof of a conjecture on the Estrada index, (2009)

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$$M_{2k}(P_n) \leq M_{2k}(T) \leq M_{2k}(S_n)$$

$$M_{2k}(P_n) < M_{2k}(T) < M_{2k}(S_n), \quad \text{for} \quad k \geq 2$$

$$ER(G) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{M_k(G)}{n^k}$$

Theorem 3.

Let G be a graph of order n and $\phi(G,\lambda) = \det(\lambda I_n - A(G))$ ne its characteristic polynomial. Then

$$ER(G) = \frac{\phi'(G,n)}{\phi(G,n)}, \tag{12}$$

where $\phi'(G,\lambda)$ is the first derivative of $\phi(G,\lambda)$.

Theorem 4.

Let G be a graph with n vertices and m edges. Then

$$\frac{n^3}{n^3-2m} \leq ER(G) \leq 1 + \frac{2m(2n-1)}{n^2(n^2-2m)}. \tag{13}$$

Equality on the left-hand side is attained if and only if $G \cong \overline{K_n}$ or (provided n as even) $G \cong \frac{n}{2}K_2$. Equality on the right-hand side is attained if and only if $G \cong \overline{K_n}$.

Estimating the resolvent energy

Proposition 1.

Let G be a graph with n vetrices, m edges, and nullity n_0. Then

$$ER(G) \geq \frac{n_0}{n} + \frac{n(n-n_0)^2}{n^2(n-n_0)-2m}. \tag{14}$$

Proposition 2.

If G is bipartite graph with n vertices and m edges where n is odd number, then $n_0 \geq 1$ and

$$ER(G) \geq \frac{1}{n} + \frac{n(n-1)^2}{n^2(n-1)-2m}. \tag{15}$$

- D. Cvetković, I. Gutman, The algebraic multiplicity of the number zero in the spectrum of a bipartite graph, (1972)

Estimating the resolvent energy

Proposition 3.

Let G be a bipartite graph with n vertices and m edges. Then

$$ER(G) \leq 1 + \frac{2m}{n(n^2 - m)}. \qquad (16)$$

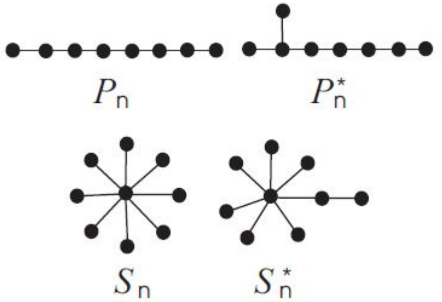
Equality is attained if and only if either $G \cong \overline{K_n}$ or $G \cong K_{a,b} \cup \overline{K}_{n-a-b}$, where $K_{a,b}$ is a complete bipartite graph such that $ab = m$.

Computational studies on resolvent energy

The ER-values of all trees and connected unicyclic, and bicyclic graphs up to 15 vertices were computed, and the structure of the extremal members of these classes was established.
First of all, the results of Theorem 2 were confirmed and extended:

Observation 1.

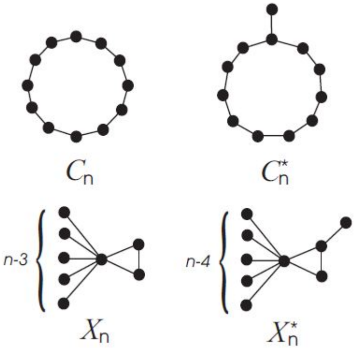
Among trees of order n, the path P_n has the smallest and the tree P_n^* second-smallest resolvent energy. Among trees of order n, the star S_n has the greatest and the tree S_n^* second-greatest resolvent energy.



Computational studies on resolvent energy

Observation 2.

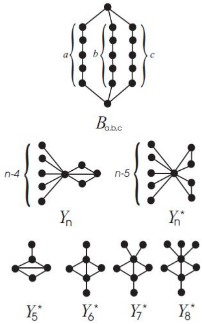
Among connected unicyclic graphs of order n, $n \geq 4$, the cycle C_n has the smallest and the graph C_n^* second-smallest reslovent energy. Among these graphs of order n, $n \geq 5$, the graphs X_n and X_n^* have, respectively, the greatest and second-greatest resolvent energy.



Obsevation 3. Among connected bicyclic graphs of order n, those with the smallest, (a), and second-smallest, (b), resolvent energy are:

$$(a)\ \begin{array}{lll} B_{p-1,p-1,p} & \textit{if} & n=3p,\ p\geq 2 \\ B_{p-1,p,p} & \textit{if} & n=3p+1,\ p\geq 1 \\ B_{p,p,p} & \textit{if} & n=3p+2,\ p\geq 1. \end{array} \qquad (b)\ \begin{array}{lll} B_{p-2,p,p} & \textit{if} & n=3p,\ p\geq 2 \\ B_{p-1,p-1,p+1} & \textit{if} & n=3p+1,\ p\geq 2 \\ B_{p-1,p,p+1} & \textit{if} & n=3p+2,\ p\geq 1. \end{array}$$

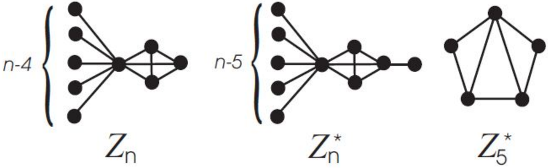
Among these graphs of order $n, n \geq 5$, the graph Y_n has the greatest resolvent energy. For $n \geq 9$, the graph Y_n^* has second-greatest resolvent energy, whereas Y_5^*, Y_6^*, Y_7^* and Y_8^* are exceptions.



Computational studies on resolvent energy

Observation 4.

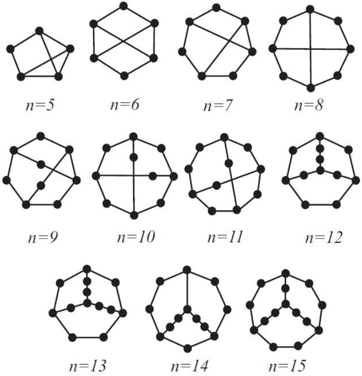
Among connected tricyclic graphs of order n, $n \geq 5$, the graph Z_n has the greatest resolvent energy. For $n \geq 6$, graph Z_n^* has second-greatest energy, whereas Z_5^* is exception.



Computational studies on resolvent energy

Observation 5.

Connected tricyclic graphs of order $n, 5 \leq n \leq 15$ with the smallest ER, are depicted in the following figure.



Computational studies on resolvent energy

Observation 6.

- The inequality $ER(S_n) < ER(C_n)$ holds for all $n \geq 4$. Consequently, any tree has smaller ER-value than any unicyclic graph of the same order.
- For $B_{a,b,c}$ specified by (a) from Observation 3, the inequality $ER(X_n) < ER(B_{a,b,c})$ holds only until $n = 6$ and is violated for all $n \geq 7$. Consequently, it is not true that any unicyclic graph has smaller ER-value than any bicyclic graph of the same order.
- The same applies also to the relation between ER of bicyclic and tricyclic graphs.
- Any unicyclic graph has smaller ER-value than any connected tricyclic graph of the same order.

Observation 7.

- Cospectral graphs have equal *ER*-values.
- There are non-cospectral graphs whose *ER*-values are different, but remarkably close. $ER(B_{3,3,3}) = 1.018571022$ whereas $ER(B_{2,3,4}) = 1.018571080$, and $ER(B_{4,4,5}) = 1.0096261837436$ whereas $ER(B_{3,5,5}) = 1.0096261837458$.

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THANK YOU FOR YOUR ATTENTION!

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